COMBINATORIAL ASPECTS OF SCHUBERT CALCULUS

INTRODUCTION

Schubert calculus is the study of the cohomology of various flag manifolds, where the points of the manifold are chains of nested vector subspaces of $\mathbb{C}^n$ of fixed dimension. The most common examples are the extreme ones, the Grassmannian of k-planes in n-space, $Gr_k(\mathbb{C}^n) = \{V \subseteq \mathbb{C}^n : \dim V = k\}$, and the full flag manifold, $\mathcal{F}l\mathbb{C}^n = \{V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{C}^n : \dim V_i = i\}$. The cohomology ring of the flag manifold is a commutative ring with a (module) basis of "Schubert classes" derived from a stratification of the manifold. In $Gr_k(\mathbb{C}^n)$ this basis is indexed by partitions of at most k parts, of size at most $n-k$ each, and in $\mathcal{F}l\mathbb{C}^n$ the basis is indexed by elements of S_n. My research examines inter-related phenomona in cohomology and quantum cohomology of various flag manifolds.

Fact. *The product of two Schubert classes can be expanded in the basis of Schubert classes in a positive way, for example that $\sigma_u \sigma_v = \sum_{w \in S_n} c_{u,v}^w \sigma_w$ with $c_{u,v}^w \geq 0$ for all u, v and w in S_n.*

These represent a modern point of view on classical questions in enumerative geometry, such as "what is the dimension of the intersection of two lines in the plane in general position?" At a simplistic level, the general coefficient $c_{u,v}^w$ can be thought of as counting the number of points in the intersection of three families of flags where each family of flags is given by the permutations u, v and w.

Similarly, if the classes σ_w are viewed as classes in torus equivariant cohomology, then the $c_{u,v}^w$ are positive combinations of the simple roots of the Lie group $(\mathbb{C}^\times)^n$, and the "intersection" is counted respecting the group action. There is a similar positivity statement here, that the $c_{u,v}^w$ are a product of positive roots.

One of the most fundamental questions in Schubert calculus is providing a method to expand the product of two cohomology classes in an obviously positive way, preferably a combinatorial way that makes it clear that the coefficients (e.g. c_{uv}^w and $c_{\lambda\mu}^\nu$) are counting something. In addition to such a formula's inherent beauty, it would tell us something about computational complexity using Mulmuley's program to prove that $P \neq NP$.

Some cases of this problem have been solved, starting with the celebrated Littlewood-Richardson rule, which solves the problem for $H^*(Gr(k, n))$ using "semistandard Young tableaux."

Theorem ([LR34]). *For classes σ_λ and σ_μ in $Gr(k, n)$, $\sigma_\lambda \cdot \sigma_\mu = \sum c_{\lambda,\mu}^\nu \sigma_\nu$ where $c_{\lambda,\mu}^\nu$ is the number of tableau of shape ν/λ (ν with λ removed) with entries of weight μ (μ_1 1s, μ_2 2s, and so on) that also respect the "lattice word condition."*

Later, Knutson and Tao created puzzles in the form of fillings of an equilateral triangle with puzzle pieces that are either smaller equilateral triangles or rhombi with edges labeled in a particular pattern with 0 and 1 which they used to solve this problem in yet another combinatorial way.

Theorem ([KT03]). *$\sigma_\lambda \cdot \sigma_\mu = \sum c_{\lambda,\mu}^\nu \sigma_\nu$, where $c_{\lambda,\mu}^\nu$ is the number of puzzles with edge weights λ, μ and ν for λ, μ and ν partitions of k parts of size at most $n - k$ encoded as strings of 0s and 1s.*

This solution to the problem has been extended to other generalized cohomology theories such as equivariant cohomology by Knutson and Tao [KT03] by providing more puzzle pieces. Knutson also conjectured a puzzle rule for arbitrary flag manifolds. Sadly, this conjecture is false, however,

Buch, Kresch, Purboo and Tamvakis recently proved Knutson's conjecture for the two-step flag manifold, that is a manifold $\{V \subseteq W \subseteq \mathbb{C}^n : \dim V = k_1 \text{ and } \dim W = k_2\}$.

Theorem ([BKPT]). $\sigma_u \sigma_v = \sum c_{uv}^w \sigma_w$ *where* c_{uv}^w *is the number of* 012 *puzzles with edge weights given by encodings of* u, v *and* w *into strings of* 0s, 1s *and* 2s.

Another generalization of cohomology is quantum cohomology. The quantum cohomology ring is $QH^*(X) = H^*(X) \otimes \mathbb{Z}[q_\omega : \omega$ is a generator of $H^2(X)]$. The structure constants in the product of two cohomology classes are given by the "three point, genus zero Gromov-Witten invariants." Informally, this means that the coefficient $c_{U,V}^W$ in $[U][V] = \sum c_{U,V}^{Wd} q^d [W]$ gives the number of curves (up to homotopy) of degree d through U, V and W. Note that quantum cohomology is not a cohomology theory as it is not functorial, but it does have many properties found in cohomology.

Theorem ([BKT03]). $QH^*(Gr(k, n))$ *can be found using the ordinary cohomology of several two-step flag manifolds.*

This theorem means that in some sense Buch, Kresch, Purboo and Tamvakis have also produced a puzzle rule for quantum cohomology of $Gr(k, n)$. Buch has now also announced an equivariant two-step puzzle rule [Buc] and hence an equivariant quantum puzzle rule for the Grassmannian.

THESIS RESEARCH AND CONTINUING PROJECTS ON THE FOMIN-KIRILLOV ALGEBRA

A partial solution to the case of $\mathcal{F}l\mathbb{C}^n$ that looks especially promising for expressing the product of two basis classes in terms of basis classes in a combinatorial way is the Fomin-Kirillov algebra. In 1999, Fomin and Kirillov [FK99] introduced a non-commutative algebra with generating set $\{T_{ij} : 1 \leq i < j \leq n\}$ that mimics the covering relations of chains the Bruhat order, which is given by containment of reduced words in simple reflections s_i (switch i and $i + 1$) for elements of S_n. The elements of the Fomin-Kirillov algebra act on the set of Schubert classes by

$$T_{ij} \cdot \sigma_w = \begin{cases} \sigma_{w \cdot (ij)} & \text{if } l(w \cdot (ij)) = l(w) + 1 \\ 0 & \text{otherwise} \end{cases}$$

where (ij) is the permutations that switches i and j and leaves other numbers fixed. For example, $T_{13} \cdot \sigma_{213} = \sigma_{312}$ but $T_{13} \cdot \sigma_{123} = 0$, where (in this sentence and hereafter unless otherwise noted) permutations are written in the form $w(1)w(2) \ldots w(n)$.

Theorem ([FK99]).

$$H^*(\mathcal{F}l\mathbb{C}^n) = \mathbb{Z}[x_1, \ldots, x_n] / \langle e_i(x_1, \ldots x_n) : 1 \leq i \leq n \rangle \hookrightarrow \mathbb{Z}[T_{ij} : 1 \leq i < j \leq n]/I$$

$$x_k \mapsto \theta_k = \sum_{k<j} T_{kj} - \sum_{i<k} T_{ik}$$

where

$$I = \langle T_{ij}^2, T_{ij}T_{jk} - T_{jk}T_{ik} - T_{ik}T_{ij}, T_{jk}T_{ij} - T_{ik}T_{jk} - Tij T_{ik} : 1 \leq i < j < k \leq n \rangle$$
$$+ \langle T_{ij}T_{kl} - T_{kl}T_{ij} : i, j, k, l \text{ are distinct} \rangle.$$

Sadly, the presentation of the image of $H^*(\mathcal{F}l\mathbb{C}^n)$ given by Fomin and Kirillov is not positive in the T_{ij} and hence doesn't act on the $\{\sigma_w\}$ in a manifestly positive way. Hence, the challenge is to use the relations in I, or some other insight, to provide a positive presentation of the image of the classes σ_w.

Conjecture ([FK99]). *The image of any class* $\sigma_w \in H^*(\mathcal{F}l\mathbb{C}^n)$ *in the Fomin-Kirillov algebra can be written as a positive sum of products of the* T_{ij} *using the relations given in* I.

One positive result is that of Postnikov [Pos99], who provides a quantum version of Pieri's formula in $QH^*(\mathcal{Fl}\mathbb{C}^n)$, using a quantum deformation of the Fomin-Kirillov algebra suggested in the original paper by Fomin and Kirillov by changing the relation $T_{ij}^2 = 0$ to be $T_{ij}^2 = 0$ if $|i - j| > 1$ but $T_{ii+1}^2 = q_i$. This result tells us how to multiply by a class $\sigma_{c[k,m]}$, where $c[k, m] = s_{m-k}s_{m-k+1}\cdots s_m$ (written in cycle notation $(k - m, k - m + 1, \ldots, m, m + 1)$).

Theorem ([Pos99]). *$\sigma_{c[k,m]}$ can be represented in the Fomin Kirillov algebra as $\sum T_{a_1 b_1} \cdots T_{a_k b_k}$ where the sum is over all distinct $a_1, \ldots a_k$ such that $a_i \leq m$ and $m < b_1 \leq \cdots \leq b_k$.*

Postnikov provides a separate, "dual" formula to multiply by a class $r[k, m] = s_{m+k}s_{m+k-1}\cdots s_m$ (written in cycle notation $(m + k, m + k - 1, \ldots, m)$). I have rephrased Postnikov's results in terms of RC sequences (see [BB93]), $\alpha_1 \leq \cdots \leq \alpha_{l(\pi)}$, each dependent on a reduced word $s_{i_1} \ldots s_{i_{l(\pi)}}$ with $\alpha_j \leq i_j$ and $\alpha_j < \alpha_{j+1}$ if $j_i < (j+1)_i$, as

Theorem (B). *If π is a permutation of the form $c[k, m]$ or $r[k, m]$, then π can be represented in the Fomin-Kirillov algebra as*

$$\sum_Q \sum_\alpha \sum_f \sum_p T_{\alpha_{p(1)} f(1)} \cdots T_{\alpha_{p(l(\pi))} f(l(\pi))}$$

where the sums are over all reduced words Q for π, associated RC sequences α, functions $f : [l(\pi)] \to [n]$ such that $f(i) \geq d$ where d is the unique descent of π and if $i < j$ with $Q_i \geq Q_j$ $f(i) \neq f(j)$ and all permutations p of $\{1, \ldots l(\pi)\}$ such that $f \circ p = f$.

This result has the advantage of combining both of Postnikov's formulae into one formula. Since under the isomorphism $H^*(\mathcal{Fl}\mathbb{C}^n) \simeq \mathbb{Z}[x_1, \ldots x_n]/\langle e_1(x_1, \ldots, x_n), \ldots, e_n(x_1, \ldots, x_n)\rangle$ a representative of the image of σ_w is given by $\sum_Q \sum_\alpha x_{\alpha_1} \cdots x_{\alpha_{l(w)}}$, where the sum is over all reduced words and associated RC sequences, I feel that this approach ought to yield more general results.

Recently, Mészáros has proved a similar result to Postnikov's for classes corresponding to permutations of the form $\lambda(s, t - 1, m) = 1 \ldots (m - t)(m - t + 2) \ldots m(m + s)(m - t + 1)(m + 1) \ldots (m + s - 1)(m + s + 1) \ldots n$.

Theorem ([M]).

$$\sigma_{\lambda(s,t,k)} \sigma_w = \sum_{(*)} c^{\lambda(s,t-1,m)}_{(a_1,b_1),\ldots,(a_{s+t-1},b_{s+t-1})} T_{a_1 b_1} \cdots T_{a_{s+t-1} b_{s+t-1}} \sigma_w$$

where the sum $(*)$ is over all $a_1, \ldots, a_{t-1}$ distinct and at most k, $m < b_1 \leq \cdots \leq b_{t-1}$, $a_t \leq \cdots \leq a_{s+t} \leq m$ and $b_t, \ldots b_{s+t}$ distinct and at least $m + 1$, with a condition limiting the occurrences of the form $T_{ab}T_{a+1b}$ and $T_{ab}T_{ab+1}$ and $c^{\lambda(s,t-1,m)}_{(a_1,b_1),\ldots,(a_{s+t-1},b_{s+t-1})}$ is a combinatorial coefficient.

Also recently, Assaf, Bergeron and Sottile have announced a result computing all $\{c^w_{u,v(\lambda,k)}\}$ for fixed u and w and all classes $v(\lambda, k)$ lifted from $Gr_k(\mathbb{C}^n)$ using the machinery of "dual equivalence" and counting certain chains in the Bruhat order [ABS]. This approach has the advantage of being combinatorial and hence positive, but the disadvantage of not providing a function that can take as inputs u and $v(\lambda, k)$ and return a list of $c^w_{u,v(\lambda,k)}$ where w ranges over all of S_n. Hence, an extention of the Pieri rule using the Fomin-Kirillov algebra that applies to all permutations lifted from a Grassmannian, let alone a more general result for multiplying any pair of Schubert classes σ_v and σ_u, using T_{ij} would still be worthwhile. Further, Assaf, Bergeron and Sottile's result appears less likely to generalize to all Schubert classes, because the classes lifted from $Gr(k, n)$ form a subring of $H^*(\mathcal{Fl}\mathbb{C}^n)$ that can be represented by a ring of symmetric polynomials in the variables $x_1, \ldots, x_k$, to which the machinery of dual equivalence applies, but this is not the case for other classes.

My work approaches further study of combinatorial aspects of the multiplication of classes in generalized cohomology theories using the Fomin-Kirillov algebra in four principal directions: working to broaden the state of the art in known positivity results, extending (currently known

and potential new) positivity results to other generalized cohomology theories e.g. K-theory, extending known results about the Fomin-Kirillov algebra to cohomology rings of flags of other Lie types, and extending the current theory to cohomology rings of more general (partial) flag manifolds.

New Positivity Results. In creating new positivity results, a first step will be to try to rephrase Assaf, Bergeron and Sottile's results in terms of the Fomin-Kirillov algebra using my formulation of Postnikov's result as inspiration. Since my version of Postnikov's result is phrased generally, rather than using the specific structure of the particular permutations involved (as Postnikov's original phrasing did, and the phrasing of Mészáros' new result also does) this seems a promising avenue. There is even some hope that this idea can expand enough to allow for multiplication of arbitrary Schubert polynomials, producing arbitrary $c_{u,v}^w$ in a clearly positive combinatorial way. The first step would be to generalize my formula to also encompass the results of Mészáros and of Assaf, Bergeron and Sottile so as to have a formula that sums over RC sequences and is valid for all classes lifted from Grassmannians. Early computational evidence suggests that such a formula might be generalizable to all classes in $H^*(\mathcal{F}l\mathbb{C}^n)$, not just those lifted from the Grassmannian. I further expect that such a formulation might well extend to quantum cohomology, as the theorems of Postnikov and Mészáros do by design, by using the quantum deformation of the Fomin-Kirillov algebra given in Fomin-Kirillov's original paper [FK99].

Extension to More Generalized Cohomology Theories. One approach is to extend the action of Fomin-Kirillov algebra to encompass (potentially quantum) equivariant Schubert calculus, that is to a cohomology ring that respects the symmetry of the natural torus action on the flag manifold. In [KM10] Kirillov and Maeno formulated an extension of the Fomin-Kirillov algebra that applies to equivariant Schubert polynomials and computes the Pieri rule for multiplying by the class corresponding to the permutation $s_{m-k+1} \cdots s_m$. The next step is to attempt to extend to a formula like this for other classes of permutations with only one descent, such as $r[k, m]$, the hook Schurs from Mészáros' formula, or to all Schurs, using the Assaf, Bergeron, and Sottile formula as inspiration.

Extension to Flags of General Lie Type. Another generalization is to expand the existing theory to other Lie types, that is to G/B for G an arbitrary Lie group and B the corresponding Borel. Here the definition of the T_{ij} can be re-imagined as T_β where β is any positive root of SL_n. Extending this definition to all types, we can write the quantum version of the Cheválley-Monk rule as

$$\sigma_{s_\alpha} * \sigma_w = \sum_{\beta \in \Phi^+} \lambda_\alpha(h_\beta) T_\beta(\sigma_w)$$

where $\lambda_\alpha(h_\beta)$ is height of the simple root α in the postitive root β and

$$T_\beta \cdot \sigma_w = \begin{cases} \sigma_{w \cdot (ij)} & \text{if } l(w \cdot s_\beta) = l(w) + 1 \\ q_\beta & \text{if } l(w \cdot s_\beta) = l(w) - l(\beta) \\ 0 & \text{otherwise} \end{cases}$$

where $q_\beta = q_{i_1} \cdots q_{i_l}$ for β that can be written uniquely as a sum of simple roots $\beta = \alpha_{i_1} + \cdots + \alpha_{i_l}$ (see [Pos05]). In work begun with Elizabeth Beazley, Nicole Lemire, Anne Shepler, Julianna Tymoczko, we have begun formulating a version of Postnikov's quantum Pieri rule for multiplying by special classes that applies in all types. We are currently in the process of testing and refining an early conjecture for a type-free quantum Pieri rule which subsumes Monk's rule and Postnikov's Pieri rule for permutations of the form $c[k, m]$. So far, we have checked the formula for small rank groups of types B and C.

Extension to General G/P**.** One more potential generalization of the Fomin-Kirillov algebra is to find a subalgebra that is isomorphic to the cohomology ring of G/P for P an arbitrary parabolic. A non-positive formula given by Mihalcea [Mih07] suggests that perhaps merely using a subalgebra of the T_β where the β only range over the roots in the radical of P and the quantum deformation of this algebra suggested in [FK99] will allow for a positive presentation of the multiplication operation in $QH^*(G/P)$. Such a result would be amenable to combination with the other avenues of research outlined above because of the similar combinatorial techniques used.

OTHER ON-GOING RESEARCH PROJECTS

Quantum Kirwan Map. In joint work with Tara Holm and Kaisa Taipale, we have begun using the the Kirwan Map, $H_T^*(X) \twoheadrightarrow H^*(X//T)$ [Kir84], and its kernel which was computed by Tolman and Weitsman [TW03] to try to understand the quantum cohomology of spaces that can be obtained as GIT quotients of other spaces with known quantum cohomology. Our first example is computing the quantum cohomology of polygon space, $\{n - \text{gons} \subseteq \mathbb{C}^3\}/SO_3$, which we view as a quotient of $(\mathbb{P}^1)^n$. The idea is that the quantum portion of quantum cohomology comes from the generators $H^2(x)$ and so by studying the Kirwan map specifically in degree 2, we can discern enough information to form a "quantum Kirwan map" and thus calculate the quantum cohomology of the GIT quotient.

Equivariant Rim Hook Rule to Calculate $QH_T^*(Gr(k,n))$**.** With Elizabeth Beazley and Kaisa Taipale, I have begun a project to extend the rim hook rule of Bertram, Ciocan-Fontanine and Fulton [BCFF99] to calculate products in $QH^*(Gr(k,n))$ using the product structure in $H^*(Gr(k, 2n - k))$ to the equivariant setting. We conjecture a similar rule, to produce the product structure on $QH_T^*(Gr(k,n))$ from the product structure on $QH_T^*(Gr(k, 2n - k))$. This requires a (graded) map from the natural torus weights for $Gr(k, 2n - k)$ to the natural torus weights of $Gr(k, n)$. We have proved our conjecture in the Pieri case (when multiplying by a class of codimension 1) and early computation evidence suggests our conjecture is true in general.

REFERENCES

[ABS]   Sami Assaf, Nantel Bergeron, and Frank Sottile, *Multiplying schuberts by schurs (in preparation)*.
[BB93]   Nantel Bergeron and Sara Billey, *RC-graphs and Schubert polynomials*, Experiment. Math. **2** (1993), no. 4, 257–269. MR 1281474 (95g:05107)
[BCFF99] Aaron Bertram, Ionuţ Ciocan-Fontanine, and William Fulton, *Quantum multiplication of Schur polynomials*, J. Algebra **219** (1999), no. 2, 728–746. MR 1706853 (2000k:14042)
[BKPT]   Anders Buch, Kresch, Kevin Purboo, and Harry Tamvakis, *Two step puzzle conjecture (in preperation)*.
[BKT03]   Anders Skovsted Buch, Andrew Kresch, and Harry Tamvakis, *Gromov-Witten invariants on Grassmannians*, J. Amer. Math. Soc. **16** (2003), no. 4, 901–915 (electronic). MR 1992829 (2004h:14060)
[Buc]   Anders Buch, *Quantum equivariant puzzle rule (in preperation)*.
[FK99]   Sergey Fomin and Anatol N. Kirillov, *Quadratic algebras, Dunkl elements, and Schubert calculus*, Advances in geometry, Progr. Math., vol. 172, Birkhäuser Boston, Boston, MA, 1999, pp. 147–182. MR 1667680 (2001a:05152)
[Kir84]   Frances Clare Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes, vol. 31, Princeton University Press, Princeton, NJ, 1984. MR 766741 (86i:58050)
[KM10]   A. N. Kirillov and T. Maeno, *Extended quadratic algebra and a model of the equivariant cohomology ring of flag varieties*, Algebra i Analiz **22** (2010), no. 3, 155–176. MR 2729944 (2012c:14100)
[KT03]   Allen Knutson and Terence Tao, *Puzzles and (equivariant) cohomology of Grassmannians*, Duke Math. J. **119** (2003), no. 2, 221–260. MR 1997946 (2006a:14088)
[LR34]   D.E. Littlewood and A.R. Richardson, *Group characters and algebra*, Phil. Trans. A **233** (1934), 99–141.
[MÍ]   Karola Mészáros, *The fomin-kirillov conjecture in the case of a hook (in preperation)*.
[Mih07]   Leonardo Constantin Mihalcea, *On equivariant quantum cohomology of homogeneous spaces: Chevalley formulae and algorithms*, Duke Math. J. **140** (2007), no. 2, 321–350. MR 2359822 (2008j:14106)
[Pos99]   Alexander Postnikov, *On a quantum version of Pieri's formula*, Advances in geometry, Progr. Math., vol. 172, Birkhäuser Boston, Boston, MA, 1999, pp. 371–383. MR 1667687 (99m:14096)

[Pos05] _____ , *Quantum Bruhat graph and Schubert polynomials*, Proc. Amer. Math. Soc. **133** (2005), no. 3, 699–709 (electronic). MR 2113918 (2005i:05195)

[TW03] Susan Tolman and Jonathan Weitsman, *The cohomology rings of symplectic quotients*, Comm. Anal. Geom. **11** (2003), no. 4, 751–773. MR 2015175 (2004k:53140)