Lec 17: Inverse of a matrix and Cramer's rule

We are aware of algorithms that allow to solve linear systems and invert a matrix. It turns out that determinants make possible to find those by explicit formulas. For instance, if A is an $n \times n$ invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}. \quad (1)$$

Note that the (i,j) entry of matrix (1) is the cofactor A_{ji} (not A_{ij}!). In fact the entry is $\frac{A_{ji}}{\det(A)}$ as we multiply the matrix by $\frac{1}{\det(A)}$. [We can divide by $\det(A)$ since it is not 0 for an invertible matrix.] Curiously, in spite of the simple form, formula (1) is hardly applicable for finding A^{-1} when n is large. This is because computing $\det(A)$ and the cofactors requires too much time for such n. Notice that $\det(A)$ can be found as soon as we know the cofactors, because of the cofactor expansion formula.

Example. Find the inverse, if it exists, for

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -2 & 3 & -1 \\ 4 & 0 & 1 \end{bmatrix}.$$

We have:

$$A_{11} = \begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix} = 3, \quad A_{12} = -\begin{vmatrix} -2 & -1 \\ 4 & 1 \end{vmatrix} = -2, \quad A_{13} = \begin{vmatrix} -2 & 3 \\ 4 & 0 \end{vmatrix} = -12.$$

Find the determinant by the expansion along the first row:

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 0 \cdot 3 + 1 \cdot (-2) + 2 \cdot (-12) = -26.$$

Since $\det(A) \neq 0$, we conclude that A is invertible, and we can continue computing cofactors[1]:

$$A_{21} = -\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -1, \quad A_{22} = \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = -8, \quad A_{23} = -\begin{vmatrix} 0 & 1 \\ 4 & 0 \end{vmatrix} = 4,$$

$$A_{31} = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -7, \quad A_{32} = -\begin{vmatrix} 0 & 2 \\ -2 & -1 \end{vmatrix} = -4, \quad A_{33} = \begin{vmatrix} 0 & 1 \\ -2 & 3 \end{vmatrix} = 2.$$

By formula (1)

$$A^{-1} = -\frac{1}{26} \begin{bmatrix} 3 & -1 & -7 \\ -2 & -8 & -4 \\ -12 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{26} & \frac{1}{26} & \frac{7}{26} \\ \frac{1}{13} & \frac{4}{13} & \frac{2}{13} \\ \frac{6}{13} & -\frac{2}{13} & -\frac{1}{13} \end{bmatrix}.$$

The method of finding A^{-1} using the augmented matrix $[A|I_3]$ seems to be faster for the previous example.

It worth mentioning that in case of 2×2 matrix A formula (1) is especially simple:

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } \det(A) = ad - bc \neq 0, \text{ then } A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

[1] If the determinant were 0, we would stop here and say that A is singular (there is no need to find rest cofactors).

Make sure that $AA^{-1} = I_2$ (thus you will prove formula (1) for the case $n = 2$). For example,

$$\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix}.$$

Now describe the Cramer's rule for solving linear systems $A\bar{x} = \bar{b}$. It is assumed that A is a square matrix and $\det(A) \neq 0$ (or, what is the same, A is invertible). Then, as we know, the linear system has a unique solution. The rule says that this solution is given by the formula

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \dots, \quad x_n = \frac{\det(A_n)}{\det(A)}, \tag{2}$$

where A_i is the matrix obtained from A by replacing the i^{th} column of A by $\bar{b}$. [Don't confuse with cofactors A_{ij}!]

Example. Solve the linear system

$$\begin{aligned} 3x_1 + x_2 - 2x_3 = 4 \\ -x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + x_2 + 4x_3 = -2. \end{aligned}$$

We have (check all calculations!)

$$\det(A) = \begin{vmatrix} 3 & 1 & -2 \\ -1 & 2 & 3 \\ 2 & 1 & 4 \end{vmatrix} = 35$$

Since $\det(A) \neq 0$, we can use the Cramer's rule. Let's find determinants of A_1, A_2, A_3:

$$\det(A_1) = \begin{vmatrix} 4 & 1 & -2 \\ 1 & 2 & 3 \\ -2 & 1 & 4 \end{vmatrix} = 0, \ \det(A_2) = \begin{vmatrix} 3 & 4 & -2 \\ -1 & 1 & 3 \\ 2 & -2 & 4 \end{vmatrix} = 70, \ \det(A_3) = \begin{vmatrix} 3 & 1 & 4 \\ -1 & 2 & 1 \\ 2 & 1 & -2 \end{vmatrix} = -35.$$

Now by formula (2):

$$x_1 = \frac{0}{35} = 0, \quad x_2 = \frac{70}{35} = 2, \quad x_3 = -\frac{35}{35} = -1.$$

Thus $0, 2, -1$ is the solution to our system.

As before, in case of the linear system with two equations and two variables the solution is particularly simple. Consider the system

$$\begin{aligned} ax + by = e \\ cx + dy = f \end{aligned}$$

with unknowns x and y. If $ad - bc \neq 0$, then by Cramer's rule

$$x = \frac{de - bf}{ad - bc}, \quad y = \frac{af - ce}{ad - bc}.$$

Make sure that these satisfy to the above system (thus you will prove Cramer's rule for 2×2 case). For example, the system

$$\begin{aligned} x + 3y = 0 \\ 2x + 7y = 1 \end{aligned}$$

has the solution $x = -\frac{3}{1} = -3$, $y = \frac{1}{1} = 1$.