Sections 13.4, 13.5

Name: Solutions

Math 1920 - Andres Fernandez

September 4, 2018

Summary of the Sections

(1) $\mathbf{u}\times\mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, has length $\|\mathbf{u}\|\|\mathbf{v}\|sin(\theta)$ (1) and direction given by the right hand rule.

(2) *Formula for Cross Product*:

$$\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$ (2)

(3) $\|u \times v\|$ is the area of the parallelogram spanned by u and v

(4) The equation of the plane through $P_0 = (x_0, y_0, z_0)$ with non-zero normal vector $\mathbf{n} = \langle a, b, c\rangle$ can be written in the following forms:

  (a) **Vector Form:** $\mathbf{n}\cdot\langle x, y, z\rangle = d$, with $d = \mathbf{n}\cdot\langle x_0, y_0, z_0\rangle)$

  (b) **Scalar Form:** $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

(5) Two planes are parallel if they have parallel normal vectors.

(6) To find the equation of the plane through three points P, Q, and R, compute a normal vector as a cross product $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$. Then use $\mathbf{n}$ and one of the points.

(7) The intersection of a plane P with a coordinate plane or a plane parallel to a coordinate plane is a trace. The trace in the yz-plane is obtained by setting $x = 0$ in the equation of the plane, the other traces are found similarly.

Problems

(1) Let $A = (4, 10, 7), B = (-1, 4, 1)$ and $C = (3, 7, 2)$. Compute the angle $\widehat{ABC}$ and the area of the triangle formed by the three points. Compute the area of the parallelogram $ABCD$ formed by adding one more point D.

Solution:

(a) We want to find the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$. First we compute $\overrightarrow{BA} = A - B = \langle 5, 6, 6\rangle$ and $\overrightarrow{BC} = \langle 4, 3, 1\rangle$. Then, we use the formula:

$$\theta = arcos\left(\frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\|\overrightarrow{BA}\|\|\overrightarrow{BC}\|}\right) = arcos\left(\frac{44}{\sqrt{2522}}\right) = 28.82°$$

(b) In order to compute these areas we need $\|\overrightarrow{BA} \times \overrightarrow{BC}\|$, and we can do this in two ways. We could use the cross product formula to get a vector and then compute its length. A shorter way, since we already now the angle, is:

$$\|\overrightarrow{BA} \times \overrightarrow{BC}\| = \|\overrightarrow{BA}\|\|\overrightarrow{BC}\| \sin(28.82°) = \sqrt{97}\sqrt{26}\sin(28.82°) = 24.21$$

And therefore we get:

$$A_{Parallelogram} = \|\overrightarrow{BA} \times \overrightarrow{BC}\| = 24.21$$

$$A_{Triangle} = \frac{\|\overrightarrow{BA} \times \overrightarrow{BC}\|}{2} = 12.1$$

(2) Find the volume of the parallelepiped spanned by $\langle 5, 8, 4\rangle$, $\langle -2, 3, 5\rangle$ and $\langle 4, 0, 4\rangle$.

Solution: The volume of the parallelepiped spanned by three vectors **a**, **b** **c** is given by the triple product, which is computed by the following determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

In this case we are given the three vectors, so we can just plug them in to get:

$$\begin{vmatrix} 5 & 8 & 4 \\ -2 & 3 & 5 \\ 4 & 0 & 4 \end{vmatrix} = 236$$

(3) Calculate the net torque shown on the board.

Solution: In this kind of problems one is presented with a force $\mathbf{F}$ applied at a point P. Usually we have to compute the torque with respect to the origin, which is given by $\|\mathbf{r} \times \mathbf{F}\| = \|\mathbf{r}\|\|\mathbf{F}\| sin(\theta)$, where $\mathbf{r}$ is the position vector $\overrightarrow{OP}$.

(4) Find the equation of the plane:

(a) With normal vector $\mathbf{n} = \langle 2, -5, 3\rangle$ containing the point $P = (10, 7, 0)$.
(b) Parallel to the plane $3x - 5z = 3$ and containing the point $P = (6, -2, 5)$.
(c) Containing the points $P = (12, 6, 7), Q = (-16, 3, 8)$ and $R = (4, 9, 1)$.
(d) Through the point $P = (-14, 7, 1)$ and containing the line $l(t) = \langle 4 - t, 3 + 5t, -t\rangle$.

Solution: (Sketch, all of these were solved in discussion)

(a) For this you just have to use the equation given in part (1) of the summary above. (make sure that ou understand how to do this, everything related to planes will rest on this kind of computation)

(b) Since it is parallel to $3x - 5z = 3$, it must have the same normal vectors. In particular, we can use $\mathbf{n} = \langle 3, 0, -5\rangle$ as a normal vector. Since we are also given a point P, we can use the equation of part (1) in the summary.

(c) We can take two of the vectors joining them, say $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Both of this must lie in the plane, and therefore both are perpendicular to the normal. We can use the cross product to obtain a normal vector $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$ and then use one of the points (say P) to obtain the equation of the plane.

(d) Choose your favorite two values of t and plug in to get two points on the line. For example we can take $Q = l(0)$ and $P = l(1)$. Now we have three points P, Q, R and we can proceed as in part (d).

(5) Are the planes $6x - 14y + 8t = 13$ and $-21x + 49y - 28t = 0$ parallel?

SOLUTION:

The planes are parallel when their normal vectors are parallel. In this case the normal vectors are $\mathbf{n_1} = \langle 6, -14, 8\rangle$ and $\mathbf{n_2} = \langle -21, 49, -28\rangle$. we can try to solve the equation $\langle 6, -14, 8\rangle = \lambda\langle -21, 49, -28\rangle$. By using the first coordinate or by inspection we see that λ must be 7, and it is easy to check that $\lambda = 7$ indeed works. Therefore the planes are parallel.

(6) The intersection of two nonparallel planes is always a line. Find the parametric and vector equations of the intersection of the planes $3x - 8y + 5z = 4$ and $7x + 4y + 3z = 17$.

SOLUTION: The normal vectors for the planes are $\mathbf{n_1} = \langle 3, -8, 5\rangle$ and $\mathbf{n_2} = \langle 7, 4, 3\rangle$. In this case a direction vector of the line of intersection must be perpendicular to both $\mathbf{n_1}$ and $\mathbf{n_2}$, since it lies in both planes. Hence, we can get a direction vector $\mathbf{v}$ by taking the cross product $\mathbf{v} = \mathbf{n_1} \times \mathbf{n_2} = \langle -44, 26, 68\rangle$.

Now we just need a point of intersection. In order to get one, set $z = 0$ in the equations to both planes to get:

$$\begin{cases} 3x - 8y = 4 \\ 7x + 4y = 17 \end{cases}$$

This is a system with two variables and two unknowns. We can solve to get $x = \frac{38}{17}, \quad y = \frac{23}{68}$. We also have $z = 0$ by assumption, and so we get a point of intersection $P = (\frac{38}{17}, \frac{23}{68}, 0)$. Now using the vector parametrization of the line with direction vector $\mathbf{v}$ and through a point P we get:

$$\mathbf{l}(t) = \langle \frac{38}{17} - 44t, \frac{23}{68} + 26t, 68t\rangle$$

(7) Determine the general equation for all planes with the given trace and specify which trace it corresponds to (which coordinate plane)

(a) The line in the yz-plane given by $8y - 7z = 6$

SOLUTION: This is the yz trace, which is obtained by setting $x = 0$ in the equation of the plane. Given a general plane $Ax + By + Cz = D$ for some constants A, B, C, D, its yz trace is then given by

$$By + Cz = D$$

In order to have the right intersection with the yz plane, this must match with the equation $8y - 7z = 6$. So we conclude that $B = 8, C = -7, D = 6$, and the general equation for a plane satisfying this is given by

$$Ax + 8y - 7z = 6$$

(b) $l(t) = \langle 3-5t, 0, 2t \rangle$

SOLUTION:

This line satisfies $y = 0$ everywhere, and so it lives in the xz plane. Let us find its equation in the xz plane only in terms of x, z. We have:

$$\begin{cases} x = 3 - 5t \\ z = 2t \end{cases}$$

We can, for example, solve for t in terms of z to get $t = \frac{1}{2}z$, and so $x = 3 - \frac{5}{2}z$. This will give the xz trace, and a reasoning similar to part (a) (make sure that you understand this!) gives us that the equation for the general plane with the given trace is $x + By + \frac{5}{2}z = 3$.

(8) At which point does the plane $x - 3y = 9$ intersect the line $r(t) = \langle -7t, 4+8t, 9-t \rangle$?

SOLUTION: Using the equation of the line, we can see that at time t we have $x = -7t, \quad y = 4 + 8t$. Hence, plugging in this into the equation of the plane, we get:

$$-7t - 3(4+8t) = 9 \Longrightarrow t = \frac{-21}{31}$$

We can use this value of t in the equation for $r(t)$ to get the point $P = (\frac{147}{31}, \frac{-44}{31}, \frac{300}{31})$. It should be checked that this point does indeed lie in the plane given.

(9) (Challenge) Show that if $a \perp b$ then there is a vector c with $a \times c = b$