Sections 16.5 and 16.6

Name: Solutions

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Summary of the Sections

(1) Total mass of regions $\mathcal{D} \subset \mathbb{R}^2$ and $\mathcal{W} \subset \mathbb{R}^3$:

$$M = \iint_{\mathcal{R}} \delta(x,y)\mathrm{d}A, \quad M = \iiint_{\mathcal{W}} \delta(x,y,z)\mathrm{d}V,$$

where δ is the mass density. The same formula can be used to compute the total charge of a region if δ is the charge density.

(2) Moments:

$$\textbf{In } \mathbb{R}^2: \quad M_x = \iint_{\mathcal{D}} y\delta(x,y)\mathrm{d}A, \quad M_y = \iint_{\mathcal{D}} x\delta(x,y)\mathrm{d}A$$

$$\textbf{In } \mathbb{R}^3: \quad M_{yz} = \iint_{\mathcal{W}} x\delta(x,y,z)\mathrm{d}V, \quad M_{xz} = \iint_{\mathcal{W}} y\delta(x,y,z)\mathrm{d}V, \quad M_{xy} = \iint_{\mathcal{W}} z\delta(x,y,z)\mathrm{d}V.$$

(3) Coordinates of the *center of mass*:

$$\textbf{In } \mathbb{R}^2: \quad x_{CM} = \frac{M_y}{M}, \quad y_{CM} = \frac{M_x}{M}.$$

$$\textbf{In } \mathbb{R}^3: \quad x_{CM} = \frac{M_{yz}}{M}, \quad y_{CM} = \frac{M_{xz}}{M} \quad z_{CM} = \frac{M_{xy}}{M}.$$

(4) Random variables X and Y have joint probability density function $p(x,y)$ if

$$P(a \le X \le b; c \le Y \le d) = \int_{x=a}^{b} \int_{y=c}^{d} p(x,y)\mathrm{d}y\mathrm{d}x.$$

(5) A joint probability density function must satisfy $p(x,y) \ge 0$ and

$$\int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} p(x,y)\mathrm{d}y\mathrm{d}x = 1.$$

(6) Let $G(u,v) = (x(u,v), y(u,v))$ be a mapping. The *Jacobian* of G is the determinant

$$\mathrm{Jac}(G) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}.$$

(7) $\mathrm{Jac}(G) = [\mathrm{Jac}(F)]^{-1}$, where $F = G^{-1}$.

(8) Change of Variables Formula: If $G : \mathcal{D}_0 \to \mathcal{D}$ has component functions with continuous partial derivatives and one-to-one on the interior of $\mathcal{D}_0$, and if f is continuous, then

$$\iint_{\mathcal{D}} f(x,y)\mathrm{d}x\mathrm{d}y = \iint_{\mathcal{D}_0} f(x(u,v), y(u,v)) \frac{\partial(u,v)}{\partial(x,y)} \mathrm{d}u\mathrm{d}v.$$

Problems

(1) Compute the total mass of the plate in Figure 1 assuming a mass density of $f(x, y) = \frac{x^2}{(x^2+y^2)}$ g/cm^2.

SOLUTION: $\frac{50}{3}\pi + 25\sqrt{3}$.

(2) Find the z_{CM} (z coordinate of the center of mass) of a cylinder of radius 2 and height 4 and mass density e^{-z}, where z is the height above the base. Can you find the x_{CM} and y_{CM} without explicitly computing any integral?

SOLUTION: $x_{CM} = y_{CM} = 0$, $z_{CM} = \frac{(1-5e^{-4})}{1-e^{-4}}$.

(3) Find a constant C such that

$$p(x, y) = \begin{cases} Cxy & \text{if } 0 \leq x \text{ and } 0 \leq y \leq 1 - x \\ 0 & \text{otherwise} \end{cases}$$

is a joint probability density function, then calculate $P(X \geq Y)$.

SOLUTION: $C = 24$, $P(X \geq Y) = \frac{1}{2}$.

(4) Compute $\iint_{\mathcal{D}} (x + 3y) \mathrm{d}x\mathrm{d}y$, where $\mathcal{D}$ is the shaded region in Figure 3. *Hint:* Use the map $\Phi(u, v) = (u - 2v, v)$.

SOLUTION: 80.

(5) Calculate $\int\int_{\mathcal{D}} e^{9x^2+4y^2}\, dx\, dy$, where $\mathcal{D}$ is the interior of the ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 \leq 1$

(6) Use the Change of Variables Formula to prove that the volume of the ellipsoid $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1$ is the volume of the unit sphere times abc.

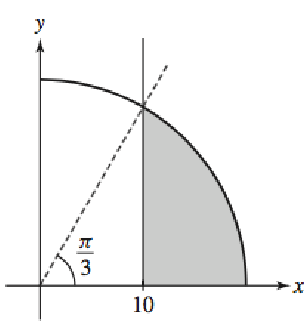

Figure 1: Problem 1

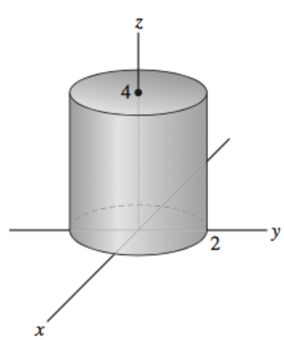

Figure 2: Problem 2

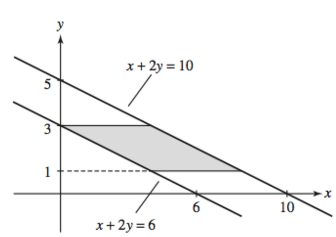

Figure 3: Problem 4