Section 18.3

Math 1920 - Andres Fernandez

Name: Solutions

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Review

(1) Divergence of a vector field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$:

$$\operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

(2) **Divergence Theorem:** Let $\mathcal{S}$ be a closed surface that encloses a region $\mathcal{W}$ in $\mathbb{R}^3$. Assume that $\mathcal{S}$ is piecewise smooth and is oriented by normal vectors pointing to the outside of $\mathcal{W}$. Let $\mathbf{F}$ be a vector field whose domain contains $\mathcal{W}$. Then

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\, dV.$$

(3) If $\operatorname{div}(\mathbf{F}) = 0$, then $\mathbf{F}$ has zero flux through the boundary $\partial\mathcal{W}$ of any $\mathcal{W}$ contained in the domain of $\mathbf{F}$.

(4) The divergence $\operatorname{div}(\mathbf{F})$ is interpreted as "flux per unit volume", which means that the flux through a small closed surface containing a point P is approximately equal to $\operatorname{div}(\mathbf{F})(P)$ times the enclosed volume.

(5) Basic operations on functions and vector fields:

$$\underset{\text{function}}{f} \xrightarrow{\nabla} \underset{\text{vector field}}{\mathbf{F}} \xrightarrow{\text{curl}} \underset{\text{vector field}}{\mathbf{G}} \xrightarrow{\text{div}} \underset{\text{function}}{g}$$

(6) The result of two consecutive operations is zero:

$$\operatorname{curl}(\nabla(f)) = \mathbf{0}, \qquad \operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0.$$

(7) The inverse-square field $\mathbf{F}_{\text{IS}} = \mathbf{e}_r / r^2$, defined for $r \neq 0$, satisfies $\operatorname{div}(\mathbf{F}_{\text{IS}}) = 0$. The flux of $\mathbf{F}_{\text{IS}}$ through a closed surface $\mathcal{S}$ is 4π if $\mathcal{S}$ contains the origin and is zero otherwise.

Problems

(1) Which of the following is correct (**F** is a continuously differentiable vector field defined everywhere)?

(a) The flux of curl(**F**) through all surfaces is zero.

Solution: This is false, since the surface is not specified to be closed. We can only used the divergence theorem if the surface is closed.

(b) If $\mathbf{F} = \nabla \phi$, then the flux of **F** through all closed surfaces is zero.

Solution: This is false. We are given that **F** has a scalar potential, and so the only thing that we know is that the line integral through all closed paths is 0. We don't know anything about the higher dimensional analogue (closed surface integrals). For this higher dimensional analogue we would need to know that **F** has a vector potential, as shown below in part c.

(c) The flux of curl(**F**) through all closed surfaces is zero.

Solution: This is true, you can prove it using the divergence theorem as follows:

$$\int\int_S curl(\mathbf{F}) \cdot d\mathbf{S} = \int\int\int_{V enclosed} div(curl(\mathbf{F}))\, dV = 0$$

Since $div(curl(\mathbf{F})) = 0$ by the formula in part 6 in the first page.

(2) How does the Divergence Theorem imply that the flux of $\mathbf{F} = \langle x^2, y - e^z, y - 2zx \rangle$ through a closed surface is equal to the enclosed volume?

Solution: An easy computation shows that $div(\mathbf{F}) = 1$ (check it). Therefore, the divergence theorem tells us that:

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int\int_{V enclosed} div(\mathbf{F})\, dV = \int\int\int_{V enclosed} 1\, dV = \text{Volume enclosed}$$

(3) Apply the Divergence Theorem to evaluate the flux $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$.

(a) $\mathbf{F} = \langle xy^2, yz^2, zx^2 \rangle$, $\mathcal{S}$ is the boundary of the cylinder given by $x^2 + y^2 \leq 4$, $0 \leq z \leq 3$.

Solution: First we compute $div(\mathbf{F}) = y^2 + z^2 + x^2$. Since the surface is closed (it is the boundary of a solid volume), we can apply the divergence theorem, and we get:

$$\int\int_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \int\int\int_{V enclosed} div(\mathbf{F})\, dV = \int\int\int_{V enclosed} x^2 + y^2 + z^2\, dV$$

In order to set up the triple integral, it is easier to work with cylindrical coordinates here. Changing everything to cylindrical coordinates and finding the bounds, we get (remember that $dV = r\, dz\, dr\, d\theta$):

$$\int_0^{2\pi} \int_0^2 \int_0^3 (r^2 + z^2)\, r\, dz\, dr\, d\theta = 60\pi$$

I omit the computation of the integral, as it only involves integration of polynomials.

(b) $\mathbf{F} = \langle x+y, z, z-x\rangle$, $\mathcal{S}$ is the boundary of the region between the paraboloid $z = 9 - x^2 - y^2$ and the xy-plane.

SOLUTION:

First we do $div(\mathbf{F}) = 1+0+1 = 2$. As in part (a), we can use the divergence theorem to conclude:

$$\int\int_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \int\int\int_{V enclosed} div(\mathbf{F})\ dV = \int\int\int_{V enclosed} 2\ dV$$

It is again advantageous to use cylindrical coordinates. Notice that we are given $0 \leq z \leq 9 - r^2$, and we can proceed as usual (try to get the values of r when the first bound for z makes sense, and remembering that $0 \leq r$) to get

$$\int_0^{2\pi}\int_0^3\int_0^{9-r^2} 2\ r\, dz\, dr\, d\theta = 81\pi$$

Again the integration is not very difficult per se, it only involves polynomials.

(4) Calculate the flux of the vector field $\mathbf{F} = 2xy\mathbf{i} - y^2\mathbf{j} + \mathbf{k}$ through the surface $\mathcal{S}$ in figure 1. (*Hint:* Apply the Divergence Theorem to the closed surface consisting of $\mathcal{S}$ and the unit disk).

SOLUTION: This is similar to a problem from the previous handout, but here we have to use the divergence theorem. Let $\mathcal{S}_1$ be the surface shown in the picture, and let $\mathcal{S}_2$ be the (filled) unit circle in the xy-plane. Then the union of $\mathcal{S}_1$ and $\mathcal{S}_2$ is a closed surface. Therefore we can use the divergence theorem.

Notice that $div(\mathbf{F}) = 2y - 2y + 0 = 0$, and therefore by divergence thm:

$$\int\int_{\mathcal{S}_1} \mathbf{F}\cdot d\mathbf{S} + \int\int_{\mathcal{S}_2} \mathbf{F}\cdot d\mathbf{S} = \int\int\int_{V enclosed} div(\mathbf{F})\ dV = 0$$

Here the normal is always chosen to point outwards (as stated in the divergence theorem).

Hence it suffices to find the flux through $\mathcal{S}_2$ (with normal pointing downwards, out of the closed surface). The integral can be done in two ways. First, notice that the unit normal to $\mathcal{S}_2$ is $\mathbf{n} = \langle 0, 0, -1\rangle$, as can be seen in the picture (the unit vector pointing downwards). Therefore,

$$\int\int_{\mathcal{S}_2} \mathbf{F}\cdot d\mathbf{S} = \int\int_{\mathcal{S}_2} \mathbf{F}\cdot\mathbf{n}\ dS = \int\int_{\mathcal{S}_2} \langle 2xy, -y^2, 1\rangle \cdot \langle 0, 0, -1\rangle\ dS = -\int\int_{\mathcal{S}_2} dS$$

The last integral is just $-Area(\mathcal{S}_2) = -\pi$, since $\mathcal{S}_2$ is just a unit circle. We conclude that

$$\int\int_{\mathcal{S}_1} \mathbf{F}\cdot d\mathbf{S} = -\int\int_{\mathcal{S}_2} \mathbf{F}\cdot d\mathbf{S} = -(-\pi) = \pi$$

Alternatively you can parametrize the unit circle $S(r,\theta) = \langle r\cos(\theta), r\sin(\theta), 0\rangle$ with $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. If you set up the integral with this parametrization (compute $\mathbf{N} = S_r \times S_\theta$ and make sure that you get the downward pointing normal, etc.) you should get $\int\int_{\mathcal{S}_2} \mathbf{F}\cdot d\mathbf{S} = -\pi$.

(5) Let $I = \iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S}$, where

$$\mathbf{F}(x,y,z) = \left\langle \frac{2yz}{r^2}, -\frac{xz}{r^2}, -\frac{xy}{r^2}\right\rangle$$

($r = \sqrt{x^2+y^2+z^2}$) and $\mathcal{S}$ is the boundary of a region $\mathcal{W}$.

(a) Check that **F** is divergence-free.

SOLUTION: There is not much to say about this. You are asked to prove that $div(\mathbf{F}) = 0$, substitute $r = \sqrt{x^2 + y^2 + z^2}$ and compute the derivatives to see that this is true.

(b) Show that $I = 0$ if $\mathcal{S}$ is a sphere centered at the origin. Explain, however, why the Divergence Theorem cannot be used to prove this.

SOLUTION:

Notice that the Divergence Theorem cannot be used because **F** is not defined at the origin, which is inside the ball with boundary $\mathcal{S}$. We will have to do it by hand.

We can parametrize the sphere using spherical coordinates (here $\rho = 1$ will be constant). The usual parametrization is $S(\theta, \phi) = \langle \cos(\theta)\sin(\phi), \sin(\phi)\sin(\theta), \cos(\phi)\rangle$, with $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$. The normal will be $\mathbf{N} = S_\theta \times S_\phi = \langle \cos(\theta)\sin(\phi), \sin(\phi)\sin(\theta), \cos(\phi)\rangle \sin(\phi)$.

We can convert everything to spherical coordinates to get:

$$\int_0^{2\pi} \int_0^{\pi} \mathbf{F} \cdot \langle \cos(\theta)\sin(\phi),\, \sin(\phi)\sin(\theta),\, \cos(\phi)\rangle\ \sin(\phi)\ d\phi\, d\theta$$

Where $\mathbf{F} = \left\langle \frac{2\sin(\theta)\sin(\phi)\cos(\phi)}{\rho^2}, -\frac{\cos(\theta)\sin(\phi)\cos(\phi)}{\rho^2}, -\frac{\cos(\theta)\sin(\phi)\sin(\theta)\sin(\phi)}{\rho^2} \right\rangle$.

A tedious computation shows that the dot product inside the integral $\mathbf{F} \cdot \mathbf{N} = 0$.

Faster way: An alternative way to see this is the following: the normal to the sphere is always parallel to the radial vector $\mathbf{r} = \langle x, y, z\rangle$. So you don't need to do all that work to get the result, just notice that $\mathbf{F} \cdot \mathbf{r} = 0$ and so the flux will be 0 (because $\mathbf{F} \cdot \mathbf{N} = 0$).

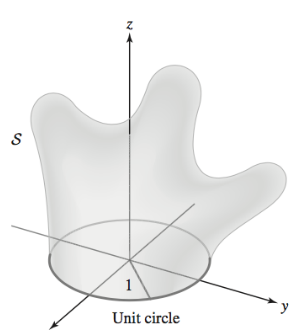

Figure 1: Problem 4.