SECTION 15.6, 15.7 NAME: SOLUTIONS

Math 1920 - Andres Fernandez October 4, 2018

SUMMARY OF THE SECTIONS

(1) If f is a function of x, y, z and if x, y, z depend on two other variables, say, s and t, then

$$f(x, y, z) = f(x(s,t), y(s,t), z(s,t))$$

is the composite function of s and t. We refer to s and t as the *independent variables.*

(2) The *Chain Rule* expresses the partial derivatives with respect to the independent variables:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s},$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}.$$

(3) The Chain Rule (for a function f in n variables) can be expressed as a dot product:

$$\frac{\partial f}{\partial t_k} = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle \cdot \left\langle \frac{\partial x_1}{\partial t_k}, \frac{\partial x_2}{\partial t_k}, \dots, \frac{\partial x_n}{\partial t_k} \right\rangle.$$

(4) *Implicit differentiation* is used to find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where z is defined implicitly by an equation $F(x, y, z) = 0$:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z},$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

(5) We say that a point $P = (a, b)$ is a *critical point* of $f(x, y)$ if $f_x(a, b) = 0$ or does not exist, and $f_y(a, b) = 0$ or does not exist.

(6) The local minimum or maximum values of f occur at critical points.

(7) The *discriminant* of $f(x, y)$ at $P = (a, b)$ is

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b).$$

(8) *Second Derivative Test:* If $P = (a, b)$ is a critical point of $f(x, y)$, then

$$D(a,b) > 0, \quad f_{xx}(a,b) > 0 \Rightarrow f(a,b) \text{ is a local miminum}$$

$$D(a,b) > 0, \quad f_{xx}(a,b) < 0 \Rightarrow f(a,b) \text{ is a local maximum}$$

$$D(a,b) < 0 \Rightarrow \text{ saddle point}$$

$$D(a,b) = 0 \Rightarrow \text{ test inconclusive}$$

(9) If f is continuous on a closed and bounded domain $\mathcal{D}$, then f takes on both a minimum and maximum value on $\mathcal{D}$. The extreme values occur either at critical points in the interior of $\mathcal{D}$ or at points on the boundary of $\mathcal{D}$.

Problems

(1) Use the Chain Rule to calculate the partial derivatives. Express the answer in terms of the independent variables.

(a) $\frac{\partial h}{\partial t_2}$; $\quad h(x,y) = \frac{x}{y}, \quad x = t_1 t_2, \quad y = t_1^2 t_2.$

Solution: $\frac{\partial h}{\partial t_2} = 0.$

(b) $\frac{\partial F}{\partial y}$; $\quad F(u,v) = e^{u+v}, \quad u = x^2, \quad v = xy.$

Solution: $\frac{\partial F}{\partial y} = xe^{x^2+xy}.$

(2) Suppose that z is defined implicitly as a function of x and y by the equation $F(x,y,z) = xz^2 + y^2z + xy - 1 = 0$.

(a) Calculate F_x, F_y, F_z.

Solution: $F_x = z^2 + y$, $F_y = 2yz + x$, $F_z = 2xz + y^2$.

(b) Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution: $\frac{\partial z}{\partial x} = -\frac{z^2+y}{2xz+y^2}$, $\frac{\partial z}{\partial y} = -\frac{2yz+x}{2xz+y^2}$.

(3) Find the critical points of the function $f(x,y) = x^4 + xy + y^2$, and use the second derivative test to determine the nature of said points (maximum, minimum, saddle or inconclusive)

(4) Three positive numbers add up to 70. What is the biggest their product can be? When does this happen?

(5) Find the maximum of the functions in the given regions:

(a) $f(x,y) = y^2 + xy - x^2$ on the region given by $0 \le x \le 2$ and $0 \le y \le 2$

(b) $f(x,y) = e^{y^2+x^2}$ on the unit disc $r \le 1$

(c) $f(x,y) = e^{y^2+x^2}$ on triangle $\quad 0 \le x, y, \quad x + y \le 1$

(6) Find critical points of the following functions. Determine whether they are local maxima, minima or saddle points.

(a) $(x-y)e^{x^2-y^2}$

(b) $x^3 + y^4 - 6x - 2y^2$