SECTION 17.2, 17.3 — NAME: SOLUTIONS

Math 1920 - Andres Fernandez — November 1, 2018

PROBLEMS

(1) Compute the following line integrals (notice that some of the functions are scalars and others are vectors)

(a) $\int_{\mathcal{C}} z^2\, ds$, where $\mathbf{r}(t) = \langle 3t, 1-t, 4t\rangle$ for $1 \leq t \leq 5$

(b) $\int_{\mathcal{C}} 5\, ds$, where $\mathbf{r}(t) = \langle e^t, \sqrt{2}t, e^{-t}\rangle$ for $0 \leq t \leq 2$

(c) $\int_{\mathcal{C}} \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$, where $\mathcal{C}$ is the line segment from (1,0) to (0,1)

(d) The integral of $\mathbf{F} = \langle x^2, xy\rangle$, along the part of the circle $x^2+y^2 = 9$ with $x \leq 0$, $y \geq 0$ oriented clockwise

SOLUTION:

a

b Both a and b are the same. I will do the second one as an example. The first step is always to compute $\mathbf{r}'(t)$, which in this case is $\mathbf{r}'(t) = \langle e^t, \sqrt{2}, e^{-t}$. Then, since we are integrating a scalar (scalar line integral) we compute the magnitude

$$\|\mathbf{r}'(t)\| = \sqrt{e^{2t}+2+e^{-2t}} = \sqrt{(e^t+e^{-t})^2} = e^t + e^{-t}$$

Here the only tricky thing was recognizing the square (completing the square). Now we can plug in the values given by the parametrization ($x = e^t,\ y = \sqrt{2}t,\ z = e^{-t}$) to get everything in terms of t.

Well in this case we are in luck, since the function we are integrating is just the constant 5, so there is no plugging in needed, and we get (using the bounds for t given)

$$\int_0^2 5e^t + 5e^{-t} dt$$

An integral you should be able to do in your sleep :)

c (Done in class)The main thing here is the parametrization of the path. We know that it is a line from (1,0) to (0,1). Therefore the equation should be of the form $\mathbf{r}(t) = P + t\mathbf{v}$ where P is the starting point and $\mathbf{v} = \overrightarrow{PQ}$ is the dircetion vector. we conclude that $P = (1,0)$ and $\mathbf{v} = \langle 0,1\rangle - \langle 1,0\rangle = \langle -1,1\rangle$.

Our path is then $\mathbf{r}(t) = \langle 1-t, t\rangle$, we just have to find the bounds for t. We have $\mathbf{r}(t) = (1,0)$ when $t = 0$. Also $\mathbf{r}(t) = (0,1)$ when $t = 1$. Therefore the bounds are $0 \leq t \leq 1$. We just now need to find $\mathbf{r}'(t)$ and perform the integral as usual.

Notice that this will be a vector line integral ,as indicated by the dx and dy. When you set up the integral you will have to replace dx by the first coordinate in $\mathbf{r}'(t)$ and dy with the second coordinate of $\mathbf{r}'(t)$. This amounts to computing a dot product (as explained in class this is just alternative notation for a vector line integral with the dot product expanded out).

d Again, the main difficulty is the parametrization. Recall that **circles are always parametrized using** sin **and** cos. In this case since the radius is $\sqrt{9} = 3$, so the parametrization is $\langle 3\cos(t), 3\sin(t)\rangle$. For the bounds, we want $x \leq 0$ and $y \geq 0$, which translates as $\cos(t) \leq 0$ and $\sin(t) \geq 0$. You can draw the unit circle and check what happens on the quadrants, it turns out that these inequalities are satisfied for the second quadrant given by $\frac{\pi}{2} \leq t \leq \pi$.

The next step is to compute the integral, notice that this is a vector line integral. You are integrating a vector field, so you will have to do the dot product $\mathbf{F} \cdot \mathbf{r}'(t)$ and then integrate that.

Now we would be done after taking the integral if the circle was oriented counterclockwise (the standard parametrization given above is counterclockwise). However we want the opposite orientation: clockwise. That is easy enough, once you compute the integral, just multiply by -1, since the integral in the opposite orientation is the minus the original integral. This is a useful trick if you don't want to memorize how to parametrize curves in both orientations, you can always multiply by -1 to get the orientation you want.

(2) Find the mass for the wire given by $\mathbf{r}(t) = \langle \sin(t), \cos(t), \sin^2(t)\rangle$ for $0 \leq t \leq \frac{\pi}{8}$ assuming that the density is $\delta = xy(y^2 - z)$

Solution: The problem is basically asking us to compute the following scalar line integral along the given path:

$$\int_C xy(y^2 - z)ds$$

In order to do this, we compute first $\mathbf{r}'(t) = \langle \cos(t), -\sin(t), 2\sin(t)\cos(t)\rangle$. Next, we compute the magnitude

$$\|\mathbf{r}'(t)\| = \sqrt{\cos^2(t) + (-\sin(t))^2 + (2\sin(t)\cos(t))^2} = \sqrt{1 + \sin^2(2t)}$$

We can plug in the coordinates of the path ($x = \sin(t)$, $y = \cos(t)$, $z = \sin^2(t)$) in the integrand to put everything in terms of t. Using the given bounds for t, we conclude that the integral is:

$$\int_0^{\frac{\pi}{8}} \sin(t)\cos(t)(\cos^2(t) - \sin^2(t))\sqrt{1 + \sin^2(2t)}dt = \int_0^{\frac{\pi}{8}} \frac{1}{2}\sin(2t)\cos(2t)\sqrt{1 + \sin^2(2t)}dt$$

(Notice that we have used the double angle formulas to make everything look a little better. In this case they are not essential, you can do the same u sub I am about to make without changing everything into double angles, it is just more difficult to see what is going on.)

Now we can do a u substitution $u = 1 + \sin^2(2t)$. The new bounds will be $1 + \sin^2(2\frac{\pi}{8}) = \frac{3}{2}$ and $1 + \sin^2(0) = 1$, so we end up with the following integral:

$$\int_1^{\frac{3}{2}} \frac{1}{8}u^{\frac{1}{2}}du$$

which is a straightforward one variable integral.

(3) What is the integral of $\mathbf{r}$ along any circle centered at the origin? Does this mean that $\mathbf{r}$ is conservative?

Solution: If we draw the vector field, we can see that it is everywhere perpendicular to the given paths (circles centered at the origin). This is because $\mathbf{r}$ points radially away from the origin.

This alone is **not** enough to determine if the vector field is conservative. In order to prove that is conservative we would have to prove that the integral along any closed path is zero. Here we are only checking this for circles centered at the origin, but there are many crazy closed path that are not necessarily circles.

(In fact this is a conservative vector field after all, what si the potential?)

(4) Compute the following integral along the ellipse $5x^2 + 3y^2 = 1$, $z = 0$ oriented clockwise

$$\oint_{\mathcal{C}} \cos(x)dx - \sin(z)dy - y\cos(z)dz$$

SOLUTION: There exist a potential function $f = \sin(x) - y\sin(z)$, you can check that $\nabla f = \mathbf{F}$ in this case. Hence the integral along the closed path is 0.

Alternatively, since the integral is not too difficult, you can parametrize the path and actually do the computation. If you encounter a very difficult integral, you might want to check just in case the vector field you are trying to integrate is conservative, that can save lots of time via he fundamental theorem $\int_{\mathcal{C}} \nabla f \cdot ds = f(\text{end point}) - f(\text{starting point})$.

(5) Find the integral of the following vector fields along the path given in the board, or explain why it cannot be done with the information given:

(a) $\langle z\sec^2(x), z, y + \tan(x)\rangle$
(b) $\langle y, e^{xy}, e^z\rangle$
(c) $\langle 2yx\cos(z) + 3x^2, x^2\cos(z) - 3y^2\sin(z), -x^2y\sin(z) + 3z - y^3\cos(z)\rangle$
(d) $\langle \frac{\tan(x)\cos(2y^2)}{x^2+y^2}, \frac{tan(x)+3x^2-y}{x^2+y^2}, \frac{2y^2-5\tan(y)+3zx^4}{x^2+y^2}\rangle$

SOLUTION:

In order to find the integral along the path given, it is necessary to find a potential function. Once we have that we can use the fundamental theorem of conservative vector fields:
$\int_{\mathcal{C}} \nabla f \cdot ds = f(\text{end point}) - f(\text{starting point})$, so we can just plug in the endpoints of the path given and will be done.

I will explain the computation of the vector potentials:

(a) We integrate each of the coordinates with respect to the different variables we get:
  i. $\frac{\partial}{\partial x} f = z\sec^2(x) \longrightarrow f = z\tan(x) + C_1(y, z)$
  ii. $\frac{\partial}{\partial y} f = z \longrightarrow f = zy + C_2(x, z)$
  iii. $\frac{\partial}{\partial z} f = y + \tan(x) \longrightarrow f = z\tan(x) + zy + C_3(y, x)$

  We conclude that we must have $f = zy + z\tan(x)$, and an easy check shows that this has the right partials.

(b) We can try to find a potential and we will see after integrating that the functions don't match up. This is because the given functions are very different (we have a y in the first coordinate, but an exponential in the second). We can guess that the the following quantity $\frac{\partial}{\partial x}\mathbf{F}_2 - \frac{\partial}{\partial y}\mathbf{F}_1$ is not zero. In fact we have $\frac{\partial}{\partial x}\mathbf{F}_2 - \frac{\partial}{\partial y}\mathbf{F}_1 = ye^{xy} - 1 \neq 0$. Since this correspond to the third coordinate of the curl, the vector field is not conservative.

(c) We integrate each of the coordinates with respect to the different variables we get:
  i. $\frac{\partial}{\partial x} f = 2yx\cos(z) + 3x^2 \longrightarrow f = x^2y\cos(z) + x^3 + C_1(y, z)$
  ii. $\frac{\partial}{\partial y} f = x^2\cos(z) - 3y^2\sin(z) \longrightarrow f = x^2y\cos(z) - y^3\sin(z) + C_2(x, z)$

iii. $\frac{\partial}{\partial z} f = -x^2 y \sin(z) + 3z - y^3 \cos(z) \longrightarrow f = x^2 y \cos(z) + \frac{3}{2} z^2 - y^3 \sin(z) + C_3(y, x)$

We conclude that $f = x^2 y \cos(z) + -y^3 \sin(z) + x^3 + \frac{3}{2} z^2$

(d) This one I put for fun, it is not very easy to see what is going on. Notice that the first coordinate does not depend on z, so when we take $\frac{\partial}{\partial z}\mathbf{F}_1$ this will be 0. However, the third coordinate does depend on x, so when we look at $\frac{\partial}{\partial z}\mathbf{F}_1 - \frac{\partial}{\partial x}\mathbf{F}_3 = -\frac{\partial}{\partial x}\mathbf{F}_3$ we will get a nonzero thing, which means that the curl will not be 0 (this corresponds to the second coordinate of the curl). Therefore the vector field will not be conservative.