Sections 14.3, 15.1

Math 1920 - Andres Fernandez

Name: Solutions

September 14, 2017

Summary of the Sections

(1) A level curve is the set of points given by an equation $f(x,y) = c$ with c a fixed number.

(2) A contour map shows level curves $f(x,y) = c$ for equally spaced value of c.

(3) The closest two level curves are at a given point, the steepest the graph of f is there.

(4) The direction of steepest ascent is always perpendicular to the level curve, and points towards higher altitudes.

(5) The limit of multivariable functions satisfies the following rules:

   (a) $\lim_{(x,y)\to P}(f(x,y) + g(x,y)) = \lim_{(x,y)\to P} f(x,y) + \lim_{(x,y)\to P} g(x,y)$

   (b) $\lim_{(x,y)\to P}(f(x,y) \cdot g(x,y)) = \big(\lim_{(x,y)\to P} f(x,y)\big) \cdot \big(\lim_{(x,y)\to P} g(x,y)\big)$

(6) A function $f(x,y)$ is continuous at $P = (a,b)$ if we have $\lim_{(x,y)\to P} f(x,y) = f(a,b)$

(7) In order to show that a limit does not exist, it suffices to show that the limits along different paths (approaching the point) are not equal.

Problems

(1) Refer to the contour map drawn on the board:

   (a) What is the average rate of change of altitude from point A to point B?
   (b) What is the direction of steepest ascent in each of the points?
   (c) Is the mountain steeper at point A or at point B?

(2) Referring to the same diagram, determine whether the limit of $f(x, y) = z$ exist and find its value:

   a $\lim_{(x,y)\to C} f(x, y)$
   b $\lim_{(x,y)\to A} f(x, y)$

   Solution:

   The solution to both of these problems was provided in recitation.

(3) Draw a contour map of $f(x, y) = xy$ with an appropriate contour interval.

   Solution: The solution to this was also provided in recitation. The contour lines will look like hyperbolas $y = \frac{c}{x}$.

(4) If the atmospheric pressure at certain region is given by $P(x, y, z) = (x-4)^2 + \frac{25}{4}y^2 + 4(z-3)^2$, describe the isobars with pressures 3 and 0.

   Solution: The isobar with pressure 3 is given by the equation $3 = (x - 4)^2 + \frac{25}{4}y^2 + 4(z - 3)^2$. Notice that this is going to be an ellipsoid, according to our standard equations for quadratic surfaces in 3-space.

   The isobar with pressure 0 is going to be given by the equation $0 = (x-4)^2 + \frac{25}{4}y^2 + 4(z-3)^2$. Notice that all the terms in the right of the equation are nonnegative numbers always (as they are squares times positive constants). Therefore, this equation will be satisfied only if $(x-4)^2 = y^2 = (z-3)^2 = 0$, and this happens when $(x, y, z) = (4, 0, 3)$. We conclude that the corresponding isobar consists of only one point, namely $(4, 0, 3)$

(5) Determine if the following limits as $(x, y) \to (0, 0)$ exist:

   (a) $\frac{x^2+y^2}{\sqrt{x^2+y^2}}$
   (b) $\frac{x^2}{x^2+y^2}$
   (c) $\frac{yx^2}{y^3+x^3}$
   (d) $\frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$
   (e) $\frac{5x^3+4yx}{y^4}$
   (f) (Challenge) $cos(\frac{1+y^2x^3}{x^2})$

   Solution:

   a Here we could cancel the square root and easily see that the limit is going to be 0. However, let's illustrate how to solve it in a way that will apply to every limit we will encounter in this course.

   The idea is the following: if I want to prove that a limit exists as I approach the origin, I can change to polar coordinates and show that there is a limit as $r \to 0$ uniformly in θ. Here it will look like this:

i. We change to polar: $\frac{x^2+y^2}{\sqrt{x^2+y^2}} = \frac{r^2}{r} = r$

ii. Now we try to take the limit as $r \to 0$. Here we get $\lim_{r\to 0} r = 0$. Notice that this limit converges no matter what θ is, and so we conclude that the limit exists and is 0.

b We can do the following:

i. We change to polar: $\frac{x^2}{x^2+y^2} = \frac{r^2\cos^2\theta}{r^2} = \cos^2\theta$

ii. Now we try to take the limit as $r \to 0$. Here we get $\lim_{r\to 0}\cos^2\theta = \cos^2\theta$. Here I get different values depending on what theta is. For example, I can plug in $\theta = 0$ and $\theta = \frac{\pi}{2}$ to get two different limits. We can conclude that the limit does not exist.

As a remark, there is an equivalent method of proving that a limit does not exist. Instead of choosing different value of θ and letting $r \to 0$, we could instead set $y = mx$ for different values of m and take the limit as $x \to 0$, and check if the value that you get depends on the choice of m.

These two approaches are the same (you might want to think about this, here there is a correspondence $m = \tan(\theta)$). However, if you want to show that a limit **does exists** then you have to use polar coordinates as in part a, so in case of doubt I would always use polar coordinates to do these kind of problems.

c i. Again we convert to polar coordinates and we get $\frac{\sin\theta\cos^2\theta}{\sin^3\theta+\cos^3\theta}$.

ii. There are no r in the expression above, so there is nothing to do when we take the limit: $\lim_{r\to 0}\frac{\sin\theta\cos^2\theta}{\sin^3\theta+\cos^3\theta} = \frac{\sin\theta\cos^2\theta}{\sin^3\theta+\cos^3\theta}$. Check that if you plug in $\theta = 0$ and $\theta = \frac{\pi}{4}$ you get two different possible limits, and so the limit cannot exist.

(Alternatively, set $y = x$ and $y = 5x$ to see that you get two different answers as $x \to 0$).

d i. Again we convert to polar coordinates and we get $\frac{r^2}{\sqrt{r^2+1}-1}$.

ii. Now we take the limit (by cancelling radicals)

$$\lim_{r\to 0}\frac{r^2}{\sqrt{r^2+1}-1} = \lim_{r\to 0}\frac{r^2(\sqrt{r^2+1}+1)}{(\sqrt{r^2+1}-1)(\sqrt{r^2+1}+1)} = \lim_{r\to 0}\frac{r^2(\sqrt{r^2+1}+1)}{r^2} = \lim_{r\to 0}\sqrt{r^2+1}+1 = 2$$

There is no dependence of θ anywhere, so we have shown that the limit is 2.

e The same polar coordinate technique works, but here perhaps it is a little less confusing to set $y = mx$ and take the limit. I will take this as an opportunity to explicitly show an example of how you do this;

So set $y = x$ and take the limit as $x \to 0$. We try:

$$\lim_{x\to 0}\frac{5x^3+4x^2}{x^4}$$

Notice that this tends to ∞ (look at the exponents), and so the limit cannot exist.