# VECTORS AND DOT PRODUCT (13.1, 13.2, 13.3) NAME: SOLUTIONS

Math 1920 - Andres Fernandez August 28, 2018

## SUMMARY OF THE SECTIONS

(1) Components of $\mathbf{v}=\overrightarrow{PQ}$, where $P=(a_1,b_1)$ and $Q=(a_2,b_2)$ are: $\mathbf{v}=\langle a_2-a_1, b_2-b_1\rangle$.

(2) The length $\|\mathbf{v}\|$ of $\mathbf{v}=\langle a,b\rangle$ is equal to $\boxed{\sqrt{a^2+b^2}}$ (1).

(3) *Vector addition:* $\langle v_1,v_2\rangle+\langle w_1,w_2\rangle=\boxed{\langle v_1+w_1, v_2+w_2\rangle}$ (2).

(4) *Scalar multiplication:* $\|\lambda\mathbf{v}\|=|\lambda|\|\mathbf{v}\|$ for $\lambda$ real.

(5) $\mathbf{v}$ and $\mathbf{w}$ are *parallel* if, for some scalar $\lambda$, $\boxed{\mathbf{w}=\lambda\mathbf{v}}$ (3).

(6) If $\mathbf{v}=\langle v_1,v_2\rangle$ makes an angle $\theta$ with the positive $x$-axis, then $v_1=\|\mathbf{v}\|\cos\theta$ and $v_2=\|\mathbf{v}\|\sin\theta$.

(7) *Equation of the sphere of radius R and center (a,b,c):* $\boxed{(x-a)^2+(y-b)^2+(z-c)^2=R^2}$ (4).

(8) *Equation of the cylinder of radius R and vertical axis through (a,b,0):* $(x-a)^2+(y-b)^2=R^2$.

(9) *Equations for the line passing through* $P_0=(x_0,y_0,z_0)$ *with direction vector* $\mathbf{v}=\langle a,b,c\rangle$*:*

   (a) *Vector parametrization:* $\mathbf{r}(t)=\overrightarrow{OP_0}+t\mathbf{v}=\boxed{\langle x_0,y_0,z_0\rangle+t\langle a,b,c\rangle}$ (5).

   (b) *Parametric equation:* $x=\boxed{x_0+at}$ (6), $y=\boxed{y_0+bt}$ (7), $z=\boxed{z_0+ct}$ (8).

(10) Let $\mathbf{a}=\langle a_1,a_2,a_3\rangle$, $\mathbf{b}=\langle b_1,b_2,b_3\rangle$. Then $\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3$

(11) If we are given the lengths $\|\mathbf{u}\|$, $\|\mathbf{v}\|$ and angle $\theta$ between two vectors, then $\mathbf{u}\cdot\mathbf{v}=\boxed{\|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta)}$ (9)

(12) Distributivity: $(\mathbf{w}+\mathbf{v})\cdot\mathbf{u}=\mathbf{w}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{u}$.

(13) Two vectors $\mathbf{u},\mathbf{v}$ are orthogonal when $\boxed{\mathbf{u}\cdot\mathbf{v}=0}$ (10).

(14) The angle between two vectors $\mathbf{u}$ and $\mathbf{v}$ is given by $\theta=arcos\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|\|\mathbf{u}\|}\right)$

(15) Given $\mathbf{a},\mathbf{b}$ the components of $\mathbf{a}$ with respect to $\mathbf{b}$ are:

-Tangential Component: $\mathbf{a}_{||\mathbf{b}}=\boxed{\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\right)\mathbf{b}}$ (11)

-Normal (Perpendicular) Component: $\mathbf{a}_{\perp\mathbf{b}}=\boxed{\mathbf{a}-\mathbf{a}_{||\mathbf{b}}}$ (12)

## Problems

(1) Let $R = (-2, 7)$. Calculate the following:

(a) The length of $\overrightarrow{OR}$.
SOLUTION: $\sqrt{53}$.

(b) The components of $\mathbf{u} = \overrightarrow{PR}$, where $P = (1, 2)$.
SOLUTION: $\langle -3, 5\rangle$.

(c) The point $P$ such that $\overrightarrow{PR}$ has components $\langle -2, 7\rangle$.
SOLUTION: $P = (0, 0)$.

(d) The point $Q$ such that $\overrightarrow{RQ}$ has components $\langle 8, -3\rangle$.
SOLUTION: $Q = (6, 4)$

(2) Find the vector:

(a) Unit vector $\mathbf{e_v}$ where $\mathbf{v} = \langle 3, 4\rangle$.
SOLUTION: $\langle \frac{3}{5}, \frac{4}{5}\rangle$.

(b) Vector of length 4 in the direction of of $\mathbf{u} = \langle -1, 1\rangle$.
SOLUTION: $\langle -2\sqrt{2}, 2\sqrt{2}\rangle$

(c) Vector of length 2 in the direction of $\mathbf{v} = \mathbf{i} - \mathbf{j}$.
SOLUTION: $\langle \sqrt{2}, -\sqrt{2}\rangle$

(d) Unit vector in the direction opposite to $\mathbf{v} = \langle -2, 4\rangle$.
SOLUTION: $\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\rangle$.

(e) Vector $\mathbf{v}$ of length 2 making an angle of 30° with the $x$-axis.
SOLUTION: $\langle \sqrt{3}, 1\rangle$.

(3) Determine whether or not the two vectors are parallel:

(a) $\mathbf{u} = \langle 1, -2, 5\rangle$, $\mathbf{v} = \langle -2, 4, -10\rangle$.
SOLUTION: By definition the two vectors are parallel if there exists a scalar $\lambda$ such that $u = \lambda v$. In other words, we need

$$\langle 1, -2, 5\rangle = \langle -2\lambda, 4\lambda, -10\lambda\rangle$$

Two vectors are the same iff all their components are the same. Setting the first components equal we get $1 = 2\lambda \implies \lambda = -\frac{1}{2}$. If the vectors are indeed parallel, this is the only value of $\lambda$ that can work.

Plugging in this value in the vector equality above we see that it works, so the vectors are parallel.

(b) $\mathbf{u} = \langle 4, 2, -6\rangle$, $\mathbf{v} = \langle 2, 1, 3\rangle$.
SOLUTION: We proceed as above. The equation that we have to solve is

$$\langle 4, 2, -6\rangle = \langle 2\lambda, \lambda, 3\lambda\rangle$$

Again, we set the first components equal to get $4 = 2\lambda \implies \lambda = 2$. Plugging $\lambda = 2$ in the vector equality above we get $\langle 4, 2, -6\rangle = \langle 4, 2, 6\rangle$. Notice that the third components do not match, so the vectors are not parallel.

(4) Find a vector parametrization for the line with the given description:

(a) Passes through $P = (1, 2, -8)$, direction vector $\mathbf{v} = \langle 2, 1, 3\rangle$.
SOLUTION: Given a point and a vector we can use the vector parametrization of a line as in the summary above: $\mathbf{r}(t) = \langle 1 + 2t, 2 + t, -8 + 3t\rangle$.

(b) Passes through $P = (-2, 0, -2)$ and $Q = (4, 3, 7)$.
SOLUTION: We are given two points. In order to do the same as above we need a direction vector, which is for example given by $\overrightarrow{PQ} = \langle 6, -2, 9\rangle$. Hence, using the usual vector parametrization of a line: $\mathbf{r}(t) = \langle -2 + 6t, 3t, -2 + 9t\rangle$.

(c) Passes through $P = (1, 1, 1)$ parallel to the line through $Q = (2, 0, -1)$ and $R = (4, 1, 3)$.

SOLUTION: We are given a point $P$. We can find a direction vector by $\overrightarrow{QR} = \langle 2, 1, 4\rangle$. Finally, use the vector parametrization to get $\mathbf{r}(t) = \langle 1 + 2t, 1 + t, 1 + 4t\rangle$.

(5) Justify the following statement: in order to know the number of intersections of two lines in the plane we only need to know their directions. Is the same true in 3-space?

SOLUTION: Technically we need the implicit hypothesis that the lines are distinct. Assuming this, there are only two possibilities in the plane:

(a) The lines are parallel (their directions vectors are parallel). In this case they do not intersect.

(b) The lines are not parallel (their directions vectors are not parallel). In this case there is always a unique intersection point.

In 3-space this is not the case, you can have two nonparallel lines that do not intersect (make sure that you can visualize an example of this).

(6) Two airplanes are traveling in an alternative time line where navigation systems have not yet been invented. If we choose as the origin some random point in the sky, their positions are given by $\mathbf{r}_1(t) = \langle -1, 1, 0\rangle + t\langle 2, 4, 1\rangle$ and $\mathbf{r}_2(t) = \langle 2, 1, \frac{-15}{8}\rangle + t\langle -1, 6, 4\rangle$. Do the airplanes collide? Do their geometric trajectories intersect? If so, where?

SOLUTION:

(a) The two airplanes collide when they are in **the same point in space at the same time**. Suppose that this hypothetical time is $t$, mathematically they are in the same point in space when $r_1(t) = r_2(t)$, that is:

$$\langle -1, 1, 0\rangle + t\langle 2, 4, 4\rangle = \langle 2, 1, \frac{-15}{8}\rangle + t\langle -1, 6, 4\rangle \Longrightarrow$$

$$\langle -1 + 2t, 1 + 4t, t\rangle = \langle 2 - t, 1 + 6t, \frac{-15}{8} + 4t\rangle$$

Now two points in space are equal iff. all of their coordinates are equal. Setting the first coordinates equal we get: $-1 + 2t = 2 - t \Longrightarrow t = 1$. So this must be the time they collide, but when we plug $t = 1$ in the vector equality above we get $\langle 1, 5, 1\rangle = \langle 1, 7, \frac{17}{15}\rangle$. Since the second and third coordinates are not the same, such a time does not exist. **Therefore the airplanes do not collide**.

(b) The trajectories of the airplanes intersect (this is the same as saying that the two lines intersect in $\mathbb{R}^3$) when **they are at the same point at two (possibly different) times**. Say those times are $t$ and $q$. Then, we want to solve $r_1(t) = r_2(q)$, or more explicitly:

$$\langle -1 + 2t, 1 + 4t, 4\rangle = \langle 2 - q, 1 + 6q, \frac{-15}{8} + 4q\rangle$$

The three coordinates must be equal, so we get a system of three equations in two unknowns:

$$\begin{cases} -1 + 2t = 2 - q \\ 1 + 4t = 1 + 6q \\ t = \frac{-15}{8} + 4q \end{cases}$$

Now you should know how to solve such a system. What you usually want to do is to multiply the equations by some numbers, and then subtract the equations to get rid of variables. **Please work this out**. Once this is done you get something like this:

$$\begin{cases} 16t = 18 \\ 6 = 8q \\ 5t = \frac{45}{8} \end{cases}$$

Notice that we do not get any contradictions. We get $t = \frac{9}{8}$ and $q = \frac{3}{4}$. We can plug these times into each equation and see that they indeed give us the same point. So the lines intersect, and the point of intersections is: $(\frac{5}{4}, \frac{11}{2}, \frac{9}{8})$.

(7) Let $v$ and $w$ be vectors with $\|v\| = 3$ and $\|w\| = 5$.

(a) If we know that $v \cdot w = -1$, what is $\|v + w\|$?
SOLUTION: $\sqrt{32}$

(b) Now we only know that the angle between the vectors is $60°$, what is $\|2v - w\|$.
SOLUTION: $\sqrt{31}$

(c) There is a vector $u$ that is perpendicular to $v$. If $\|3v + u\| = 15$, what is the length of $u$?
SOLUTION: 12

(d) Suppose that $\|v - 2w\| = \sqrt{109}$. What is the angle between $u$ and $w$.
SOLUTION: $90°$

(8) Let $u = \langle -3, 6, 1\rangle$ and $v = \langle 2, -7, 4\rangle$. What are the tangential and normal components of $u$ with respect to $v$.
SOLUTION: $u_{||v} = \langle \frac{-88}{69}, \frac{308}{69}, \frac{-176}{69}\rangle$ $u_{\perp v} = \langle \frac{-119}{69}, \frac{206}{69}, \frac{245}{69}\rangle$

(9) Let $v = \langle a, 4, 7\rangle$, $w = \langle -1, 2, 3\rangle$ and $u = \langle b^2, 5, b\rangle$. If $u$ is orthogonal to $w$ and $v$ is orthogonal to $w$, find $a, b, \|v\|$.
SOLUTION:

(a)

$$\begin{aligned} v \perp w \Longrightarrow v \cdot w &= 0 \\ -a + 8 + 21 &= 0 \\ a &= -29 \end{aligned}$$

(b)

$$\begin{aligned} u \perp w \Longrightarrow u \cdot w &= 0 \\ -b^2 + 10 + 3b &= 0 \\ b &= 5, -2 \end{aligned}$$

(c)

$$\|v\| = \sqrt{(-29)^2 + 4^2 + 7^2} = \sqrt{309}$$