## Section 17.1, 17.2

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## Problems

(1) Sketch the following vector fields:

(a) $\langle y, -3\rangle$

(b) $\nabla f, \quad$ with $f = y^2 - x$

(c) $\frac{\mathbf{e_r}}{r}$

(2) Prove the following:

(a) $div(f\mathbf{F}) = f \quad div(\mathbf{F}) + \nabla f \cdot \mathbf{F}$

(b) $curl(f\mathbf{F}) = f \quad curl(\mathbf{F}) + \nabla f \times \mathbf{F}$

(c) $curl(\nabla f) = 0$

Solution:

(a)

(b) (Done in class) The proof of parts $a, b$ is conceptually the same, you have to expand both sides of the equality above using the product rule, and you will get the same expressions. I do not recommend trying this as an exercise before the prelim, since it is long and not very illuminating.

(c) This is a very important equality for $f$ smooth (or at least $C^2$). It tells us that **if the curl is not 0, then the vector field is not conservative**. The proof goes as follows,

$$curl(\nabla f) = curl\left(\left\langle \frac{\partial}{\partial x}f, \frac{\partial}{\partial y}f, \frac{\partial}{\partial z}f\right\rangle\right) = \left\langle \frac{\partial}{\partial y}\frac{\partial}{\partial z}f - \frac{\partial}{\partial z}\frac{\partial}{\partial y}f,\ \frac{\partial}{\partial z}\frac{\partial}{\partial x}f - \frac{\partial}{\partial x}\frac{\partial}{\partial z}f,\ \frac{\partial}{\partial x}\frac{\partial}{\partial y}f - \frac{\partial}{\partial y}\frac{\partial}{\partial x}f\right\rangle$$

By commutativity of the mixed partials, we get that all three coordinates above cancel, and so $curl(\nabla f) = \langle 0, 0, 0\rangle$.

(3) Show that if $F_1, F_2, F_3$ are functions of one variable, then $curl\left(\langle F_1(x), F_2(y), F_3(z)\rangle\right) = 0$

Solution:

Basically the information that we are give is that some partials are zero (the coordinates of our vector field do not depend on all of the variables). We can write down the curl to get:

$$curl(\mathbf{F}) = \left\langle \frac{\partial}{\partial y}\mathbf{F}_3(z) - \frac{\partial}{\partial z}\mathbf{F}_2(y),\ \frac{\partial}{\partial z}\mathbf{F}_1(x) - \frac{\partial}{\partial x}\mathbf{F}_3(z), \frac{\partial}{\partial x}\mathbf{F}_2(y) - \frac{\partial}{\partial y}\mathbf{F}_1(x)\right\rangle$$

Notice that all partials here vanish, since the variables of the derivative and the functions never match, so we get that $curl(\mathbf{F}) = \mathbf{0}$.

(4) Find potentials for the following vector fields, or show that one does not exists:

(a) $F = \langle 2xyz, x^2z, x^2yz\rangle$

(b) $F = \langle yz^2, xz^2, 2xyz\rangle$

(c) $F = \frac{\mathbf{e_r}}{r^3}$

(d) $F = \frac{\mathbf{e_r}}{r^4}$

(e) $\langle e^y - 2, x^2 - 3, z \rangle$

SOLUTION:

(a) You can start to try to find a potential, and you will see that it is impossible, since the functions don't match up. This means that such a potential does not exists, and once we are in this situation we have to look at the curl and show that is is not zero. In order to do this it suffices to look at one of the coordinates of the curl.

In this case you can try them all, since the functions involved are not too complicated. You will see that the first coordinate of the curl is $\frac{\partial}{\partial y}\mathbf{F}_3 - \frac{\partial}{\partial z}\mathbf{F}_2 = x^2z - x^2 \neq 0$.

Therefore the vector field is not conservative.

(b) This looks very similar to the previous one. However if we try to find a potential, we will see that things indeed match up. If we integrate each of the coordinates as usual:

i. $\frac{\partial}{\partial x} f = yz^2 \longrightarrow f = xyz^2 + C_1(y,z)$

ii. $\frac{\partial}{\partial y} f = xz^2 \longrightarrow f = xyz^2 + C_2(x,z)$

iii. $\frac{\partial}{\partial z} f = 2xyz \longrightarrow f = xyz^2 + C_3(y,x)$

Therefore $f = xyz^2$, and the field is conservative.

(c) Using in two dimensions (three dimensions will work in the same way) $\mathbf{e_r} = \frac{1}{\sqrt{x^2+y^2}}\langle x, y\rangle$ and $r = \sqrt{x^2+y^2}$, we get

$$\frac{\mathbf{e_r}}{r^3} = \frac{1}{(x^2+y^2)^2}\langle x, y\rangle$$

We can take the integrals of the corresponding coordinates to get:

i. $\frac{\partial}{\partial x} f = \frac{x}{(x^2+y^2)^2} \longrightarrow f = \frac{-1}{2(x^2+y^2)} + C_1(y)$

ii. $\frac{\partial}{\partial y} f = \frac{y}{(x^2+y^2)^2} \longrightarrow f = \frac{-1}{2(x^2+y^2)} + C_2(x)$

Therefore we get $f = \frac{-1}{2(x^2+y^2)}$.

(d) This computation is going to be very similar to the one above, we have $\frac{\mathbf{e_r}}{r^4} = \frac{1}{(x^2+y^2)^{\frac{5}{2}}}\langle x, y\rangle$. So we get:

i. $\frac{\partial}{\partial x} f = \frac{x}{(x^2+y^2)^{\frac{5}{2}}} \longrightarrow f = \frac{-1}{3(x^2+y^2)^{\frac{3}{2}}} + C_1(y)$

ii. $\frac{\partial}{\partial x} f = \frac{y}{(x^2+y^2)^{\frac{5}{2}}} \longrightarrow f = \frac{-1}{3(x^2+y^2)^{\frac{3}{2}}} + C_2(x)$

Therefore $f = \frac{-1}{3(x^2+y^2)^{\frac{3}{2}}}$ is the required potential.

(e) Notice that the first coordinate is an exponential, whereas the second coordinate is a polynomial in $x$. This can lead you to suspect that the corresponding partials when we compute the curl won't cancel out. In fact this will be the case, and the third coordinate of the curl will be:

$$\frac{\partial}{\partial x}\mathbf{F}_2 - \frac{\partial}{\partial y}\mathbf{F}_1 = 2x - e^y \neq 0$$

Hence the vector field is not conservative.

(5) Compute the line integrals of the following vectors along the given path:

(a) $F = \langle z, x^2, y \rangle, \quad \mathbf{r}(t) = \langle \cos(t), \tan(t), t \rangle \quad$ with $0 \leq t \leq \frac{\pi}{4}$

(b) $F = \langle e^x, e^y, xyz \rangle, \quad \mathbf{r}(t) = \langle t^2, t, \frac{t}{2} \rangle \quad$ for $0 \leq t \leq 1$

SOLUTION: See the next handout for line integrals.