Sections 13.6, 13.7, 14.1, 14.2 — Name: Solutions

Math 1920 - Andres Fernandez — September 11, 2018

Summary of the Sections

(1) Conversion from rectangular coordinates to cylindrical/spherical coordinates:

  (a) **Cylindrical:** $r = \sqrt{x^2 + y^2}$ $\quad \tan(\theta) = \frac{y}{x}$ $\quad z = z$

  (b) **Spherical:** $\rho = \sqrt{x^2 + y^2 + z^2}$ $\quad \tan(\theta) = \frac{y}{x}$ $\quad \cos(\phi) = \frac{z}{\rho}$

  With $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$.

(2) Conversion from cylindrical/spherical to rectangular:

  (a) **Cylindrical:** $x = r\cos(\theta)$ $\quad y = r\sin(\theta)$ $\quad z = z$

  (b) **Spherical:** $x = \rho\cos(\theta)\sin(\phi)$ $\quad y = \rho\sin(\theta)\sin(\phi)$ $\quad z = \rho\cos(\phi)$

(3) A vector valued function is a function of the form

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

(4) The underlying curve C traced by $\mathbf{r}(t)$ is the set of points $(x(t), y(t), z(t))$ in space for t in the domain of $\mathbf{r}(t)$.

(5) Every curve can be parametrized in infinitely many ways.

(6) The projection of $\mathbf{r}(t)$ onto the xy-plane is the curve traced by $\langle x(t), y(t), 0\rangle$. The projections onto the rest of the plane are found similarly.

(7) Limits, differentiation, and integration of vector-valued functions are performed componentwise.

(8) Differentiation rules:

  (a) Sum rule: $(\mathbf{r_1}(t) + \mathbf{r_2}(t))' = \mathbf{r_1}'(t) + \mathbf{r_2}'(t)$

  (b) Constant Multiple Rule: $(c\mathbf{r}(t))' = c\mathbf{r}'(t)$.

  (c) Chain Rule: $\frac{d}{dt}\mathbf{r}(g(t)) = g'(t)\mathbf{r}'(g(t))$

(9) Product Rules:

  (a) Scalar times a vector: $\frac{d}{dt}(f(t)\mathbf{r}(t)) = f'(t)\mathbf{r} + f(t)\mathbf{r}'(t)$

  (b) Dot product: $\frac{d}{dt}(\mathbf{r_1}(t) \cdot \mathbf{r_2}(t)) = \mathbf{r_1}'(t) \cdot \mathbf{r_2}(t) + \mathbf{r_1}(t) \cdot \mathbf{r_2}'(t)$

  (c) Cross Product: $\frac{d}{dt}(\mathbf{r_1}(t) \times \mathbf{r_2}(t)) = \mathbf{r_1}'(t) \times \mathbf{r_2}(t) + \mathbf{r_1}(t) \times \mathbf{r_2}'(t)$

(10) The tangent line to $\mathbf{r}(t)$ at $\mathbf{r}(t_0)$ is given by the vecotr parametrization:

$$\mathbf{L}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0)$$

(11) The general solution to $\mathbf{R}'(t) = \mathbf{r}(t)$ is given by $\mathbf{R}(t) = \int \mathbf{r}(t)dt + \mathbf{c}$, where integration is done componentwise.

Problems

(1) Express the following points in cylindrical and spherical coordinates:

(a) $(3, 2, 0)$

(b) $(-2, 1, 1)$

Solution:

a
  i. Cylindrical: $r = \sqrt{11}$, $z = 0$, and $\theta = \arctan\left(\frac{2}{3}\right) \approx 0.588$ rad (with θ chosen so that $x, y \geq 0$)
  ii. Spherical: $\rho = \sqrt{11}$, $\theta \approx 0.588$ rad, $\phi = \pi$

b
  i. Cylindrical: $r = \sqrt{5}$, $z = 1$, and $\theta = \arctan\left(\frac{1}{-2}\right) \approx 5.81$ rad (with θ chosen so that $x \geq 0$)
  ii. Spherical: $\rho = \sqrt{6}$, $\theta \approx 5.81$ rad, and $\phi = \arctan\frac{\sqrt{5}}{\sqrt{6}} \approx 0.7399$ rad (here remember that we always have $0 \leq \phi \leq \pi$)

(2) Express the following constraints (regions in space) in spherical coordinates:

(a) The (filled) unit ball

(b) The space between two spheres of radii 4 and 5 respectively

(c) The region given by $x^2 + y^2 + z^2 \geq 4$, $y = \sqrt{3}x$ and $y \leq 0$.

Solution:

a $\rho \leq 1$

b $4 \leq \rho \leq 5$

c $\rho \geq 2$, $\theta = \frac{4\pi}{3}$, $0 \leq \phi \leq \pi$

(3) Draw the region given in spherical coordinates by $0 \leq \theta \leq \frac{\pi}{3}$ and $\rho \leq 3$.

(4) How can you express the surface given by $6x^2 - 8z^{\frac{1}{3}} + 5y^2 - 15 = 0$ in cylindrical coordinates?

Solution: Here the only thing to do is to substitute the equations for x, y, z in cylindrical coordinates and then try to simplify. If you do this you will get:

$$6(r\cos\theta)^2 - 8z^{\frac{1}{3}} + 5(r\sin\theta)^2 - 15 = 0$$

You can combine some of the squared sines and cosines to get

$$r^2\cos^2\theta - 8z^{\frac{1}{3}} + 5r^2 - 15 = 0$$

(5) Find the radius, center and plane containing the following circles:

(a) $\mathbf{r}(t) = 7\mathbf{i} + (12\cos t)\mathbf{j} + (12\sin t)\mathbf{k}$.
Solution: The plane is $x = 7$, the center is $(7, 0, 0)$ and the radius is 12.

(b) $\langle \sin t, 0, 4 + \cos t \rangle$.
Solution: Radius 1, center $(0, 0, 4)$, and it is contained in the xz-plane.

(6) Let $\mathbf{r_1} = \langle 3t^2-2,\ ln(5t^4+4),\ 4t+5\rangle$ and $\mathbf{r_2} = \langle 5t+3,\ e^{6t-7},\ 7t+2\rangle$ be two vector valued functions. Do they collide? Do they intersect? If so find the points of collision/intersection.

SOLUTION:

Let's start with intersection. The two trajectories intersect if they are at the the same point for some (not necessarily equal) values of the parameters. Essentially, we have to solve $\mathbf{r_1}(t) = \mathbf{r_2}(s)$. Setting the equality in each of the coordinates, we get the following system of equations:

$$\begin{cases} 3t^2 - 2 = 5s + 3 \\ \ln(5t^4 + 4) = e^{6s-7} \\ 4t + 5 = 7s + 2 \end{cases}$$

Using the last equation we get $s = \frac{4}{7}t + \frac{3}{7}$. We can plug in the first equation to get the quadratic $3t^2 - \frac{20}{7}t - \frac{50}{7} = 0$. You can solve to get two values for t, in this case it turns out that the numbers are quite ugly, so you might want to get numerical values using a calculator. Then you can use $s = \frac{4}{7}t + \frac{3}{7}$ to get the corresponding values for s. It remains to see if the second equation is satisfied for these values. You can use a calculator to see that in either case you don't get numerical equality in the second equation. Therefore, they don't intersect.

Since they don't intersect, they won't collide (remember that collision is more strict: it means that they two objects should be at the same point at exactly the same time).

(7) Determine the intersection of the plane of the ellipsoid $x^2 - 4x + 3y^2 + 5z^2 = 40$ with:

(a) The plane $y = h$, with h a fixed parameter. For which h is the intersection empty?

(b) The plane $x - y = h$ for h some parameter. (Challenge:when is the intersection empty?)

SOLUTION:

(a) We set $y = h$ in the equation for the ellipsoid above, and so we get that the intersection is given by the two equations:

$$\begin{cases} x^2 - 4x + 3h^2 + 5z^2 = 40 \\ y = h \end{cases}$$

We can complete squares in the first equation to get the equation of an ellipse in standard form:

$$\begin{cases} (x-2)^2 + 5z^2 = 44 - 3h^2 \\ y = h \end{cases}$$

Now it is easy to see that the intersection is going to be empty only if $44 - 3h^2 < 0$ (think carefully why this is the case). Solving this inequality (do the computation) we get that the intersection is empty when $h > \sqrt{\frac{44}{3}}$ or $h < -\sqrt{\frac{44}{3}}$.

(b) Plugging in $y = x - h$ we get that the intersection is given by:

$$\begin{cases} x^2 - 4x + 3(x-h)^2 + 5z^2 = 40 \\ x + y = h \end{cases}$$

We get again an ellipse, something we can see easily by completing the square:

$$\begin{cases} 4(x - \frac{1}{2} - \frac{3}{4}h)^2 + 5z^2 = 40 - h^2 + (1 + \frac{3}{2}h)^2 \\ x + y = h \end{cases}$$

Again it can be seen that this will be empty when the right hand side is negative, and a similar inequality should be solved in this case. (solve it)

(8) Let $\mathbf{r}_1(t) = \langle t^2, t^3, t\rangle$ and $\mathbf{r}_2(t) = \langle e^{3t}, e^{2t}, e^t\rangle$. Evaluate the derivative by using the appropriate product rule.

(a) $\frac{d}{dt}(\mathbf{r}_1(t) \cdot \mathbf{r}_2(t))$.

(b) $\frac{d}{dt}(\mathbf{r}_1(t) \times \mathbf{r}_2(t))$.

SOLUTION:

a $(3t^2 + 2t)e^{3t} + (2t^3 + 3t^2)e^{2t} + (t + 1)e^t$

b $[(t^3 + 3t^2)e^t - (2t + 1)e^{2t}]\mathbf{i} + [(3t + 1)e^{3t} - (t^2 + 2t)e^t]\mathbf{j} + [(2t^2 + 2t)e^{2t} - (3t^3 + 3t^2)e^{3t}]\mathbf{k}$

(9) Solve the following initial value problem:

$$\mathbf{r}''(t) = \left\langle 3 - t,\ sin(9\pi t),\ \frac{-7}{t^2} \right\rangle, \qquad \mathbf{r}'(1) = \langle 2, 0, 1\rangle, \qquad \mathbf{r}(1) = \langle 4, 0, 2\rangle$$

SOLUTION: By integrating we get:

$$\mathbf{r}'(t) = \int \mathbf{r}''(t)\, \mathrm{d}t\ + \mathbf{C} = \left\langle 3t - \frac{1}{2}t^2,\ -\frac{1}{9\pi}\cos 9\pi t,\ \frac{7}{t} \right\rangle + \mathbf{C}$$

We can plug in $t = 1$ above and use $\mathbf{r}'(1) = \langle 2, 0, 1\rangle$ to get $\mathbf{C} = \left\langle -\frac{1}{2},\ -\frac{1}{9\pi},\ -6 \right\rangle$. So we end up with $\mathbf{r}'(t) = \left\langle 3t - \frac{1}{2}t^2 - \frac{1}{2},\ -\frac{1}{9\pi}\cos 9\pi t - \frac{1}{9\pi},\ \frac{7}{t} - 6\right\rangle$. We can integrate once again to get:

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\, \mathrm{d}t\ + \mathbf{B} = \left\langle \frac{3}{2}t^2 - \frac{1}{6}t^3 - \frac{1}{2}t,\ -\frac{1}{81\pi^2}\sin 9\pi t - \frac{1}{9\pi}t,\ 7\ln(t) - 6t \right\rangle$$

The computation for $\mathbf{B}$ is very similar to the one above. You plug in $t = 1$ and use the value of $\mathbf{r}(1)$ given above.

(10) A fighter plane, which can shoot a laser beam straight ahead, travels along the path

$$\mathbf{r}(t) = \langle t - t^3, 12 - t^2, 3 - t\rangle.$$

Show that the pilot cannot hit any target on the x-axis.

SOLUTION: Suppose that the fighter plane shoots the laser at time t_0. After that, we are told that the path of the laser will be a straight line, we want to parametrize this line. So we need a starting point and a direction vector.

The starting point is clear, it is whatever the position of the plane is at time t_0, in this case $P_0 = \left(t_0 - t_0^3,\ 12 - t_0^2,\ 3 - t_0\right)$. The direction vector will be the tangent to the path of the plane at time t_0, that is $\mathbf{r}'(t_0)$. Taking the derivative, we get $\mathbf{v} = \mathbf{r}'(t_0) = \left\langle 1 - 3t_0^2,\ -2t_0,\ -1\right\rangle$. Hence the equation fo motion for the laser is $\mathbf{l}(t) = P_0 + t\mathbf{v} = \left\langle t_0 - t_0^3 + t(1 - 3t_0^2),\ 12 - t_0^2 + t(-2t_0),\ 3 - t_0 - t\right\langle$. Here the variable t is the time that passes after the laser has been shot.

Now, if the fighter plane could hit the x-axis, that would mean that $\mathbf{l}(t)$ intersects the x-axis. Remember that the x-axis is given by $y = z = 0$. This would mean that the last two coordinates of $\mathbf{l}(t)$ would be zero for some choice of times t_0 and t. This gives us the system:

$$\begin{cases} 12 - t_0^2 + t(-2t_0) = 0 \\ 3 - t_0 - t = 0 \end{cases}$$

From the second equation we get $t = 3 - t_0$. Plugging this into the first equation yields $t_0^2 - 6t_0 + 12 = 0$. This quadratic has no real solution, so there is no time when the fighter plane can hit a target in the x-axis.