Orbits of flat group schemes

Andres Fernandez Herrero

Abstract

We show that orbits of a flat group scheme of finite type over a base are representable by a locally closed immersions under the assumption that the stabilizer is flat. We also provide an example to show that this condition on the stabilizer is necessary. This discussion arose from a question of User JJH on Mathoverflow [JJH21].

Setup: Let S be a locally Noetherian scheme; we will work over S (all unlabeled fiber products will be over S). Fix $X \to S$ locally of finite type, and let $G \to S$ be an S-flat group scheme of finite type over S. Suppose that G acts on X, with action morphism $\rho : G \times X \to X$.

Fix a section $x : S \to X$. We can define the scheme theoretic stabilizer of the section x as the S-group scheme G_x defined by the fiber diagram

$$\begin{array}{ccc} G_x & \longrightarrow & S \\ \downarrow & & \downarrow {\scriptstyle (x,x)} \\ G & \xrightarrow{\rho} & X \times X \end{array}$$

Note that G_x admits a monomorphism of S-group schemes $G_x \hookrightarrow G$. Since $X \to S$ is locally of finite type, we know that $G_x \to S$ is of finite type. We shall assume that G_x is S-flat. Then, the classifying stack BG_x is algebraic. Also, we can form the fppf quotient G/G_x as a sheaf, and it is represented by an algebraic space [Sta21, Tag 06PH]. We have the following diagram of algebraic stacks where all squares are Cartesian (cf. [Alp13, §2.1])

$$\begin{array}{ccccc} G & \longrightarrow & G/G_x & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ S & \longrightarrow & BG_x & \longrightarrow & X/G \end{array}$$

By [Alp13, Prop. 2.2], the morphism $BG_x \to X/G$ is a monomorphism. This means that the morphism of algebraic spaces $G/G_x \to X$ is a monomorphism. We set $O := G/G_x \hookrightarrow X$, and call it the orbit of x. The main proposition that we want to show in this note is

Proposition 1. *Let X, G, S as in the setup above. above, and choose a section $x : S \to X$. Suppose that the scheme-theoretic stabilizer G_x is flat over S. Then,*

(a) The orbit O is represented by an S-flat scheme of finite type over S. All geometric S-fibers of O are Cohen-Macaulay and equidimensional.

(b) If the group scheme G is smooth over S, then O is also smooth over S.

(c) The monomorphism $O \hookrightarrow X$ is a locally closed immersion.

The main content of the proposition is part (c). This does not necessarily hold if we don't assume that the stabilizer G_x is flat over S, as the next example shows.

Example 2. *Let k be a field. Let $S = \mathbb{A}^1_k = Spec(k[x])$ and $X = \mathbb{A}^2_k = Spec(k[x, y])$. The constant additive group $\mathbb{G}_a \times S$ over S acts on X by the equation*

$$t \cdot (x, y) = (x, y + xt)$$

The (set-theoretic) orbit of the 0-section $\phi : S \to X$ defined by $\phi(x) = (x, 0)$ consists of the the union $V \cup (0, 0)$, where V is the open complement of the y-axis $(x = 0)$, and $(0, 0) \in \mathbb{A}^2$ is the origin. This subset is not locally closed inside $\mathbb{A}^2$.

In order to show Proposition 1, we will make use of three lemmas.

Lemma 3. *Let $f : Z \hookrightarrow Y$ be a quasicompact monomorphism of schemes locally of finite type over S. Then, the following are equivalent*

(1) The morphism f is a locally closed immersion.

(2) For any discrete valuation ring R and any morphism $Spec(R) \to S$, the base change $f_R : Z_{Spec(R)} \to Y_{Spec(R)}$ is a locally closed immersion.

Proof. The implication (1) ⇒ (2) is clear, since being a locally closed immersion is preserved by arbitrary base-change on S. For the converse, we shall use [Gro66, Cor. 15.7.6]. This says that f satisfies the valuative criterion for local properness (explained below) for all points in the set-theoretic image $f(Z)$ if and only if it factors as $f = h \circ g$, where $h : U \to Y$ is an open immersion and $g : Z \to U$ is proper. In such a factorization, we would have that g is a proper monomorphism, and hence a closed immersion. Since $f : Z \to Y$ is quasicompact, this would imply (by taking the schematic image of f) that f is indeed a locally closed immersion. Therefore, it suffices to show that f satisfies the valuative criterion for local properness at points of $f(Z)$.

Let R be a discrete valuation ring with fraction field K. Suppose that we are given a morphism $Spec(R) \to Y$ that factors set theoretically through $f(Z)$. Then, the valuative criterion stipulates that for every section $Spec(K) \to Y \times_Z Spec(K)$ we must show that there exists a unique extension $Spec(R) \to Y \times_Z Spec(R)$. In order to check this we can, without loss of generality, use the morphism $Spec(R) \to Y \to S$ to base change $f_R : Z_{Spec(R)} \to Y_{Spec(R)}$ and assume that the base S

is $Spec(R)$. Then, by the assumption (2) we know that f_R is a locally closed immersion. In particular it admits a factorization $f_R = h \circ g$ with h an open immersion and g a proper morphism. By [Gro66, Cor. 15.7.6], this implies that the valuative criterion for local properness is satisfied for points in the image of f_R, as desired. $\square$

Lemma 4. *Let S be a locally Noetherian scheme, and let Y, Z be flat S-schemes which are locally of finite type over S. Let $f : Z \to Y$ be a quasicompact surjective morphism such that for all $s \in S$ the morphism of fibers $Z_s \to Y_s$ is a closed immersion. Then, the morphism $Z \to Y$ is a closed immersion.*

Proof. The claim can be checked Zariski locally on the target Y; after passing a finite cover of Y we can assume that both Y and Z are of finite type over S. By assumptions, the morphism $Z \to Y$ is universally injective, so by part (4) in [Sta21, Tag 01S4] it is automatically separated. Since $Z \to Y$ is unramified and universally injective, it suffices to show that it is proper in order to prove that it is a closed immersion [Sta21, Tag 04XV]. We can check this using the valuative criterion for properness. It is enough to check existence for the valuative criterion, because we already know that $Z \to Y$ is separated. Let R be a discrete valuation ring, we will denote by s (resp. η) the special (resp. generic) point of $Spec(R)$. Fix a morphism $Spec(R) \to Y$ and a section $\phi : \eta \to Z \times_Y \eta$. We want to show that there is an extension $\widetilde{\phi} : Spec(R) \to Z \times_Y Spec(R)$. In order to show this, we can safely base-change using the composition $Spec(R) \to Y \to S$ in order to assume that $S = Spec(R)$, so $Spec(R) \to Y$ is a section. We shall therefore assume that S is the spectrum of a DVR.

We know that $Z_\eta \to Y_\eta$ is a closed immersion. By replacing Y with the scheme theoretic closure X of Z_η in Y, we can assume that the morphism $Z \to Y$ induces an isomorphism of generic fibers. Indeed, since Z is flat we know that Z is the scheme theoretic closure of its generic fiber, so the morphism $Z \to Y$ factors through X. Moreover, the scheme closure X is automatically flat over the base $Spec(R)$, since R is a DVR. Since $Z \to Y$ is quasifinite and separated, we can use Zariski's main theorem to get a factorization $Z \hookrightarrow W \to Y$, where $Z \hookrightarrow W$ is open and $W \to Y$ is finite. Since $Z_\eta \to Y_\eta$ is an isomorphism and $W \to Y$ is separated, we see that $Z_\eta \to W_\eta$ is a closed immersion. After replacing W with the scheme theoretic closure of Z_η in W, we can assume that W is flat over R and $Z_\eta \to W_\eta \to Y_\eta$ are all isomorphisms.

Now we return to our section $\phi : \eta \to Z \times_Y \eta$. The image is a closed point p of the generic fibers $Z_\eta \cong W_\eta \cong Y_\eta$. Let V denote the scheme theoretic closure of p in W. We get a commutative diagram of morphisms that are flat over the base

$$\begin{array}{ccc} V & \hookrightarrow & W \\ \downarrow & & \downarrow \\ Spec(R) & \hookrightarrow & Y \end{array}$$

The morphism $V \to Spec(R)$ is finite and flat, and it induces an isomorphism at the generic fiber. By flatness, $\mathcal{O}_V \subset \mathrm{Frac}(R)$. Since R is normal, we must have that $V \to Spec(R)$ is an isomorphism. We denote by v_η and v_s the generic and special points of V inside W. By going up, every point in the special fiber of $W \times_Y Spec(R) \to Spec(R)$ is a specialization of v_η, and so it is contained in the special fiber of V. We conclude that v_s is the unique point in the special fiber of $W \times_Y Spec(R)$. Since $Z \to Y$ is surjective, there must be some point in Z that maps to v_s. In other words, the closed subscheme $V \hookrightarrow W$ is contained in the open $Z \hookrightarrow W$. This means that the fiber product $Z \times_W V \to V$ is an isomorphism. The morphism

$$\widetilde{\phi} : Spec(R) = V \xrightarrow{\sim} Z \times_W V \hookrightarrow Z \times_Y Spec(R)$$

is the extension we were looking for, thus concluding the proof. □

Remark 5. *Lemma 4 is not true if we don't assume surjectivity. See the inclusion of the orbit in Example 7 below.*

Lemma 6. *Let k be a field and let H be a group scheme of finite type over k. Then H is Cohen-Macaulay and equidimensional.*

Proof. The equidimensionality of H is shown in [Sta21, Tag 045X]. In order to show that H is Cohen-Macaulay, we can suppose without loss of generility that k is algebraically closed [Sta21, Tag 0352]. We consider two cases depending on the characteristic of k. If k has characteristic 0, then H is smooth over k [Sta21, Tag 047N], and so it is Cohen-Macaulay. Otherwise, if the characteristic of k is a prime p, then for each n we have a short exact sequence of group schemes

$$1 \to H_n \to H \xrightarrow{q} H/H_n \to 1$$

where H_n is the zero-dimensional n^{th} Frobenius kernel as in [Jan03, §9]. For n big enough, the quotient H/H_n is reduced. Since k is algebraically closed, this means that H/H_n is smooth [Sta21, Tag 047P]. The short exact sequence above makes H a H_n-principal bundle over H/H_n. Let x be a closed point of H, and consider the image $y = q(x)$ in H/H_n. Since the map of local rings $\mathcal{O}_{H/H_n,y} \to \mathcal{O}_{H,x}$ is flat with fiber dimension 0, both local rings have the same dimension d. Since $\mathcal{O}_{H/H_n}$ is regular, there exists a regular sequence $x_1, x_2, \ldots x_d \in \mathcal{O}_{H/H_n}$ of length d inside the maximal ideal of $\mathcal{O}_{H/H_n,y}$. The image of this sequence of elements in $\mathcal{O}_{H,x}$ remains a regular sequence, because the morphism $\mathcal{O}_{H/H_n,y} \to \mathcal{O}_{H,x}$ is flat. This shows that $\mathrm{depth}(\mathcal{O}_{H,x}) \geq d = \dim(\mathcal{O}_{H,x})$, and therefore the local ring $\mathcal{O}_{H,x}$ is Cohen-Macaulay. □

Now we are ready for the proof of the Proposition 1.

Proof of Proposition 1.

Proof of (a): By assumption G is S-flat and of finite type. By Lemma 6, the fibers of G are Cohen-Macaulay and equidimensional. Since being of finite type, Cohen-Macaulay and S-flat can be checked flat locally, the fppf quotient morphism

$G \to G/G_x$ shows that O is of finite type and flat over S, and the fibers are Cohen-Macaulay. Since it is a $G \to G_x$ is a G_x principal bundle and the fibers of both G and G_x are equidimensional, we can use [Sta21, Tag 02JS] to conclude that the fibers of O are equidimensional. Since X is a scheme, we can apply [Sta21, Tag 03XX] to the monomorphism $O \hookrightarrow X$ to conclude that O is a scheme.

Proof of (b): If $G \to S$ is smooth, then the fppf morphism $G \to G/G_x$ shows that O is smooth.

Proof of (c): We will freely use the results from part (a). Note that the formation of the orbit $O \hookrightarrow X$ commutes with arbitrary base-change on S. By Lemma 3, we can assume that S is $Spec(R)$ for a discrete valuation ring R. We will denote by η and s the generic and special points of S respectively. By S-flatness, both equidimensional fibers O_s and O_η have the same dimension n.

Take the scheme theoretic image of $Z \subset X$ of the morphism $\phi : O \hookrightarrow X$. Since S is a DVR and O is S-flat, it follows that Z is automatically flat over X. It can be checked that G still acts on Z (as the scheme theoretic image of a quasicompact G-equivariant morphism), so we might as well replace X with Z and assume that X is flat of finite type over S, and O is scheme-theoretically dense in X. Now taking scheme closure commutes with flat base change, so the generic fiber $O_\eta \hookrightarrow X_\eta$ is scheme theoretically dense. The usual argument for orbits over fields shows that $O_\eta \hookrightarrow X_\eta$ is a scheme-theoretically dense open immersion. Since O_η is equidimensional of dimension n, we must have that X_η is equidimensional of dimension n, and the boundary $B_\eta = X_\eta \setminus O_\eta$ has strictly smaller dimension. By flatness, the dimension of X_s is the same as the dimension of X_η, and so we must have that X_s and O_s have the same dimension. Again, the usual argument for fields implies that $O_s \hookrightarrow X_s$ is locally closed, and since it is full dimension and equidimensional this means that the image of $O_s \subset X_s$ is open (but note that X_s could be nonreduced, so we don't know yet that $O_s \to X_s$ is an open immersion).

Take the set-theoretic closure in X of the boundary $B_\eta \to X_\eta$, let's call it B. The closed subset B is G-stable, and the fibers have smaller dimension than the fibers of O. This means that B_s must be disjoint from the open orbit O_s of bigger dimension. Hence the closed subset B is disjoint from the image of $\phi : O \hookrightarrow X$, and we can replace X with $X \setminus B$. Hence we can assume that the generic fiber $\phi_\eta : O_\eta \to X_\eta$ is an isomorphism. Finally, by removing the closed complement $X_s \setminus O_s$ inside the special fiber X_s, we can assume that the morphism of R-flat schemes $\phi : O \to X$ becomes an surjective closed immersion when restricted to the special fiber. Since the restriction to the generic fiber is also a surjective closed immersion (isomorphism actually), by Lemma 4 we have that $O \hookrightarrow X$ is a closed immersion. Since O is schematically dense inside X, this means that $O \hookrightarrow X$ must be an isomorphism. In summary, we have shown that $O \to X$ induces an open immersion into its scheme theoretic closure. This implies that $O \to X$ is a locally closed immersion. □

Example 7. *The scheme theoretic image Z of the locally closed immersion $O \hookrightarrow X$ does not need to commute with passing to fibers. In other words, the fibers of the orbit O do not need to be dense in the fibers of the closure Z. Therefore one cannot* *in principle conclude properties of the fibers of Z in terms of properties of O.*

Here is an example. Let R be a DVR with uniformizer π and consider the scheme $X = Spec(R[s,t]/(st-\pi))$. We can let $\mathbb{G}_m$ act on X with weight -1 on t and weight 1 on s. Consider the section $(t,s) = (1,\pi)$. The stabilizer of this section is trivial, and the orbit O is the open immersion $O \hookrightarrow X$ with closed complement the vanishing locus of t. At the generic fiber we have an isomorphism $O_\eta \cong X_\eta$ (there is a single orbit at the generic fiber), but at the special fiber the complement $X_s \setminus O_s$ is the vanishing locus of t, which contains two orbits: another open orbit of $(t,s) = (0,1)$, and the closed orbit at the origin $(t,s) = (0,0)$.

References

[Alp13] Jarod Alper. Good moduli spaces for Artin stacks. *Ann. Inst. Fourier (Grenoble)*, 63(6):2349–2402, 2013.

[Gro66] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III. *Inst. Hautes Études Sci. Publ. Math.*, (28):255, 1966.

[Jan03] Jens Carsten Jantzen. *Representations of algebraic groups*, volume 107 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2003.

[JJH21] User JJH. Mathoverflow question. `https://mathoverflow.net/questions/409235/smoothness-of-orbit-of-group-scheme/409924#409924`, 2021.

[Sta21] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`, 2021.