SECTION 14.3 — NAME: SOLUTIONS

Math 1920 - Andres Fernandez — September 13, 2018

SUMMARY OF THE SECTIONS

(1) A vector valued function is a function of the form $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a \leq t \leq b$ is

$$s = \int_a^b \|\mathbf{r}'(t)\| dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt$$

(2) Arc length function : $s(t) = \int_a^t \|\mathbf{r}'(u)\| du$

(3) Speed is the derivative of distance with respect time:

$$v(t) = \frac{ds}{dt} = \|\mathbf{r}'(t)\|$$

(4) We say that $\mathbf{r}(s)$ is an arc length parametrization if $\|\mathbf{r}'(s)\| = 1$ for all s.

Problems

(1) Find the arc length for $1 \leq t \leq 3$:

(a) $\mathbf{r}(t) = \left\langle t, 4t^{\frac{3}{2}}, 2t^{\frac{3}{2}} \right\rangle$

(b) $\mathbf{r}(t) = \langle 2t+1, 8, 3t-4 \rangle$

(c) $\mathbf{r}(t) = \left\langle 4t^2+3, \frac{1}{3}t^3, -3+t^3 \right\rangle$

(d) $\mathbf{r}(t) = \langle 5\sin(t)+2, t, 5\cos(t) \rangle$

Solution:

(a) We have

$$\mathbf{r}'(t) = \left\langle 1,\ 6t^{\frac{1}{2}},\ 3t^{\frac{1}{2}} \right\rangle \quad \longrightarrow \quad ||\mathbf{r}'(t)|| = \sqrt{1+45t}$$

Therefore the arc-length is:

$$s = \int_1^3 ||\mathbf{r}'(t)||\ \mathrm{d}t = \int_1^3 \sqrt{1+45t}\ \mathrm{d}t \quad = \quad \frac{1}{45}\cdot\frac{2}{3}\left(1+45t\right)^{\frac{3}{2}}\Big|_1^3 \quad = \quad \frac{1}{45}\cdot\frac{2}{3}\left(136\right)^{\frac{3}{2}} - \frac{1}{45}\cdot\frac{2}{3}\left(46\right)^{\frac{3}{2}}$$

(b) We have

$$\mathbf{r}'(t) = \langle 2,\ 0,\ 3 \rangle \quad \longrightarrow \quad ||\mathbf{r}'(t)|| = \sqrt{13}$$

Therefore the arc-length is:

$$s = \int_1^3 ||\mathbf{r}'(t)||\ \mathrm{d}t = \int_1^3 \sqrt{13}\ \mathrm{d}t \quad = \quad 2\sqrt{13}$$

(c) We have

$$\mathbf{r}'(t) = \langle 8t,\ t^2,\ 3t^2 \rangle \quad \longrightarrow \quad ||\mathbf{r}'(t)|| = t\sqrt{64+45t^2}$$

Therefore the arc-length is:

$$s = \int_1^3 ||\mathbf{r}'(t)||\ \mathrm{d}t = \int_1^3 t\sqrt{64+10t^2}\ \mathrm{d}t \quad = \quad \frac{1}{20}\cdot\frac{2}{3}\left(64+10t^2\right)^{\frac{3}{2}}\Big|_1^3 \quad = \quad \frac{1}{20}\cdot\frac{2}{3}\left(154\right)^{\frac{3}{2}} - \frac{1}{20}\cdot\frac{2}{3}\left(74\right)^{\frac{3}{2}}$$

(d) For this one

$$\mathbf{r}'(t) = \langle 5\cos t,\ 1,\ -5\sin t \rangle \quad \longrightarrow \quad ||\mathbf{r}'(t)|| = \sqrt{26}$$

Therefore the arc-length is:

$$s = \int_1^3 ||\mathbf{r}'(t)||\ \mathrm{d}t = \int_1^3 \sqrt{26}\ \mathrm{d}t \quad = \quad 2\sqrt{26}$$

(2) Find the speed of the curves at the time $t = 0$:

(a) $\mathbf{r}(t) = \langle \sin(3t),\ \sin(3t - \pi),\ t\cos(7t) \rangle$

(b) $\mathbf{r}(t) = \langle e^{t^2} + 4t,\ ln(t^3 + 1),\ \sqrt{4t^4 + 1} \rangle$

(c) $\mathbf{r}(t) = \langle 3t^5 - 6e^t,\ \tan(3t),\ t - 1 \rangle$

Solution:

(a) For these ones you have to find $\mathbf{r}(0)$ and then take the length. I will do the first one as an example:

$$\mathbf{r}'(t) = \langle 3\cos(3t),\ 3\cos(3t - \pi),\ \cos(7t) - 7t\sin(7t) \rangle \quad \longrightarrow \quad \mathbf{r}'(0) = \langle 3, -3, 1 \rangle$$

Therfore the speed at $t = 0$ is:

$$||\mathbf{r}'(0)|| = \sqrt{19}$$

The rest are similar.

(3) Consider the following equation of motion for a particle $\mathbf{l}(t) = \langle 4\sin(t)\cos(4t-1),\ 4\cos(t)\cos(4t-1),\ 4\sin(4t-1) \rangle$. Show that the particle always stays at a distance of 4 from the origin, and find the speed of the particle as a function of time. (Remark: With the information above, can you see that $\mathbf{l}(t) \cdot \mathbf{l}'(t) = 0$ without doing any computations?)

Solution: The distance to the origin is just $||\mathbf{l}(t)||$, so we calculate:

$$||\mathbf{l}(t)|| = \sqrt{(4\sin(t)\cos(4t-1))^2 + (4\cos(t)\cos(4t-1))^2 + (4\sin(4t-1))^2}$$

Expanding a little bit, and pulling out a 16 out of the square root we get:

$$||\mathbf{l}(t)|| = 4\sqrt{(\sin^2(t) + \cos^2(t))\cos^2(4t-1) + \sin^2(4t-1)} = 4\sqrt{\cos^2(4t-1) + \sin^2(4t-1)} = 4$$

And so we have shown that indeed the particle is always a distance 4 form the origin.

For the speed, we just have to take the derivative and compute the length. It is going to be quite messy given the amount of product rules that we need to apply. I don't think that the computation per se is very illuminating; the main point I wanted to make here is that computing $\mathbf{l}'(t)$ is very complicated. However, we can see that $\mathbf{l}(t) \cdot \mathbf{l}'(t) = 0$ very easily!

First, we know that $\mathbf{l}(t) \cdot \mathbf{l}(t) = ||\mathbf{l}(t)||^2 = 16$ by the computation above. Taking the derivative with respect to t in both sides, we see that

$$2\,\mathbf{l}(t) \cdot \mathbf{l}'(t) = 0$$

(For the left-hand side use the product rule for the dot product, and for the right-hand side it's zero since it is the constant function 16). So we get $\mathbf{l}(t) \cdot \mathbf{l}'(t) = 0$ right away.

(4) Let $\mathbf{r_1} = \langle 27t^3 - 24t + 4,\ 3t + 9,\ 3t \rangle$ and $\mathbf{r_2} = \langle 88t + 4,\ t^2 + 2t - 1,\ 2t \rangle$ be two vector valued functions. Do they collide? Do they intersect? If so find the points of collision/intersection.

Solution: We start with intersection. The two trajectories intersect if they are at the the same point for some (not necessarily equal) values of the parameters. Essentially, we have to solve $\mathbf{r_1}(t) = \mathbf{r_2}(s)$. Setting the equality in each of the coordinates, we get the following system of equations:

$$\begin{cases} 27t^3 - 24t + 4 = 88s + 4 \\ 3t + 9 = s^2 + 2s - 1 \\ 3t = 2s \end{cases}$$

We can substitute use the last equation to express t in terms of s in the second equation. We get

$$2s + 3 + 9 = s^2 + 2s - 1 \qquad \longrightarrow \qquad s = \pm\sqrt{13}$$

We can use the last equation to get t and so we get two possible solutions: $(t, s) = \left(\frac{2}{3}\sqrt{13}, \sqrt{13}\right)$ or $(t, s) = \left(-\frac{2}{3}\sqrt{13}, -\sqrt{13}\right)$.

Now, so far we haven't used the first equation. If you plug in the values above, you will see that the equation is satisfied in both cases, which give the points of intersection $P_1 = \left(88\sqrt{13} + 4,\ 2\sqrt{13} + 12,\ 2\sqrt{13}\right)$ and $P_2 = \left(-88\sqrt{13} + 4,\ -2\sqrt{13} + 12,\ -2\sqrt{13}\right)$

Notice that the times of intersection we got above are not the same for the particles, that is $t \neq s$. Therefore, they don't collide (if they collided at a point, there should be a solution above where $t = s$).

(5) The velocity of an airplane is given by $\mathbf{r}'(t) = \langle 3 - 4t^2, e^t - 4, 5t^4 \rangle$. If the airplane starts at $t = 0$ at the point $P = (3, 6, 0)$

(a) Where is the airplane at time T?

(b) What is the total distance the airplane has flown by time T? (Set up the integral)

Solution:

(a) This is just an initial value problem. Integrating $\mathbf{r}'(t)$ we get:

$$\mathbf{r}(t) = \left\langle 3t - \frac{4}{3}t^3,\ e^t - 4t,\ t^5 \right\rangle + \mathbf{C}$$

Plugging in at $t = 0$ we get that:

$$\langle 0, 1, 0 \rangle + \mathbf{C} = \langle 3, 6, 0 \rangle \qquad \longrightarrow \qquad \mathbf{C} = \langle 3, 5, 0 \rangle$$

And therefore we get that at time t the airplane is at the position $\mathbf{r}(t) = \langle 3t - \frac{4}{3}t^3 + 3,\ e^t - 4t + 5,\ t^5 \rangle$.

(b) The total distance traveled is the arc-length. Here I am expecting you to set up the integral (you do not need to compute it!). We know that the arc-length is given by:

$$s(T) = \int_0^T ||\mathbf{r}'(u)||\, \mathrm{d}u = \int_0^T \sqrt{(3 - 4u^2)^2 + (e^u - 4)^2 + (5u^4)^2}\, \mathrm{d}u$$