Sections 15.3, 15.4, 15.5

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Name: Solutions

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Summary of the Sections

1. For small small changes Δx, Δy we have:

$$f(a+\Delta x, b) \approx f(a,b) + f_x(a,b)\Delta x$$

$$f(a, b+\Delta y) \approx f(a,b) + f_y(a,b)\Delta y$$

2. Clairaut's theorem states that mixed partials are equal as long as all functions we are dealing with are continuous. Hence, we can take higher partial derivatives in any order we please.

3. The linearization of$f$ in two and three variables:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(y-b)$$

$$L(x,y,z) = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

4. If f_x and f_y exist and are continuous in a disk containing (a,b), then f is differetiable at (a,b).

5. **Equation fo the tangent plane** to z= f(x,y) at (a,b).

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(y-b)$$

6. The **gradient** of a function f is fiven by $\nabla f = \left\langle \frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f, \frac{\partial}{\partial z} f \right\rangle$.

7. **Chain rule for paths**: $\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f_{\mathbf{r}(t)} \cdot \mathbf{r}'(t)$

8. The directional derivative with respect to $\mathbf{u}$ a unite vector is given by $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$

9. If the angle between $\mathbf{u}$ and ∇f is θ, then $D_{\mathbf{u}} f = \|\nabla f\| \|\mathbf{u}\| \cos(\theta)$

10. The equation of the tangent plane to the level surface $F(x,y,z) = k$ at point $P = (a,b,c)$ is

$$\nabla F_P \cdot \langle x-a, y-b, z-c \rangle = 0$$

Problems

1. Suppose that we have a cylinder of radius $r = 90$cm and height $h = 6$cm. Estimate the change in volume if we increase the radius by 2cm.

   Solution:

   We know that the volume of a cylinder is given by $V = \pi r^2 h$. Therefore, the change of volume is going to be approximately:

   $$\Delta V \approx V_r \Delta r + V_h \Delta h = (2\pi r h)\Delta r + (\pi r^2)\Delta h = (2\pi \cdot 90 \cdot 6) \cdot 2 + (\pi 90^2) \cdot 0 = 2160\pi \text{ cm}^3$$

2. Find f such that:

   (a) $\frac{\partial}{\partial x} f = 6x^2 y$, and $\frac{\partial}{\partial y} f = 2x^3 - 3$

   (b) $\frac{\partial}{\partial x} f = e^x - y\sin(xy)$, and $\frac{\partial}{\partial y} f = -x\sin(xy) + 5y^4$

   Solution:

   (a) Integrating in each variable we get:

   $$\begin{cases} f_x = 6x^2 y & \longrightarrow \quad f = 2x^3 y + C_1(y) \\ f_y = 2x^3 - 3 & \longrightarrow \quad f = 2x^3 y - 3y + C_2(x) \end{cases}$$

   We conclude that $f = 2x^3 y - 3y + C$.

   (b) Same thing here:

   $$\begin{cases} f_x = e^x - y\sin(xy) & \longrightarrow \quad f = e^x + \cos(xy) + C_1(y) \\ f_y = -x\sin(xy) + 5y^4 & \longrightarrow \quad f = \cos(xy) + y^5 + C_2(x) \end{cases}$$

   We conclude that $f = e^x + \cos(xy) + y^5 + C$.

3. Find the tangent plane to the following graphs at the given point:

   (a) $f(x, y) = ln(4x^2 - y^2)$ at $(1, 1)$

   (b) $f(x, y) = e^{\frac{x}{y}}$ at $(2, 1)$

   Solution:

   (a) Using $z = ln(4x^2 - y^2) = ln(3)$, we know that the point in space is given by $(x, y, z) = (1, 1, ln(3))$. Now that we have a point for the plane, it suffices a normal vector. But we know that the normal to the surface is given by:

   $$\mathbf{n}_{tan\ plane} = \langle f_x, f_y, -1 \rangle = \left\langle \frac{8x}{4x^2 - y^2}, \frac{-2y}{4x^2 - y^2}, -1 \right\rangle$$

   Plugging in the point we get $\mathbf{n}_{tan\ plane} = \langle \frac{8}{3}, \frac{-2}{3}, -1 \rangle$, so the plane is going to be given by:

   $$0 = \frac{8}{3}(x - 1) + \frac{-2}{3}(y - 1) + (-1)(z - ln(3))$$

(b) Exactly the same thing here. Using $z = e^{\frac{x}{y}} = e^2$, we know that the point in space is given by $(x, y, z) = (2, 1, e^2)$. Now that we have a point for the plane, it suffices a normal vector. But we know that the normal to the surface is given by:

$$\mathbf{n}_{tan\ plane} = \langle f_x, f_y, -1 \rangle = \left\langle \frac{1}{y} e^{\frac{x}{y}},\ -\frac{x}{y^2} e^{\frac{x}{y}},\ -1 \right\rangle$$

Plugging in the point we get $\mathbf{n}_{tan\ plane} = \langle e^2,\ -2e^2,\ -1 \rangle$, so the plane is going to be given by:

$$0 = e^2(x-2) + (-2e^2)(y-1) + (-1)(z-e^2)$$

4. At which points is the vector $\mathbf{n} = \langle 2, 7, 2 \rangle$ normal to the tangent plane of $z = xy^3 + 8y^{-1}$?

SOLUTION: As stated before, the normal to the surface $f(x, y) = xy^3 + 8y^{-1}$ is given by:

$$\mathbf{n}_{tan\ plane} = \langle f_x, f_y, -1 \rangle = \left\langle y^3,\ 3xy^2 - \frac{8}{y^2},\ -1 \right\rangle$$

If we want $\mathbf{n} = \langle 2, 7, 2 \rangle$ to be a normal vector, it has to be parallel to the expression for $\mathbf{n}_{tan\ plane}$ obtained above (any two normal vectors to a plane in 3-space are parallel). Therefore, there is a constant λ such that $\mathbf{n}_{tan\ plane} = \lambda \langle 2, 7, 2 \rangle$. This just means:

$$\begin{cases} y^3 = 2\lambda \\ 3xy^2 - \frac{8}{y^2} = 7\lambda \\ -1 = 2\lambda \end{cases}$$

Here we can solve to get $\lambda = -\frac{1}{2}$, $x = \frac{3}{2}$ and $y = -1$. We are left to find z, and for that we use the fact that the point has to lie on the surface, in other words $z = xy^3 + 8y^{-1} = -\frac{19}{2}$.

5. Let $H(x, y) = 4x^{\frac{3}{2}}y^4$. Show that we have $\frac{\Delta H}{H} \approx \frac{3}{2}\frac{\Delta x}{x} + 4\frac{\Delta y}{y}$. If x is increased by 8% and y is decreased by 6%, what is the percentage of change of H? Does H increase or decrease?

SOLUTION: Here we just use the formula:

$$\Delta H \approx H_x \Delta x + H_y \Delta y = 6x^{\frac{1}{2}}y^4 \Delta x + 16x^{\frac{3}{2}}y^3 \Delta y$$

Now, dividing by H we get:

$$\frac{\Delta H}{H} \approx \frac{6x^{\frac{1}{2}}y^4 \Delta x + 16x^{\frac{3}{2}}y^3 \Delta y}{4x^{\frac{3}{2}}y^4} = \frac{3}{2}\frac{\Delta x}{x} + 4\frac{\Delta y}{y}$$

Finally, we are given that $\frac{\Delta x}{x} = 0.08$ and $\frac{\Delta y}{y} = -0.06$, and so we estimate using the equation above that $\frac{\Delta H}{H} \approx \frac{3}{2} \cdot 0.08 - 4 \cdot 0.06 = -0.12$. We conclude that H decreases by about 12%.

6. Determine whether the derivative of the function along the path is zero, positive or negative in each of the points A, B, C (see blackboard).

SOLUTION: Done in recitation.

7. Determine the derivative of the function along the path given:

(a) $f(x, y, z) = yx^2 - e^{xy} + ln(x)$ $\qquad$ $\mathbf{r}(t) = \langle t^2, ln(t), \sqrt{t} \rangle$

(b) $f(x, y, z) = \sin(xyz)$ $\qquad$ $\mathbf{r}(t) = \langle e^t, \cos(4t), -t \rangle$

Solution: For these two problems I am not concerned about getting the derivatives right. What I wanted to illustrate here is that there are two ways of taking the derivative of the function along the given path. I will solve each of them in a different (equivalent) way:

(a) We can plug in the coordinates to get $f\left(t^2, ln(t), \sqrt{t}\right) = t^4 ln(t) - e^{tln(t)} + 2ln(t)$. Now you can just compute $\frac{\mathrm{d}}{\mathrm{d}t}$ in the usual way to get the derivative along the path.

(b) The derivative along the path can be computed as $\nabla f \cdot \mathbf{r}'(t)$. First, we compute the gradient:

$$\nabla f = \langle yz\cos(xyz),\ xz\cos(xyz),\ xy\cos(xyz)\rangle$$

Changing the coordinates x, y, z in terms of t we get:

$$\nabla f = \left\langle -t\cos(4t)\cos\left(-te^t\cos(4t)\right),\ -te^t\cos\left(-te^t\cos(4t)\right),\ e^t\cos(4t)\cos\left(-te^t\cos(4t)\right)\right\rangle$$

We can also calculate $\mathbf{r}'(t) = \langle e^t,\ -4\sin(4t),\ -1\rangle$. Now doing $\nabla f \cdot \mathbf{r}'(t)$ is just a matter of computation.

8. Find the normal vector to the ellipsoid $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$ at $(3, 0, 0)$. When is the tangent plane normal to the vector $\langle 0, 1, 2\rangle$?

Solution: The ellipsoid is just a level surface for the function $F(x, y, z) = \frac{x^2}{9} + \frac{y^2}{4} + z^2$. What is a vector normal to the level surface? The gradient! So we can compute the gradient of F to get:

$$\mathbf{n}_{tan\ plane} = \nabla F = \left\langle \frac{2}{9}x,\ \frac{1}{2}y,\ 2z\right\rangle$$

If we plug in the point $P = (3, 0, 0)$ we get that the normal of the tangent plane at P is $\left\langle \frac{2}{3}, 0, 0\right\rangle$.

For the last part, the plane is normal to the vector if $\mathbf{n}_{tan\ plane}$ $\langle 0, 1, 2\rangle$. This happens if $\mathbf{n}_{tan\ plane} \times \langle 0, 1, 2\rangle = \mathbf{0}$. Expanding the cross product, this just means:

$$\begin{cases} y - 2z = 0 \\ -\frac{4}{9}x = 0 \\ \frac{2}{9}x = 0 \end{cases}$$

From this we get $x = 0$ and $y = 2z$. Plugging this in the equation of the ellipsoid (remember that the point has to be on the ellipsoid) we get:

$$\frac{(2z)^2}{4} + z^2 = 1 \qquad \longrightarrow \qquad z = \pm\frac{1}{\sqrt{2}}$$

And therefore, using $y = 2z$ we get two points: $\left(0, \sqrt{2}, \frac{1}{\sqrt{2}}\right)$ and $\left(0, -\sqrt{2}, -\frac{1}{\sqrt{2}}\right)$.

9. Find a function such that $\nabla f = \left\langle 2xe^y - z, y^3 + x^2e^y, -x\right\rangle$

Solution: Integrating in each variable we get:

$$\begin{cases} f_x = 2xe^y - z & \longrightarrow \quad f = x^2e^y - xz + C_1(y, z) \\ f_y = y^3 + x^2e^y & \longrightarrow \quad f = \frac{1}{4}y^4 + x^2e^y + C_2(x, z) \\ f_z = -x & \longrightarrow \quad f = -xz + C_3(x, y) \end{cases}$$

We conclude that $f = x^2e^y - xz + \frac{1}{4}y^4 + C$.