Dot and Cross Products (13.3, 13.4) — Name: Solutions

Math 1920 - Andres Fernandez — August 29, 2017

Summary of the Sections

(1) Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$. Then $\mathbf{a} \cdot \mathbf{b} = \boxed{a_1 b_1 + a_2 b_2 + a_3 b_3}^{(1)}$

(2) If we are given the lengths $\|\mathbf{u}\|$, $\|\mathbf{v}\|$ and angle θ between two vectors, then $\mathbf{u} \cdot \mathbf{v} = \boxed{\|\mathbf{u}\| \|\mathbf{v}\| \cos(\theta)}^{(2)}$

(3) Distributivity: $(\mathbf{w} + \mathbf{v}) \cdot \mathbf{u} = \boxed{\mathbf{w} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u}}^{(3)}$.

(4) Scalar multiplication: $\boxed{\lambda(\mathbf{u} \cdot \mathbf{v}) = (\lambda \mathbf{u}) \cdot \mathbf{v}}^{(4)}$

(5) Two vectors $\mathbf{u}, \mathbf{v}$ are orthogonal when $\boxed{\mathbf{u} \cdot \mathbf{v} = 0}^{(5)}$.

(6) The angle between two vectors $\mathbf{u}$ and $\mathbf{v}$ is given by $\theta = arcos\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\| \|\mathbf{u}\|}\right)$

(7) Given $\mathbf{a}, \mathbf{b}$ the components of $\mathbf{a}$ with respect to $\mathbf{b}$ are:

-Tangential Component: $\mathbf{a}_{||\mathbf{b}} = \boxed{\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\right) \mathbf{b}}^{(6)}$

-Normal (Perpendicular) Component: $\mathbf{a}_{\perp \mathbf{b}} = \boxed{\mathbf{a} - \mathbf{a}_{||\mathbf{b}}}^{(7)}$

(8) $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, has length $\boxed{\|\mathbf{u}\| \|\mathbf{v}\| sin(\theta)}^{(8)}$ and direction given by the right hand rule.

(9) *Formula for Cross Product*:

$$\boxed{\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}}^{(9)}$$

(10) $\|u \times v\|$ is the area of the parallelogram spanned by u and v

Problems

(1) Let v and w be vectors with $\|v\| = 3$ and $\|w\| = 5$.

(a) If we know that $v \cdot w = -1$, what is $\|v + w\|$?
Solution: $\sqrt{32}$

(b) Now we only know that the angle between the vectors is $60°$, what is $\|2v - w\|$.
Solution: $\sqrt{31}$

(c) There is a vector u that is perpendicular to v. If $\|3v + u\| = 15$, what is the length of u?
Solution: 12

(d) Suppose that $\|v - 2w\| = \sqrt{109}$. What is the angle between u and w.
Solution: $90°$

(2) Let $u = \langle -3, 6, 1\rangle$ and $v = \langle 2, -7, 4\rangle$. What are the tangential and normal components of u with respect to v.

Solution: $u_{||v} = \langle \frac{-88}{69}, \frac{308}{69}, \frac{-176}{69}\rangle$ $u_{\perp v} = \langle \frac{-119}{69}, \frac{206}{69}, \frac{245}{69}\rangle$

(3) Compute the length of the component of v along u as given in the board.

Solution: Solved in discussion.

(4) Let $v = \langle a, 4, 7\rangle$, $w = \langle -1, 2, 3\rangle$ and $u = \langle b^2, 5, b\rangle$. If u is orthogonal to w and v is orthogonal to w, can we conclude that v and u are orthogonal? Find $a, b, \|v\|, \|u \times w\|$.

Solution: No, we can't conclude that they are orthogonal (think this through, u and v could be the same vector).

(a)
$$\begin{aligned} v \perp w \implies v \cdot w &= 0 \\ -a + 8 + 21 &= 0 \\ a &= -29 \end{aligned}$$

(b)
$$\begin{aligned} u \perp w \implies u \cdot w &= 0 \\ -b^2 + 10 + 3b &= 0 \\ b &= 5, -2 \end{aligned}$$

(c)
$$\|v\| = \sqrt{(-29)^2 + 4^2 + 7^2} = \sqrt{309}$$

(d) We can compute $\|w\| = \sqrt{14}$. Since u, v are orthogonal

$$\|v \times w\| = \|v\|\|w\| sin(90°) = \sqrt{309} \cdot \sqrt{14} \cdot 1 = \sqrt{4326}$$

In general if we don't know the angle we would have to use the formula for cross product to get the vector, and then compute its length.

(5) Let $A = (4, 10, 7), B = (-1, 4, 1)$ and $C = (3, 7, 2)$. Compute the angle $\widehat{ABC}$ and the area of the triangle formed by the three points. Compute the area of the parallelogram $ABCD$ formed by adding one more point D.

Solution:

(a) We want to find the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$. First we compute $\overrightarrow{BA} = A - B = \langle 5, 6, 6\rangle$ and $\overrightarrow{BC} = \langle 4, 3, 1\rangle$. Then, we use the formula:

$$\theta = arcos\left(\frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\|\overrightarrow{BA}\|\|\overrightarrow{BC}\|}\right) = arcos\left(\frac{44}{\sqrt{2522}}\right) = 28.82°$$

(b) In order to compute these areas we need $\|\overrightarrow{BA} \times \overrightarrow{BC}\|$, and we can do this in two ways. We could use the cross product formula to get a vector and then compute its length. A shorter way, since we already now the angle, is:

$$\|\overrightarrow{BA} \times \overrightarrow{BC}\| = \|\overrightarrow{BA}\|\|\overrightarrow{BC}\| \sin(28.82°) = \sqrt{97}\sqrt{26}\sin(28.82°) = 24.21$$

And therefore we get:

$$A_{Parallelogram} = \|\overrightarrow{BA} \times \overrightarrow{BC}\| = 24.21$$

$$A_{Triangle} = \frac{\|\overrightarrow{BA} \times \overrightarrow{BC}\|}{2} = 12.1$$

(6) Find the volume of the parallelepiped spanned by $\langle 5, 8, 4\rangle$, $\langle -2, 3, 5\rangle$ and $\langle 4, 0, 4\rangle$.

Solution: The volume of the parallelepiped spanned by three vectors **a**, **b** **c** is given by the triple product, which is computed by the following determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

In this case we are given the three vectors, so we can just plug them in to get:

$$\begin{vmatrix} 5 & 8 & 4 \\ -2 & 3 & 5 \\ 4 & 0 & 4 \end{vmatrix} = 236$$

(7) Calculate the net torque shown on the board.

Solution: In this kind of problems one is presented with a force $\mathbf{F}$ applied at a point P. Usually we have to compute the torque with respect to the origin, which is given by $\|\mathbf{r} \times \mathbf{F}\| = \|\mathbf{r}\|\|\mathbf{F}\| sin(\theta)$, where $\mathbf{r}$ is the position vector $\overrightarrow{OP}$.

(8) (Challenge) Show that if $a \perp b$ then there is a unique vector c with $a \times c = b$