Unions of Matrix Schubert Varieties

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- **Definition** A matrix Schubert variety is the closure of the sweep of any matrix in M_n by $B_- \times B_+$.
- Since we are allowed downward row and rightward column operations, these are represented by partial permutation matrices:

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Fulton's Theorem

- **Question:** What equations cut out these orbits?
- Observe that we only need northwest rank conditions to describe these orbits.
- **Theorem (Fulton, 1992):** Only the equations from the "essential" northwest rank conditions are needed to cut out the orbit $\overline{X_\pi}$.
- **Definition** The essential set for a permutation is formed by taking the Rothe diagram of the permutation (cross out below and to the right of each 1 in the permutation matrix, the remaining boxes are the diagram) and selecting the boxes in the diagram that have no boxes immediately to the south or east.

Fulton's Theorem Example

$$I_{15432} = I(\overline{X_{15432}})$$

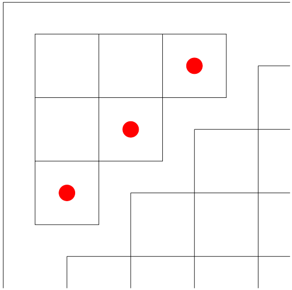


Figure: The generators for I_{15432} are the 2×2 minors of the northwest 2×4, 3×3 and 4×2 submatrices of a matrix of variables.

A Question

- What are the equations that cut out

$$\cup_{\pi \in S} \overline{X_\pi}$$

  for an arbirary S?
- That is, what is

$$I(\cup_{\pi \in S} \overline{X_\pi})$$

  for an arbirary S?

A Plan of Attack

- **Theorem (Knutson-Miller 2004)** The set of equations given by Fulton form a Gröbner basis under any "antidiagonal" term order and the initial ideal is Stanley-Riesner.
- The I_π are radical (indeed, compatibly Frobenius split).
- We know that

$$\text{init } (I_{\pi_1} \cap \cdots \cap I_{\pi_n}) = \text{init } I_{\pi_1} \cap \cdots \cap \text{init } I_{\pi_n}$$

  and further it's easy to find a generating set for this.
- So for each set of antidiagonals $A_1, \ldots, A_n$ we form a generator $g_{A_1,\ldots,A_n}$ that is in each I_{π_i} and has initial term the union of the antidiagonals $A_1, \ldots A_n$.

$g_{A_1,\ldots,A_n}$

- The generator $g_{A_1,\ldots,A_n}$ is given by

$$g_{A_1,\ldots,A_n} = \sum (-1)^{\text{sign}(f)} \prod_{b \in \cup A_i} m_{\text{row}(b),f(b)}$$

where the sum is over all possible functions

$$f : \{\cup A_i\} \to \text{ the columns}$$

subject to the restrictions:

1. For each column c, $|f^{-1}(c)| = |c \cap (\cup A_i)|$. Equivalently, every summand is the same $T \times T$ weight as init g.
2. For each column c and for each antidiagonal A_i, $|f(A_i) \cap c| \leq 1$.
3. For each box $b \in A_i$, $f(b) \leq \max\{c :$ c is is a column of an antidiagonal S containing $A_i\}$, where "antidiagonal" is the broadest sense of the word–it refers to any collection of boxes such that if $\text{row}(a) \geq \text{row}(b)$ then $\text{column}(a) < \text{column}(b)$.

Example

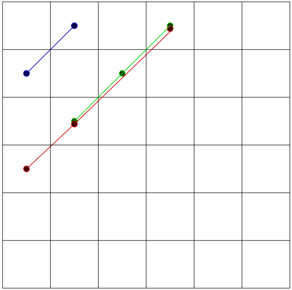
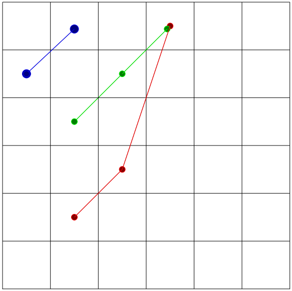


Figure: The generator for these three antidiagonals would be

$$\begin{vmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{vmatrix} \begin{vmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{vmatrix}.$$

Example



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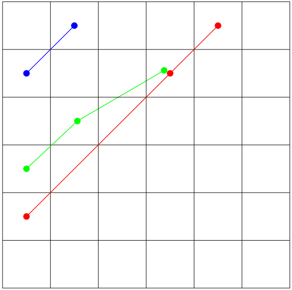

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$$\left(\begin{vmatrix} m_{2,1} & m_{2,2} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} \end{vmatrix} \begin{vmatrix} m_{1,1} & m_{1,5} \\ m_{5,1} & m_{2,2} \end{vmatrix} - \begin{vmatrix} m_{1,1} & m_{1,2} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,4} \end{vmatrix} \begin{vmatrix} m_{2,1} & m_{1,5} \\ m_{5,1} & m_{5,5} \end{vmatrix} \right.$$

$$\left. + \begin{vmatrix} m_{1,1} & m_{1,2} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,4} \end{vmatrix} \begin{vmatrix} m_{3,1} & m_{2,5} \\ m_{5,1} & m_{5,5} \end{vmatrix} - \begin{vmatrix} m_{1,1} & m_{1,2} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{4,4} \end{vmatrix} \begin{vmatrix} m_{4,1} & m_{3,5} \\ m_{5,1} & m_{5,5} \end{vmatrix} \right)$$