Sections 18.2 | Name: Solutions
Math 1920 - Andres Fernandez | November 21, 2017

## Problems

(1) Verify Stoke's theorem for $\mathbf{F} = \langle yz, 0xz\rangle$ and $\mathcal{S}$ the portion of the plane $\frac{x}{2}+\frac{y}{3}+z=1$, where $x, y, z \geq 0$.

Solution: This is not really a problem about the use of Stoke's theorem, I just included it to refresh how to compute flux/circulation along a given surface/path. The surface is going to be a triangle. Therefore for the circulation you will have to break the integral into three straight paths.

If you want to try this problem the final answer is -1, but I do not really recommend doing it as practice for Stoke's theorem.

(2) Suppose that a vector field $\mathbf{F}$ has a vector potential. What is the flux of $\mathbf{F}$ across any closed surface? Use this to compute the flux of $\mathbf{F} = \langle y, z, 0\rangle$ along the given surface if we assume that this has a vector potential.

Solution: The Figure shown on the board was some horrible surface with boundary given by the unit circle in the xy-plane. I have included a picture below from the next handout with a very similar surface (for our purpose the same).

Let $\mathcal{S}_1$ be the surface shown in the picture, and let $\mathcal{S}_2$ be the unit circle in the xy-plane. Then The union of $\mathcal{S}_1$ and $\mathcal{S}_2$ is a closed surface. Since $\mathbf{F}$ has vector potential, we know (by Stoke's or by divergence, you can prove this using either theorem) that:

$$\int\int_{\mathcal{S}_1} \mathbf{F}\cdot d\mathbf{S} + \int\int_{\mathcal{S}_2} \mathbf{F}\cdot d\mathbf{S} = 0$$

Where the normal has to be consistent (always pointing outward). Therefore, we get

$$\int\int_{\mathcal{S}_1} \mathbf{F}\cdot d\mathbf{S} = -\int\int_{\mathcal{S}_2} \mathbf{F}\cdot d\mathbf{S} = -\int\int_{\mathcal{S}_2} \mathbf{F}\cdot \mathbf{n} dS$$

Notice that the normal for $\mathcal{S}_2$ is $\mathbf{n} = \langle 0, 0, -1\rangle$ (it has to point outwards, you can see it geometrically in the picture below). Hence, $\mathbf{F}\cdot\mathbf{n} = 0$, and so the integral over $\mathcal{S}_2$ above is 0.

Alternatively you can parametrize the unit circle $S(r,\theta) = \langle r\cos(\theta), r\sin(\theta), 0\rangle$ and set up the integral with this parametrization (compute $\mathbf{N} = S_r \times S_\theta$ etc.)

Either way when you set up the integral you get that the final answer is 0.

(3) Let $I$ be the flux of $\mathbf{F} = \langle e^y, 2xe^{x^2}, z^2\rangle$ through the upper hemisphere $\mathcal{S}$ of the unit sphere.

(a) Let $\mathbf{G} = \langle e^y, 2xe^{x^2}, 0\rangle$. Find a vector field $\mathbf{A}$ such that $\text{curl}(\mathbf{A}) = \mathbf{G}$.

Solution:

This part is not very illuminating and is not going to be relevant for the final. I explained briefly in discussion what is the reasoning for getting the vector potential in this specific case. If you want to you can check that $\mathbf{A} = \langle 0, 0, e^y - e^{x^2}\rangle$ works.

(b) Use Stokes' theorem to show that the flux of $\mathbf{G}$ through $\mathcal{S}$ is zero. *Hint:* Calculate the circulation of $\mathbf{A}$ around $\partial S$.

Solution:

Stoke's theorem applied to $\mathbf{A}$ gives us:

$$\oint_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{s} = \int\int_{\mathcal{S}} curl(\mathbf{A}) \cdot d\mathbf{S} \qquad \left(= \int\int_{\mathcal{S}} \mathbf{G} \cdot d\mathbf{S}\right)$$

Where the boundary of $\mathcal{S}$ is denoted by $\mathcal{C}$ and is the unit circle in the xy-plane. Hence in order to compute the flux of $\mathbf{G}$ it suffice to compute the circulation of $\mathbf{A}$ along $\mathcal{C}$.

We know how to parametrize $\mathcal{C}$, since it is a circle (with $r = 1$). One possible parametrization is $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0\rangle$ with $0 \leq t \leq 2\pi$. Therefore $\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0\rangle$. And so the circulation is:

$$\oint_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{A} \cdot \mathbf{r}'(t)\, dt = \int_0^{2\pi} \left\langle 0, 0, e^y - e^{x^2} \right\rangle \cdot \langle -\sin(t), \cos(t), 0\rangle\, dt = 0$$

(c) Calculate $I$. *Hint:* Use (b) to show that $I$ is equal to the flux of $\langle 0, 0, z^2\rangle$ through $\mathcal{S}$.

Solution:

We know that $\mathbf{F} = \mathbf{G} = \langle 0, 0, z^2\rangle$. Therefore by part (b):

$$\int\int_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \int\int_{\mathcal{S}} \mathbf{G} \cdot d\mathbf{S} + \int\int_{\mathcal{S}} \langle 0, 0, z^2 \cdot d\mathbf{S} = 0 + \int\int_{\mathcal{S}} \langle 0, 0, z^2 \cdot d\mathbf{S}$$

So we are left to compute the flux of a much easier vector field. We can parametrize the upper unit hemisphere using spherical coordinates: $S(\theta, \phi) = \langle \cos(\theta)\sin(\phi), \sin(\phi)\sin(\theta), \cos(\phi)\rangle$, with $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \frac{\pi}{2}$.

The normal can be computed to get $\mathbf{N} = S_\theta \times S_\phi = \langle \cos(\theta)\sin(\phi), \sin(\phi)\sin(\theta), \cos(\phi)\rangle \sin(\phi)$. Therefore, we end up with the integral:

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left\langle 0, 0, \cos^2(\phi)\right\rangle \cdot \langle \cos(\theta)\sin(\phi),\ \sin(\phi)\sin(\theta),\ \cos(\phi)\rangle\ \sin(\phi)\, d\phi\, d\theta$$

Expanding everything out and doing a $u$ substitution we get:

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos^3(\phi)\sin(\phi)\, d\phi\, d\theta = \frac{\pi}{2}$$

(4) Let $\mathbf{F} = \langle y^2, x^2, z^2\rangle$. Prove that the circulation along any two curves lying on the cylinder with central axis the z-axis (as in the board) is the same.

Solution: I have included a picture below with basically the same image I drew on the board. Let's apply Stoke's theorem to the area in the cylinder bounded by the two curves. Notice that $\mathcal{C}_1$ has the right orientation, but $\mathcal{C}_2$ doesn't (make sure you understand this, when you walk along a curve with the right orientation the surface should be to your left). Therefore, Stoke's theorem gives us:

$$\oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} - \oint_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} = \int\int_{\mathcal{S}} curl(\mathbf{F}) \cdot d\mathbf{S}$$

Now, a computation shows that $curl(\mathbf{F}) = \langle 0, 0, 2x - 2y\rangle$. Hence, we have:

$$\int\int_{\mathcal{S}} curl(\mathbf{F}) \cdot d\mathbf{S} = \int\int_{\mathcal{S}} \langle 0, 0, 2x - 2y\rangle \cdot \mathbf{n}\, dS$$

Now, $\mathcal{S}$ is in the cylinder, so its normal vector $\mathbf{n}$ is the normal vector to the cylinder. Using the picture below, you can reason geometrically to see that $\mathbf{n}$ is going to be parallel to the xy-plane, and therefore the z-component of $\mathbf{n}$ is 0. This means that the normal is of the form $\mathbf{n} = \langle n_x, n_y, 0\rangle$. And so:

$$\int\int_{\mathcal{S}} curl(\mathbf{F}) \cdot d\mathbf{S} = \int\int_{\mathcal{S}} \langle 0, 0, 2x - 2y\rangle \cdot \langle n_x, n_y, 0\rangle\, dS = 0$$

Alternatively, instead of reasoning geometrically you can parametrize the cylinder (here r=1 is constant) $C(\theta, z) = \langle \cos(\theta), \sin(\theta), z \rangle$ and compute the normal directly $\mathbf{N} = C_\theta \times C_z$ to see that the dot product in the integral above is 0.

Either way, we conclude that the circulation along the paths is the same, since

$$\oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} - \oint_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s} = \int\int_{\mathcal{S}} curl(\mathbf{F}) \cdot d\mathbf{S} = 0$$

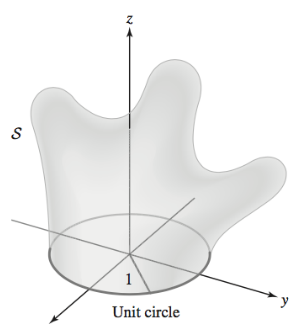


Figure 1: Problem 2.

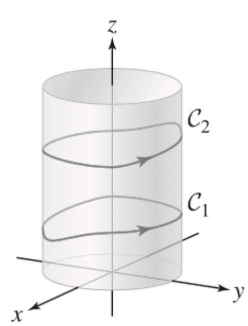


Figure 2: Problem 4.