REVIEW **NAME: SOLUTIONS**

Math 1920 - Andres Fernandez November 30, 2017

PROBLEMS

(1) Use Green's Theorem to evaluate the line integral around the given closed curve.

(a) $\oint_{\mathcal{C}} xy^3\,dx + x^3y\,dy$, where $\mathcal{C}$ is the rectangle $-1 \le x \le 2$, $-2 \le y \le 3$, oriented counterclockwise.

SOLUTION:

We use Green's theorem with $\mathbf{F} = \langle xy^3, x^3y\rangle$, and we get:

$$\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s} = \int_{-1}^{2}\int_{-2}^{3} \frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1\; dxdy = \int_{-1}^{2}\int_{-2}^{3} 3x^2y - 3y^2x\; dxdy$$

The double integral only involves integration of polynomials, after a simple computation the final answer should be -30.

(b) $\oint_{\mathcal{C}} y^2\,dx - x^2\,dy$, where $\mathcal{C}$ consists of the arcs $y = x^2$ and $y = \sqrt{x}$, $0 \le x \le 1$, oriented clockwise.

SOLUTION:

Notice that the orientation is clockwise, so we have to put a negative sign when using Green's theorem. This time we set $\mathbf{F} = \langle y^2, -x^2\rangle$ and so:

$$\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s} = -\int_{0}^{1}\int_{x^2}^{\sqrt{x}} \frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1\; dxdy = -\int_{0}^{1}\int_{x^2}^{\sqrt{x}} -2x - 2y\; dxdy$$

For the bounds in the double integral, notice that we are given the bounds for x ($0 \le x \le 1$) right away. Also we know that y is between x^2 and $\sqrt{x}$. By drawing the curves or plugging in one value say $\frac{1}{2}$ we can see that $x^2 \le \sqrt{x}$ on the domain given, and so $x^2 \le y \le \sqrt{x}$.

Again doing the integral should not entail any conceptual difficulties. You just have to remember how to integrate powers ($\int x^a = \frac{x^{a-1}}{a}$). The final answer should be $\frac{3}{5}$.

(2) Compute the flux $\oint_{\partial D} \mathbf{F}\cdot\mathbf{n}\,ds$ of the following vector fields across the respective paths (with normal vector pointing outwards):

(a) $\mathbf{F}(x,y) = \langle x^3, yx^2\rangle$, $\mathcal{D}$ is the unit square.

SOLUTION: We have to use the **vector form** of Green's theorem. This is the equivalent to the divergence theorem, but in 2-dimensions. Since the normal is pointing outwards, we have the right normal and so we don't need to add any negatives. We get:

$$\oint_{\partial D} \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^1\int_0^1 \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2\; dxdy = \int_0^1\int_0^1 3x^2 + x^2\; dxdy = \frac{4}{3}$$

(b) $\mathbf{F}(x,y) = \langle x^3 + 2x, y^3 + y\rangle$ across the circle $\mathcal{D}$ given by $x^2 + y^2 = 4$.

SOLUTION: We proceed as before:

$$\oint_{\partial D} \mathbf{F}\cdot\mathbf{n}\,ds = \int\int_{\mathcal{D}} \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2\; dA = \int\int_{\mathcal{D}} 3x^2 + 2 + 3y^2 + 1\; dA = \int\int_{\mathcal{D}} 3(x^2+y^2) + 3\; dA$$

Now since $\mathcal{D}$ is a circle, it makes the most sense to integrate using polar coordinates. Remember that we have to add an extra r because $dA = r\,dr\,d\theta$. The integral finally becomes:

$$\int_0^{2\pi}\int_0^2 (3r^2+3)\,r\,dr\,d\theta = 36\pi$$

(3) Let I be the flux of $\mathbf{F} = \langle e^y, 2xe^{x^2}, z^2\rangle$ through the upper hemisphere $\mathcal{S}$ of the unit sphere.

(a) Let $\mathbf{G} = \langle e^y, 2xe^{x^2}, 0\rangle$. Find a vector field $\mathbf{A}$ such that $\mathrm{curl}(\mathbf{A}) = \mathbf{G}$.

SOLUTION:

This part is not very illuminating and is not going to be relevant for the final. I explained briefly in discussion what is the reasoning for getting the vector potential in this specific case. If you want to you can check that $\mathbf{A} = \langle 0, 0, e^y - e^{x^2}\rangle$ works.

(b) Use Stokes' theorem to show that the flux of $\mathbf{G}$ through $\mathcal{S}$ is zero. *Hint:* Calculate the circulation of $\mathbf{A}$ around ∂S.

SOLUTION:

Stoke's theorem applied to $\mathbf{A}$ gives us:

$$\oint_{\mathcal{C}} \mathbf{A}\cdot d\mathbf{s} = \int\int_{\mathcal{S}} curl(\mathbf{A})\cdot d\mathbf{S} \qquad \left(= \int\int_{\mathcal{S}} \mathbf{G}\cdot d\mathbf{S}\right)$$

Where the boundary of $\mathcal{S}$ is denoted by $\mathcal{C}$ and is the unit circle in the xy-plane. Hence in order to compute the flux of $\mathbf{G}$ it suffice to compute the circulation of $\mathbf{A}$ along $\mathcal{C}$.

We know how to parametrize $\mathcal{C}$, since it is a circle (with $r = 1$). One possible parametrization is $\mathbf{r}(t) = \langle\cos(t), \sin(t), 0\rangle$ with $0 \le t \le 2\pi$. Therefore $\mathbf{r}'(t) = \langle-\sin(t), \cos(t), 0\rangle$. And so the circulation is:

$$\oint_{\mathcal{C}} \mathbf{A}\cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{A}\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi} \left\langle 0, 0, e^y - e^{x^2}\right\rangle\cdot\langle-\sin(t), \cos(t), 0\rangle\,dt = 0$$

(c) Calculate I. *Hint:* Use (b) to show that I is equal to the flux of $\langle 0, 0, z^2\rangle$ through $\mathcal{S}$.

SOLUTION:

We know that $\mathbf{F} = \mathbf{G} = \langle 0, 0, z^2\rangle$. Therefore by part (b):

$$\int\int_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \int\int_{\mathcal{S}} \mathbf{G}\cdot d\mathbf{S} + \int\int_{\mathcal{S}} \langle 0, 0, z^2\cdot d\mathbf{S} = 0 + \int\int_{\mathcal{S}} \langle 0, 0, z^2\cdot d\mathbf{S}$$

So we are left to compute the flux of a much easier vector field. We can parametrize the upper unit hemisphere using spherical coordinates: $S(\theta,\phi) = \langle\cos(\theta)\sin(\phi), \sin(\phi)\sin(\theta), \cos(\phi)\rangle$, with $0 \le \theta \le 2\pi$ and $0 \le \phi \le \frac{\pi}{2}$.

The normal can be computed to get $\mathbf{N} = S_\theta \times S_\phi = \langle\cos(\theta)\sin(\phi), \sin(\phi)\sin(\theta), \cos(\phi)\rangle\sin(\phi)$. Therefore, we end up with the integral:

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left\langle 0, 0, \cos^2(\phi)\right\rangle\cdot\langle\cos(\theta)\sin(\phi),\ \sin(\phi)\sin(\theta),\ \cos(\phi)\rangle\ \sin(\phi)\ d\phi\,d\theta$$

Expanding everything out and doing a u substitution we get:

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \cos^3(\phi)\sin(\phi)\ d\phi\,d\theta = \frac{\pi}{2}$$

(4) Let $\mathcal{S}$ be the portion of the plane $z = -x$ contained in the half-cylinder of radius R. Use Stokes' theorem to calculate the circulation of $\mathbf{F} = \langle z, x, y + 2z \rangle$ around the boundary of $\mathcal{S}$ (a half-ellipse) in the counterclockwise direction when viewed from above.

Solution:

If you walk along the curve in the direction shown, the surface will be to your left, so we have the right orientations for Stoke's theorem with upper pointing normal. First we have that $curl(\mathbf{F}) = \langle 1, 1, 1 \rangle$. Using Stoke's we have:

$$\oint \mathbf{F} \cdot d\mathbf{s} = \int\int_S curl(\mathbf{F}) \cdot d\mathbf{S}$$

Now we just need to parametrize the surface. We can use x, y as parameters. Notice that since S lives in a plane, the parametrization is not going to be that bad. Using $z = -x$ we get $S(x, y) = \langle x, y, -x \rangle$ as a parametrization. We just need to know what the range for x, y is, that is, what is the projection of S into the xy-plane. Looking at the figure below, it is not difficult to see that the projection is going to be the half circle of radius R with $x \leq 0$ as given (this is just the projection of the half cylinder). Let's denote this half circle $\mathcal{D}$. Then the flux is going to be:

$$\int\int_S curl(\mathbf{F}) \cdot d\mathbf{S} = \int\int_{\mathcal{D}} curl(\mathbf{F}) \cdot \mathbf{N}\, dx\, dy = \int\int_{\mathcal{D}} \langle 1, 1, 1 \rangle \cdot \langle 1, 0, 1 \rangle\, dx\, dy$$

Here for the computation of the normal vector $\mathbf{N}$ you can do the usual cross product or you can just directly find the normal of the plane $z = -x$. Notice that it is important that we get upper normal (see beginning paragraph above). Otherwise we would have to multiply by -1 to get the right $\mathbf{N}$. We end up with:

$$\int\int_{\mathcal{D}} 2\, dx\, dy = 2\int\int_{\mathcal{D}} dA = 2Area(\mathcal{D}) = \pi R^2$$

(5) Show that the circulation of $\mathbf{F}(x, y, z) = \langle x^2, y^2, z(x^2 + y^2) \rangle$ around any curve $\mathcal{C}$ on the surface of the cone $z^2 = x^2 + y^2$ is equal to zero.

Solution:

Stoke's theorem tells us that for any path $\mathcal{C}$ we have

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int\int_S curl(\mathbf{F}) \cdot \mathbf{n}\, dS$$

Where S is the surface in the enclosed by $\mathcal{C}$. Every time you want to show that circulation is 0, you should expect $curl(\mathbf{F}) \cdot \mathbf{n}$ to be 0. Since the surface lies in the cone, $\mathbf{n}$ will always be the normal to the cone (you can think why this is the case geometrically). So it suffices to parametrize the cone, find the normal $\mathbf{N}$ and then show that the dot product is always 0.

First notice that $curl(\mathbf{F}) = \langle 2yz, -2xz, 0 \rangle$. Also, we are given the equation of the cone in the figure: $x^2 + y^2 = z^2$. Therefore if we use x, y as parameters, we get the parametrization $G(x, y) = \langle x, y, \sqrt{x^2 + y^2} \rangle$ (notice that $z \geq 0$ in the picture).

Now you know the usual process, we can compute $\mathbf{N} = G_x \times G_y = \left\langle -\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1 \right\rangle$. Once you have this, it should not be too difficult to show that $curl(\mathbf{F}) \cdot \mathbf{N} = 0$.

(6) Compute the flux of $\mathbf{F} = \langle xyz + xy, \frac{1}{2}y^2(1 - z) + e^x, e^{x^2+y^2} \rangle$ through $\mathcal{S}$ with outer pointing normal, where $\mathcal{S}$ is the boundary of the solid bounded by the cylinder $x^2 + y^2 = 16$ and the planes $z = 0$ and $z = y - 4$.

Solution: Since the surface is closed (it is the boundary of a solid volume), we can use the divergence theorem:

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} = \int\int\int_{V enclosed} div(\mathbf{F})\, dV = \int\int\int_{V enclosed} yz+y+y(1-z)\, dV = \int\int\int_{V enclosed} 2y\, dV$$

Here, since we have some determined bounds for z and some type of cylindrical bounds for x, y, it is going to be better to use cylindrical coordinates. Remember that here $dV = r\,dr\,d\theta\,dz$ and so we are left with the integral:

$$\int_0^{2\pi}\int_0^4\int_{r\sin(\theta)-4}^0 2r\sin(\theta)\, r\,dz\,dr\,d\theta$$

Here notice that $y - 4 \leq z \leq 0$, since $y \leq 4$ on the cylinder given (you can also draw the picture). After integrating, the final answer should be -128π.

(7) Compute the flux of $\mathbf{F} = \langle \sin(yz), \sqrt{x^2+z^4}, x\cos(x-y)\rangle$, $\mathcal{S}$ is any smooth closed surface that is the boundary of a region in $\mathbb{R}^3$.

SOLUTION:

You can use the divergence theorem:

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} = \int\int\int_{V enclosed} div(\mathbf{F})\, dV$$

If you try to compute the divergence of $\mathbf{F}$ above it should be easy to see that this is 0 (you should not have to do any computations). Therefore by Divergence Theorem, the flux through any closed surface is 0 (the triple integral above is 0).

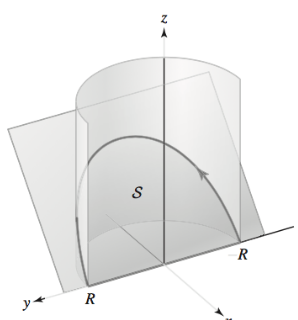

Figure 1: Problem 4.

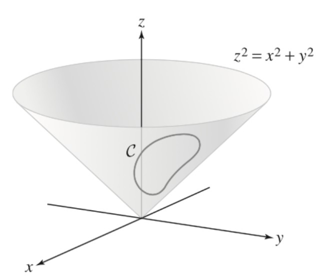

Figure 2: Problem 5.