Homotopical Algebra

Yuri Berest, Sasha Patotski

Fall 2015

These are lecture notes of a graduate course MATH 7400 currently taught by Yuri Berest at Cornell University. The notes are taken by Sasha Patotski; the updates will appear every week. If you are reading the notes, please send us corrections; your questions and comments are also very much welcome.

Emails: berest@math.cornell.edu and apatotski@math.cornell.edu.

Contents

Chapter 1

Homotopy theory of categories

1 Basics of simplicial sets

1.1 Definitions

Definition 1.1.1. *The* simplicial category $\mathbf{\Delta}$ *is a category having the finite totally ordered sets* $[n] := \{0, 1, \dots, n\}$, $n \in \mathbb{N}$, *as objects, and order-preserving functions as morphisms (i.e. the functions* $f\colon [n] \to [m]$ *such that* $i \leqslant j \Rightarrow f(i) \leqslant f(j)$*).*

Definition 1.1.2. *A* simplicial set *is a contravariant functor from* $\mathbf{\Delta}$ *to the category of sets* **Sets**. *Morphism of simplicial sets is just a natural transformation of functors.*

Notation 1.1.3. We denote the category $\mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Sets})$ of simplicial sets by **sSets**.

Definition 1.1.4. *Dually, we define* cosimplicial sets *as* covariant *functors* $\mathbf{\Delta} \to \mathbf{Sets}$. *We denote the category of cosimplicial sets by* **cSets**.

There are two distinguished classes of morphisms in $\mathbf{\Delta}$:

$$\begin{aligned} &d^i\colon [n-1] \hookrightarrow [n] && (n \geqslant 1,\ 0 \leqslant i \leqslant n) \\ &s^j\colon [n+1] \twoheadrightarrow [n] && (n \geqslant 0,\ 0 \leqslant j \leqslant n) \end{aligned}$$

called the *face maps* and *degeneracy maps* respectively. Informally, d^i skips the value i in its image and s^j doubles the value j. More precisely, they are defined by

$$d^i(k) = \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } k \geqslant i \end{cases} \qquad s^j(k) = \begin{cases} k & \text{if } k \leqslant j \\ k-1 & \text{if } k > j \end{cases}$$

Theorem 1.1.5. *Every morphism* $f \in \mathrm{Hom}_{\mathbf{\Delta}}([n], [m])$ *can be decomposed in a unique way as*

$$f = d^{i_1} d^{i_2} \cdots d^{i_r} s^{j_1} \cdots s^{j_s}$$

where $m = n - s + r$ *and* $i_1 < \cdots < i_r$ *and* $j_1 < \cdots < j_s$.

The proof of this theorem is a little technical, but a few examples make it clear what is going on. For the proof, see for example Lemma 2.2 in [GZ67].

Example 1.1.6. Let $f\colon [3] \to [1]$ be $\{0, 1 \mapsto 0; 2, 3 \mapsto 1\}$. One can easily check that $f = s^0 \circ s^2$.

Corollary 1.1.7. *For any $f \in \mathrm{Hom}_{\mathbf{\Delta}}([n], [m])$, there is a unique factorization*

$$f\colon [n] \overset{s}{\twoheadrightarrow} [k] \overset{d}{\hookrightarrow} [m]$$

where s is surjective and d injective.

Corollary 1.1.8. *The category $\mathbf{\Delta}$ can be presented by $\{d^i\}$ and $\{s^j\}$ as generators with the following relations:*

$$\begin{aligned}
d^j d^i &= d^i d^{j-1} && i < j \\
s^j s^i &= s^i s^{j+1} && i \leqslant j \\
s^j d^i &= \begin{cases} d^i s^{j-1} & \text{if } i < j \\ id & \text{if } i = j \text{ or } i = j+1 \\ d^{i-1} s^j & \text{if } i > j+1 \end{cases}
\end{aligned} \tag{1.1}$$

Corollary 1.1.9. *Giving a simplicial set $X_* = \{X_n\}_{n \geqslant 0}$ is equivalent to giving a family of sets $\{X_n\}$ equipped with morphisms $d_i\colon X_n \to X_{n-1}$ and $s_i\colon X_n \to X_{n+1}$ satisfying*

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i && i < j \\
s_i s_j &= s_{j+1} s_i && i \leqslant j \\
d_i s_j &= \begin{cases} s_{j-1} d_i & \text{if } i < j \\ \mathrm{id} & \text{if } i = j \text{ or } i = j+1 \\ s_j d_{i-1} & \text{if } i > j+1 \end{cases}
\end{aligned} \tag{1.2}$$

The relation between (1.1) and (1.2) is given by $d_i = X(d^i)$ and $s_i = X(s^i)$.

Remark 1.1.10. It is convenient (at least morally) to think of simplicial sets as a graded right "module" over the category $\mathbf{\Delta}$, and of cosimplicial set as a graded *left* "module". A standard way to write $X \in \mathrm{Ob}(\mathbf{sSets})$ is

$$\begin{tikzcd}
X_0 \arrow[r, dashed, shift left=0] & X_1 \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[r, dashed, shift left=1.5] \arrow[r, dashed, shift right=1.5] & X_2 \arrow[l, shift right=3] \arrow[l] \arrow[l, shift left=3] & \cdots
\end{tikzcd}$$

with solid arrows denoting the maps d_i and dashed arrows denoting the maps s_j.

Definition 1.1.11. *For a simplicial set X_* the elements of $X_n := X[n] \in$ **Sets** are called n-simplices.*

Definition 1.1.12. *For a simplicial set X_*, an n-simplex $x \in X_n$ is called* degenerate *if $x \in \mathrm{Im}(s_j \colon X_{n-1} \to X_n)$ for some j. The set of degenerate n-simplices is given by*

$$\{\textit{degenerate } n\textit{-simplices}\} = \bigcup_{j=0}^{n-1} s_j(X_{n-1}) \subseteq X_n \tag{1.3}$$

An alternative way to describe degenerate simplices is provided by the following

Lemma 1.1.13. *The set of degenerate n-simplices in X_* is given by*

$$\bigcup_{\substack{f \colon [n] \twoheadrightarrow [k] \\ f \neq \mathrm{id}}} X(f)\,(X_k) = \operatorname*{colim}_{f \colon [n] \twoheadrightarrow [k]} X(f)(X_k). \tag{1.4}$$

Proof. Exercise. □

Remark 1.1.14. Although formula (1.3) looks simpler, formula (1.4) has advantage over (1.3) as it can be used to define degeneracies in simplicial objects of an arbitrary category $\mathbf{C}$ (at least, when the latter has small colimits).

Definition 1.1.15. *If $X = \{X_n\}$ is a simplicial set, and $Y_n \subseteq X_n$ is a family of subsets, $\forall n \geqslant 0$, then we call $Y = \{Y_n\}$ a* simplicial subset *of X if*

- *Y forms a simplicial set,*
- *the inclusion $Y \hookrightarrow X$ is a morphism of simplicial sets.*

The definition of simplicial (and cosimplicial) sets can be easily generalized to simplicial and cosimplicial objects in any category.

Definition 1.1.16. *For any category $\mathbf{C}$ we define a* simplicial object in $\mathbf{C}$ *as a functor $\mathbf{\Delta}^{op} \to \mathbf{C}$. The simplicial objects in any category $\mathbf{C}$ form a category, which we will denote by $\mathbf{sC}$.*

Definition 1.1.17. *Dually, we define a* cosimplicial object in $\mathbf{C}$ *as a (covariant) functor $\mathbf{\Delta} \to \mathbf{C}$. The category $\mathbf{Fun}(\mathbf{\Delta}, \mathbf{C})$ of cosimplicial objects in $\mathbf{C}$ will be denoted by $\mathbf{cC}$.*

1.2 Motivation

In this section we provide some motivation for the definition of simplicial sets, which might seem rather counter-intuitive. We will discuss various examples later in section 1.3 which will hopefully help to develop some intuition.

Definition 1.2.1. *The n-*dimensional geometric simplex *is the topological space*

$$\Delta^n = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \colon \sum_{i=0}^{n} x_i = 1, x_i \geqslant 0 \right\}$$

Thus Δ^0 is a point, Δ^1 is an interval, Δ^2 is an equilateral triangle, Δ^3 is a filled tetrahedron, etc. We will label the vertices of Δ^n as $e_0, \dots, e_n$. Then any point $x \in \Delta^n$ can be written as a linear combination $x = \sum x_i e_i$, where $x_i \in \mathbb{R}$ are called the *barycentric coordinates* of x.

To any subset $\alpha \subset [n] = \{0, 1, \dots, n\}$ we can associate a simplex $\Delta_\alpha \subset \Delta^n$, which is the convex hull of vertices e_i with $i \in \alpha$. The sub-simplex Δ_α is also called α-face of Δ^n. Thus we have a bijection

$$\{\text{subsets of } [n]\} \leftrightarrow \{\text{faces of } \Delta^n\}.$$

Definition 1.2.2. *By a* finite polyhedron *we mean a topological spaces X homeomorphic to a union of faces of a simplex Δ^n:*

$$X \simeq S = \Delta_{\alpha_1} \cup \dots \cup \Delta_{\alpha_r} \subset \Delta^n.$$

The choice of such a homeomorphism is called a triangulation*.*

Infinite polyhedra arize from simplicial complexes, which we define next.

Definition 1.2.3. *A* simplicial complex *X on a set V (called the set of vertices) is a collection of non-empty finite subsets of V, closed under taking subsets, i.e.*

$$\forall \sigma \in X, \emptyset \neq \tau \subset \sigma \Rightarrow \tau \in X.$$

Note that the set V is not necessarily finite, nor $V = \bigcup_{\sigma \in X} \sigma$. To *realize* a simplicial complex X, we define an $\mathbb{R}$-vector space $\mathbb{V}$ spanned by elements of V, and for every $\sigma \in X$ we define a simplex $\Delta_\sigma \subset \mathbb{V}$ to be the convex hull of $\sigma \subset V \subset \mathbb{V}$.

Definition 1.2.4. *The* realization *$|X|$ of a simplicial complex X is the union*

$$|X| = \bigcup_{\sigma \in X} \Delta_\sigma \subset \mathbb{V}$$

equipped with induced topology from $\mathbb{V}$, i.e. $U \subset |X|$ is open if and only if $U \cap \Delta_\sigma$ is open in Δ_σ for all $\sigma \in X$.

Definition 1.2.5. *A* polyhedron *is a topological space homeomorphic to the realization $|X|$ of some simplicial complex X.*

The quotient of a simplicial complex by a simplicial subcomplex may not be a simplicial complex again. Simplicial sets are then a generalization of simplicial complexes which "capture in full the homotopy theory of spaces". We will make this statement precise later when we will discuss Quillen equivalences, and in particular the Quillen equivalence between topological spaces and simplicial sets.

To any simplicial complex one can associate a simplicial set in the following way. Suppose we have a simplicial complex (X, V) with V being totally ordered. Define $SS_*(X)$ to be a simplicial set having $SS_n(X)$ the set of ordered $(n+1)$-tuples $(v_0, \dots, v_n)$, where $v_i \in V$ are vertices such that $\{v_0, \dots, v_n\} \in X$. Notice that we do allow $v_i = v_j$ for some $i \neq j$ in the tuple $(v_0, \dots, v_n)$, and $\{v_0, \dots, v_n\}$ means the underlying set of that tuple.

If $f\colon [n] \to [m]$ is a morphism in $\mathbf{\Delta}$, the induced map $f_*\colon SS_m(X) \to SS_n(X)$ is given by $(v_0, \dots, v_n) \mapsto (v_{f(0)}, \dots, v_{f(n)})$. In particular, the face map $d_i\colon SS_n(X) \to SS_{n-1}(X)$ is given by

$$d_i\colon (v_0, \dots, v_n) \mapsto (v_0, \dots, \widehat{v_i}, \dots, v_n) \qquad \text{(omitting the } i\text{-th element)}$$

and the degeneracy map $s_j\colon SS_n(X) \to SS_{n+1}(X)$ is given by

$$s_j\colon (v_0, \dots, v_n) \mapsto (v_0, \dots, v_j, v_j, \dots, v_n) \quad \text{(repeating the } j\text{-th element)}$$

Exercise 1.

1. Show that the simplicial set $SS_*(X)$ determines X. More precisely, there is a bijection between X and the set of non-degenerate simplices in $SS_*(X)$.
2. For any (ordered) simplicial complex (X, V) (i.e. V is totally ordered) we have homeomorphism $|X| \simeq |SS_*(X)|$, where $|SS_*(X)|$ denotes the geometric realization of the simplicial set $SS_*(X)$. (see section 2.2 for the definition of geometric realization).

1.3 Examples of (co-)simplicial sets

Example 1.3.1 (Discrete simplicial sets)**.** To any set $A \in \mathbf{Sets}$ we can associate a simplicial set $A_* \in \mathbf{sSets}$ defined by $A_n = A$, $\forall n$ with all maps d_i, s_j being id_A. This gives a fully faithful embedding $\mathbf{Sets} \hookrightarrow \mathbf{sSets}$. The simplicial sets arising in this way are called *discrete*.

Example 1.3.2 (Geometric simplices)**.** Consider the n-dimensional geometric simplex $\Delta^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1}_{\geqslant 0}\colon \sum x_i = 1\}$ (see Section 1.2).

Given $f\colon [m] \to [n]$, the corresponding $\Delta^f\colon \Delta^m \to \Delta^n$ is defined to be the restriction of the linear map $\mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$, sending the vertex e_i to $e_{f(i)}$. The collection $\Delta^* = \{\Delta^n\}$ forms a cosimplicial set.

Remark 1.3.3. The assignment $[n] \mapsto \Delta^n$ defines a faithful functor $\mathbf{\Delta} \hookrightarrow \mathbf{Spaces}$ which gives a topological realization of the simplicial category. Sometimes we will identify $\mathbf{\Delta}$ with the subcategory of $\mathbf{Spaces}$ which is its image. Historically, the category $\mathbf{\Delta}$ first arose as this subcategory.

Example 1.3.4 (Singular set functor)**.** Given a topological space X, define $S_n(X) := \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, X)$. Then $S_*(X) = \{S_n(X)\}$ is naturally a simplicial set, called the set of *singular simplices* in X (or the *singular set* of X).

More generally, for any *cosimplicial* object $Y\colon \mathbf{\Delta} \to \mathbf{C}$ in a category $\mathbf{C}$ and any contravariant functor $F\colon \mathbf{C}^{op} \to \mathbf{Sets}$ the composition

$$\mathbf{\Delta}^{op} \xrightarrow{Y^{op}} \mathbf{C}^{op} \xrightarrow{F} \mathbf{Sets}$$

gives a simplicial set.

Example 1.3.5 (Standard simplices)**.** Recall that for any category $\mathbf{C}$ one has the Yoneda embedding $h\colon \mathbf{C} \hookrightarrow \mathbf{Fun}(\mathbf{C}^{op}, \mathbf{Sets})$ sending each object $X \in \mathrm{Ob}(\mathbf{C})$ to the functor $h_X = \mathrm{Hom}_C(-, X)$ represented by this object. In particular, when $\mathbf{C} = \mathbf{\Delta}$, the Yoneda functor gives an embedding $\mathbf{\Delta} \hookrightarrow \mathbf{sSets}$.

The *standard n-simplex* is a simplicial set $\Delta[n]_*$ defined as the image of $[n] \in \mathrm{Ob}(\mathbf{\Delta})$ under the Yoneda embedding.

Thus $\Delta[n]_*$ is just a simplicial set with

$$\Delta[n]_k = \mathrm{Hom}_{\mathbf{\Delta}}([k], [n]) \simeq \{(j_0, \dots, j_k) \mid 0 \leqslant j_0 \leqslant \dots \leqslant j_k \leqslant n\},$$

where $f\colon [k] \to [n]$ corresponds to the sequence of its values $(f(0), \dots, f(k))$ in $[n]$.

The face and degeneracy maps are then given by

$$d_i\colon (j_0, \dots, j_k) \mapsto (j_0, \dots, \widehat{j_i}, \dots, j_k), \qquad s_l\colon (j_0, \dots, j_k) \mapsto (j_0, \dots, j_l, j_l, \dots, j_k)$$

If $n = 0$, there is only one k-simplex in $\Delta[0]_k$ for each k, namely, $\Delta[0]_k = \{(0, 0, \dots, 0)\}$, and the only non-degenerate simplex in $\Delta[0]_*$ is the 0-simplex (0).

If $n = 1$, the set of k-simplices in $\Delta[1]_*$ is given by

$$\Delta[1]_k = \{(j_0, \dots, j_k) \mid 0 \leqslant j_0 \leqslant \dots \leqslant j_k \leqslant 1\} = \{(\underbrace{0, \dots, 0}_{i}, \underbrace{1, \dots, 1}_{k+1-i}) \mid 0 \leqslant i \leqslant k+1\},$$

so $\Delta[1]_k$ has $k+2$ simplices in it. Obviously, there are three non-degenerate simplices: two are 0-simplices (0) and (1), and one is the 1-simplex $(0, 1)$.

Another way to describe $\Delta[n]_*$ is given by the following

Lemma 1.3.6. *Simplicial set $\Delta[n]_*$ (co)represents the functor* $\mathbf{sSets} \to \mathbf{Sets}$*, $X_* \mapsto X_n$. In other words, there is a natural isomorphism of sets $\mathrm{Hom}_{\mathbf{sSets}}(\Delta[n]_*, X_*) \simeq X_n$.*

Proof. This is an immediate consequence of Yoneda lemma. □

Note that the assignment $[n] \mapsto \Delta[n]_*$ defines a fully faithful functor $\mathbf{\Delta} \to \mathbf{sSets}$, which is nothing but the (covariant) Yoneda embedding. Combined together the simplicial sets $\Delta[n]_*$ define a cosimplicial simplicial set (i.e. a cosimplicial object in the category $\mathbf{sSets}$) $\Delta[\bullet]_* \in \mathrm{Ob}(\mathbf{csSets})$.

Remark 1.3.7. A convenient way to think about cosimplicial simplicial sets is as of "graded bimodules" over $\mathbf{\Delta}$ (cf. remark 1.1.10).

There are face maps $d^i\colon \Delta[n]_* \to \Delta[n+1]_*$ and degeneracy maps $s^j\colon \Delta[n]_* \to \Delta[n-1]_*$. For example, $d^0\colon \Delta[0]_k \to \Delta[1]_k$ sends $(0, 0, \dots, 0)$ to $(1, 1, \dots, 1)$.

Example 1.3.8 (Simplicial spheres)**.** Consider the standard n-simplex $\Delta[n]_*$. The *boundary* $\partial\Delta[n]_*$ of $\Delta[n]_*$ is the simplicial subset of $\Delta[n]_*$ of the form

$$\partial\Delta[n]_* = \bigcup_{0 \leqslant i \leqslant n} d^i(\Delta[n-1]_*) \subset \Delta[n]_*.$$

The *simplicial n-sphere* $\mathbb{S}^n_*$ is the quotient simplicial set $\mathbb{S}^n_* = \Delta[n]_*/\partial\Delta[n]_*$. In other words, $\mathbb{S}^n_*$ is defined by $\mathbb{S}^n_k = \{*, \Delta[n]_k \setminus (\partial\Delta[n]_k)\}$, there is a commutative diagram of simplical sets

$$\begin{tikzcd} \partial\Delta[n]_* \arrow[r, hook] \arrow[d] & \Delta[n]_* \arrow[d] \\ * \arrow[r, hook] & \mathbb{S}^n_* \end{tikzcd}$$

where $*$ is the (discrete) simplicial set associated to the single element set $\{*\}$.

The simplicial 1-sphere is called the *simplicial circle* $\mathbb{S}^1_*$. We've seen above that $\Delta[1]_k \simeq \{(\underbrace{0,\dots,0}_{i}, \underbrace{1,\dots,1}_{k+1-i}),\ i = 0,\dots,k+1\}$. With this identification,

$$\partial\Delta[1]_k = d^0(\Delta[0]_k) \cup d^1(\Delta[1]_k) = \{(0,\dots,0),(1,\dots,1)\},$$

and $*$ is represented by $* = *_k = (0,\dots,0) \sim (1,\dots,1) \in \mathbb{S}^n_k$. There are two nondegenerate simplices in $\mathbb{S}^1_*$, the simplex (0) of degree 0 and the simplex $(0,1)$ of degree 1.

This construction of the simplicial circle arises from the cellular decomposition of the topological circle $\mathbb{S}^1$ consisting of two cells, $*$ (a point) and e (an arc), i.e. $\mathbb{S}^1 = * \cup e$. The simplicial set generated by $\{*\}$ and e is $\mathbb{S}^1_*$.

A simple but far reaching observation (due to A.Connes) is that $\mathbb{S}^1_k$ can naturally be identified with $\mathbb{Z}/(k+1)\mathbb{Z}$ by sending $s_0^k(*)$ to $0 \in \mathbb{Z}/(k+1)\mathbb{Z}$, and $s_{k-1}\dots\widehat{s_{i-1}}\dots s_0(e)$ to $i \in \mathbb{Z}/(k+1)\mathbb{Z}$. This means, in particular, that the set $\mathbb{S}^1_k$ comes equipped with an action of the cyclic group $\mathbb{Z}/(k+1)\mathbb{Z}$. The point is that $\mathbb{S}^1$ is an (abelian) topological group, and we would like to recover the group structure from its simplicial model $\mathbb{S}^1_*$. The simplicial circle $\mathbb{S}^1_*$ is a simplest example of something called a *cyclic set.*

1.4 Remarks on basepoints and reduced simplicial sets

In this section, we clarify two important notions: *basepoints* and *reduced* simplicial sets.

Recall that Lemma 1.3.6 gives for any $X_* \in \mathbf{sSets}$ and any $n \geqslant 0$ a natural isomorphism of sets $\mathrm{Hom}_{\mathbf{sSets}}(\Delta[n]_*, X_*) \simeq X_n$. Picking an element $x_0 \in X_0$ corresponds to a map of simplicial sets $f_{x_0}\colon \Delta[0]_* \to X_*$. But $\Delta[0]_k = \{(0,\dots,0)\}$ has only one element in each degree $k \geqslant 0$. Fixing any element $x_0 \in X_0$ we get a family of simplices $\{f_{x_0}(0,\dots,0) \in X_k\}_{k\geqslant 0}$, one in each degree k. Note that $f_{x_0}(0,\dots,0)$ is just $s_0^k(x_0) \in X_k$.

Definition 1.4.1. *A* basepoint *in a simplicial set X_* is a sequence $\{s_0^k(x_0) \in X_k\}$. In other words, a basepoint is a morphism of simplicial sets $* \to X_*$, where $*$ denotes the discrete simplicial set corresponding to the one element set $\{*\}$.*

Definition 1.4.2. *A* pointed *simplicial set is a simplcial set with a choice of a basepoint. We denote the category of pointed simplicial sets by* $\mathbf{sSets}_*$.

If the degree zero part X_0 of a simplicial set X_* consists of a single point $X_0 = \{*\}$, we call X_* a *reduced* simplicial set. Note that a reduced simplicial set X_* has a canonical (in fact, unique) basepoint. We denote the category of reduced simplicial sets by $\mathbf{sSets}_0$.

1.5 The nerve of a category

Let $\mathbf{C}$ be a small category. The nerve $B_*\mathbf{C}$ of the category $\mathbf{C}$ is a simplicial set defined as follows:

$$\begin{aligned}
&B_0\mathbf{C} = \mathrm{Ob}(\mathbf{C})\\
&B_1\mathbf{C} = \mathrm{Mor}(\mathbf{C})\\
&B_2\mathbf{C} = \{\text{composable } c_0 \to c_1 \to c_2\}\\
&\quad \dots\\
&B_n\mathbf{C} = \{\text{composable } c_0 \to c_1 \to \dots \to c_n\}
\end{aligned}$$

The face maps are given by

$$d_i \colon [c_0 \to \dots \to c_n] \mapsto [c_0 \to \dots \to c_{i-1} \to \widehat{c_i} \to c_{i+1} \to \dots c_n]$$

where the notation $\widehat{c_i}$ means that c_i is omitted for $i = 0, n$ and replaced by the composition $[c_{i-1} \to \widehat{c_i} \to c_{i+1}] \mapsto [c_{i-1} \to c_{i+1}]$ for $i = 1, 2, \dots, n-1$.

The degeneracy maps s_j are defined by inserting $\mathrm{id} \colon c_j \to c_j$ in the j-th position:

$$s_j \colon [c_0 \to \dots \to c_n] \mapsto [c_0 \to \dots \to c_j \to c_j \to \dots c_n]$$

There are several other ways to look at the simplicial set $B_*\mathbf{C}$. Let $s, t \colon \mathrm{Mor}(\mathbf{C}) \to \mathbf{C}$ denote the source and target map, sending a morphism $f \colon i \to j$ to i and j, respectively. Consider the fibre product

$$\begin{tikzcd}
\mathrm{Mor}(\mathbf{C}) \times_{\mathbf{C}} \mathrm{Mor}(\mathbf{C}) \arrow[r] \arrow[d] & \mathrm{Mor}(\mathbf{C}) \arrow[d, "s"] \\
\mathrm{Mor}(\mathbf{C}) \arrow[r, "t"] & \mathbf{C}
\end{tikzcd}$$

By definition, $\mathrm{Mor}(\mathbf{C}) \times_{\mathbf{C}} \mathrm{Mor}(\mathbf{C}) \subset \mathrm{Mor}(\mathbf{C}) \times \mathrm{Mor}(\mathbf{C})$ is the set of pairs $(f, g) \in \mathrm{Mor}(\mathbf{C}) \times \mathrm{Mor}(\mathbf{C})$ such that $s(f) = t(g)$. But the last condition characterizes precisely the composable morphisms in $\mathbf{C}$. Thus, $B_2\mathbf{C} = \mathrm{Mor}(\mathbf{C}) \times_{\mathbf{C}} \mathrm{Mor}(\mathbf{C})$. By the same argument, one easily sees that

$$B_n\mathbf{C} = \mathrm{Mor}(\mathbf{C}) \times_{\mathbf{C}} \mathrm{Mor}(\mathbf{C}) \times_{\mathbf{C}} \dots \times_{\mathbf{C}} \mathrm{Mor}(\mathbf{C}).$$

The face maps and degeneracy maps are exactly the structure maps defining the corresponding fibered products.

Yet another useful way of thinking about the nerve construction which can be pretty useful is the following.

Recall that any poset J gives rise to a category which we denote $\overrightarrow{J}$. The objects of $\overrightarrow{J}$ are $\mathrm{Ob}(\overrightarrow{J}) = J$, and for $i, j \in \mathrm{Ob}(\overrightarrow{J})$, $\mathrm{Hom}(i, j) = \emptyset$ if $i \not\leq j$, and has exacly one arrow $i \to j$ otherwise.

For example, $[n] = \{0 < 1 < \cdots < n\}$ is naturally a poset (in fact, a totally ordered set). Then $\overrightarrow{[n]} = \{0 \to 1 \to 2 \to \cdots \to n\}$ where the arrows represent the non-identity morphisms. It is easy to check that $[n] \mapsto \overrightarrow{[n]}$ defines a fully faithful embedding $F\colon \mathbf{\Delta} \hookrightarrow \mathbf{Cats}$ from the simplicial category $\mathbf{\Delta}$ to the category $\mathbf{Cats}$ of all small categories. Then, any $\mathbf{C} \in \mathrm{Ob}(\mathbf{Cats})$ represents the contravariant functor $h_{\mathbf{C}}\colon \mathbf{Cats}^{op} \to \mathbf{Sets}$ whose composition $\mathbf{\Delta}^{op} \hookrightarrow \mathbf{Cats}^{op} \xrightarrow{h_{\mathbf{C}}} \mathbf{Sets}$ with F^{op} gives a simplicial set, which coincides with the nerve $B_*\mathbf{C}$ of $\mathbf{C}$, i.e. $B_*\mathbf{C} = h_{\mathbf{C}} \circ F^{op}$.

Example 1.5.1. Let's take for example $\mathbf{C}$ to be $\overrightarrow{[n]}$. Let's compute its nerve using the above observation. We have $B_k\overrightarrow{[n]} = \mathrm{Hom}_{\mathbf{Cats}}(\overrightarrow{[k]}, \overrightarrow{[n]}) \simeq \mathrm{Hom}_{\mathbf{\Delta}}([k], [n]) = \Delta[n]_k$, where the isomorphism comes from the full faithfulness of the embedding $\mathbf{\Delta} \hookrightarrow \mathbf{Cats}$. Thus, $B_*\overrightarrow{[n]} = \Delta[n]_*$.

1.6 Examples of nerves

In this section we give some examples of simplicial sets arising in geometry, topology and other parts of mathematics.

Example 1.6.1 (Nerve of a covering)**.** Let X be a topological space and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of X. Define a category $X_{\mathcal{U}}$ as follows. The set of objects of $X_{\mathcal{U}}$ is

$$\mathrm{Ob}(X_{\mathcal{U}}) = \{(x, U_\alpha) \mid x \in U_\alpha \in \mathcal{U}\} = \coprod_{\alpha \in I} U_\alpha$$

and the morphisms between (x, U_α) and (y, U_β) are defined by

$$\mathrm{Hom}_{X_{\mathcal{U}}}[(x, U_\alpha), (y, U_\beta)] = \begin{cases} \emptyset, \text{ if } x \neq y \\ x \to y, \text{ if } x = y \text{ in } U_\alpha \cap U_\beta \end{cases}$$

In other words, $\mathrm{Mor}(X_{\mathcal{U}})$ is the disjoint union $\coprod U_\alpha \cap U_\beta$ of *non-empty* pairwise intersections of elements in the covering $\mathcal{U}$. Then $B_n X_{\mathcal{U}} = \coprod U_{\alpha_0} \cap \cdots \cap U_{\alpha_n}$ is the set of non-empty $(n+1)$-wise intersections of open sets from the covering, with d_i's and s_j's given by the obvious inclusions. This simplicial set is called *the nerve of the covering* $\mathcal{U}$.

As a (trivial) example, take $\mathcal{U} = \{X\}$. Then $B_*X_{\mathcal{U}} = X$ viewed as a discrete simplicial set. We will see later that $X_{\mathcal{U}}$ is an example of a *topological* category, and for a "good" (locally contractible) covering $\mathcal{U}$, the geometric realization of $B_*X_{\mathcal{U}}$ recovers the homotopy type of X.

Example 1.6.2 (Nerve of a group)**.** Let G be a (discrete) group. It can be viewed as a category with a single object $*$ and the morphism set given by G. In particular, all morphisms in the category $\underline{G}$ are isomorphisms. We will denote this category by $\underline{G}$ to distinguish it from the group G itself.

We have $B_n\underline{G} = G\times\cdots\times G = G^n$ because all morphisms are composable. The degeneracy map s_j inserts 1 into the j-th position, i.e.

$$s_j\colon (g_1,\dots,g_n)\mapsto (g_1,\dots,1,\dots,g_n)$$

and the map d_i is defined by

$$d_i\colon (g_1,\dots,g_n)\mapsto \begin{cases}(g_2,\dots,g_n), & \text{if } i=0\\ (g_1,\dots,g_ig_{i+1},\dots,g_n), & \text{if } 1\leqslant i\leqslant n-1\\ (g_1,\dots,g_{n-1}), & \text{if } i=n.\end{cases}$$

Notice that, in general, the maps d_i and s_j are not group homomorphisms. However, if G is abelian then this is the case, and so, for abelian G, the nerve of $\underline{G}$ is actually a *simplicial abelian group* $B_*\underline{G}$ (i.e. a simplicial object in the category **Ab** of abelian groups).

Example 1.6.3 (Bar construction)**.** Let X be a set (a topological space, C^∞-manifold, etc.). Construct a simplicial set E_*X as follows. Let EX be the category with $\mathrm{Ob}(EX)=X$ and $\mathrm{Mor}(EX)=X\times X$. Set $E_*X := B_*(EX)$. Explicitly, we have $E_nX = X\times\cdots\times X = X^{n+1}$. The maps d_i and s_j send $(x_0,\dots,x_n)$ to $(x_0,\dots,\widehat{x_i},\dots,x_n)$ and $(x_0.\dots,x_j,x_j,\dots,x_n)$ respectively.

Let now $X=G$ be a (discrete) group. Note that the category EG differs from $\underline{G}$ in that $\mathrm{Ob}(EG)=G$ while $\mathrm{Ob}(\underline{G})=\{*\}$. Then E_*G is a simplicial set equipped with a right G-action, and there is a projection $E_*G\twoheadrightarrow B_*G$ with the fibre G, which is actually a fibration (we will define what that is later).

Example 1.6.4 (Action groupoid)**.** Let X be a set equipped with a *left* action of a group G. Define the category $\mathbf{C}=G\ltimes X$ with $\mathrm{Ob}(\mathbf{C})=X$ and $\mathrm{Hom}_{\mathbf{C}}(x,y)=\{g\in g\mid g.x=y\}$. This category is a groupoid (i.e. every morphism is an isomorphism) called *the action groupoid.*

Then $B_n\mathbf{C}=G^n\times X$ with d_i's and s_j's defined as in the example 1.6.2 understanding that d_n sends $(g_1,\dots,g_n,x)$ to $(g_1,\dots,g_n\cdot x)$ where $g_n\cdot x$ denotes the action of g_n on $x\in X$. Then $B_*\mathbf{C}\simeq E_*G\times_G X$ is called the simplicial Borel construction.

Remark 1.6.5. We will see later when discussing geometric realization that geometric realization functor will, in some sense, just "erase" words "simplicial" from all our constructions. I.e., "simplicial circle" will become just "circle", "simplicial Borel construction" will become the classical Borel construction from equivariant cohomology, et.c.

Example 1.6.6 (Bousfield–Kan construction)**.** Let $F\colon \mathbf{C}\to\mathbf{Sets}$ be a functor from a small category $\mathbf{C}$ to **Sets**. Define a new category $\mathbf{C}_F$ as follows. Its objects are $\mathrm{Ob}(\mathbf{C}_F)=\{(c,x)\mid c\in\mathbf{C}, x\in F(c)\}$, and morphisms are given by

$$\mathrm{Hom}_{\mathbf{C}_F}\big((c,x),(c',x')\big)=\{f\in\mathrm{Hom}_{\mathbf{C}}(c,c')\mid F(f)\colon F(c)\to F(c')\text{ sends } x\mapsto x'\}$$

Definition 1.6.7. *The* homotopy colimit *of F is defined by*

$$\mathrm{hocolim}(F) := B_*\mathbf{C}_F\in\mathbf{sSets}.$$

In particular, if F is a constant functor, i.e. sending all $c \in \mathbf{C}$ to the same set and any $f \in \mathrm{Mor}(\mathbf{C})$ to id, then $\mathrm{hocolim}(F) \simeq B_*\mathbf{C}$.

For example, let's take $\mathbf{C}$ to be simply $\mathbf{C} = \{* \to *\}$. Then defining F is equivalent to picking a morphism $X_0 \to X_1$ in **Sets**. The simplicial set $\mathrm{hocolim}(F)$ in this case is called the *simplicial mapping cylinder*.

We will discull this construction in detail later, see also [BK72].

1.7 One example: the nerve of a groupoid

One of the most important examples for us will be the nerves of groupoids. These arise very often in construction of "moduli stacks " of various objects. A particularly important example (and its variations) comes from geometry: it's the moduli space of principal bundles with connections. Since we will be using this example a lot in the future, let's recall basic definitions to be precise. For the details of the geometric part of discussion below please see the book [KN96]. For a short review of the theory of connections on principal bundles see Section 1 of Appendix A.

Given a Lie group G, define a category $\mathbf{Bun}(G)$ as follows. The objects of $\mathbf{Bun}(G)$ are the triples (P, M, θ), where $P \to M$ is a principal bundle, and θ is a connection on P. The morphisms $(P', M', \theta') \to (P, M, \theta)$ are given by connection preserving bundle maps, i.e.

$$\begin{tikzcd} P' \arrow[r, "\widetilde{\varphi}"] \arrow[d] & P \arrow[d] \\ M' \arrow[r, "\bar{\varphi}"] & M \end{tikzcd}$$

such that $\widetilde{\varphi}^*\theta = \theta'$.

Let $\mathbf{Bun}_n(G)$ be a full subcategory of $\mathbf{Bun}(G)$ consisting of triples (P, M, θ) with $\dim M \leqslant n$.

Definition 1.7.1. *An object $A \in \mathbf{Bun}(G)$ is called n-*classifying *if*

(1) $A \in \mathbf{Bun}_n(G)$;

(2) $\forall B = (P, M, \theta)$ with $\dim M \leqslant n$ there exists a morphism $\varphi\colon A \to B$;

(3) any two such morphisms φ and φ' are homotopic through bundle maps.

Theorem 1.7.2 (Narasimhan-Ramanan)**.** *For each n (including $n = \infty$) there exists an n-classifying $A \in \mathbf{Bun}(G)$.*

Fix now a manifold M. Define $\mathcal{G}(M)$ to be the subcategory of $\mathbf{Bun}(G)$ with triples of the form (P, M, θ) as objects (M is fixed), and bundle maps of the form

$$\begin{tikzcd} P' \arrow[rr, "\varphi"] \arrow[dr, "\pi'"'] & & P \arrow[dl, "\pi"] \\ & M & \end{tikzcd} \tag{1.5}$$

satisfying again $\varphi^*\theta = \theta'$ as morphisms. Notice that G-equivariance of φ in the diagram (1.5) implies that all such ϕ are actually isomorphisms, so $\mathcal{G}(M)$ is a *groupoid*.

Definition 1.7.3. *Define* $B_\nabla G\colon \mathbf{Man}^{op} \to \mathbf{sSets}$ *sending* M *to* $B_*\mathcal{G}(M)$*. This is an example of a "simplicial presheaf on manifolds".*

For more details and applications we refer to [FH13]. We will talk about results of that paper in more detail further in the course.

2 Geometric realization

Geometric realization is a natural construction of a topological space from a simplicial set. It is a far reaching generalization of constructing a geometric simplex out of combinatorial data, see Section 1.2. Namely, geometric realization defines a functor $|-|\colon \mathbf{sSets} \to \mathbf{Spaces}$. Before giving a precise definition, we should make a few remarks on the category of "spaces".

2.1 General remarks on spaces

In this section we recall a "convenient" category of spaces we will work with. We could just have taken category **Top** of all topological spaces. But working with this category has certain disadvantages. For example, **Top** is not Cartesian closed: there is no natural mapping space functor $\underline{\mathrm{Hom}}(-,-)\colon \mathbf{Top}^{op} \times \mathbf{Top} \to \mathbf{Top}$ satisfying the adjunction

$$\mathrm{Hom}_{\mathbf{Top}}(Z \times X, Y) \simeq \mathrm{Hom}_{\mathbf{Top}}(Z, \underline{\mathrm{Hom}}(X, Y))$$

In topology, the standard way to remedy this problem is to restrict the category of spaces: instead of working with all topological spaces we restrict to compactly generated weakly Hausdorff spaces (CGWH). We briefly recall below what are compactly generated weakly Hausdorff spaces. For details, see [Ste67] and [Str09].

Definition 2.1.1. *A topological space* X *is called* compactly generated *if for any subset* $Z \subseteq X$*,* Z *is closed in* X *if the set* $Z \cap K$ *is closed in* K *for any compact subset* $K \subseteq X$*.*

Definition 2.1.2. *Topological space* X *is called* weakly Hausdorff *if for any compact Hausdorff space* K *and every continuous map* $f\colon K \to X$ *its image* $f(K)$ *is closed in* X*.*

Remark 2.1.3. A space X weakly Hausdorff space if and only if the image of the diagonal embedding $\mathrm{diag}\colon X \to X \times X$ is closed in $X \times X$. This is different from the usual definition of Hausdorff because $X \times X$ denotes the product in the category of CGWH spaces, which is different from the product in **Top**. The underlying set of $X \times X$ is the usual Cartesian product, but the topology is not the product topology. For details see the references above.

Notation 2.1.4. We will denoted the full subcategory of **Top** consisting of CGWH spaces simply by **Spaces**.

It doesn't matter for us too much what exactly CG and WH mean. What matters are the following two properties which CG and WH guarantee.

First of all, the natural inclusion **Spaces** $\hookrightarrow$ **Top** induces an equivalence of homotopy categories $\mathbf{Ho}(\mathbf{Spaces}) \xrightarrow{\sim} \mathbf{Ho}(\mathbf{Top})$ (we will give a precise definition of a homotopy category later in the course). The equivalence really means that if we only care about spaces up to homotopy, we might as well just work with **Spaces** without loss of information.

Second, one of the most important classes of spaces, namely the CW complexes, form a subcategory of **Spaces**. We assume the reader is familiar with CW complexes, but we still remind basic definitions, mostly to fix notation.

We denote by $\mathbb{D}^n = \{x \in \mathbb{R}^n \mid \|x\| \leqslant 1\}$ the standard n-ball, and by $\mathbb{S}^{n-1} = \partial\mathbb{D}^n$ its boundary, the $(n-1)$-sphere. Notice that $\mathbb{D}^n \simeq \Delta^n$ and $\mathbb{S}^{n-1} \simeq \partial\Delta^n$ as topological spaces.

For a topological space X and a map $f\colon \mathbb{S}^{n-1} \to X$ we define

$$X \cup_f \mathbb{D}^n = \operatorname{colim}\{\mathbb{D}^n \hookleftarrow \mathbb{S}^{n-1} \xrightarrow{f} X\} = X \sqcup \mathbb{D}^n / y \sim f(y)\ ,\ y \in \partial\mathbb{D}^n$$

The space $X \cup_f \mathbb{D}^n$ is the result of attaching an n-cell $e = \operatorname{int}(\mathbb{D}^n)$ (interior of $\mathbb{D}^n$) to X via f. In this situation we call $f = f_e$ the attaching map corresponding to the cell e.

Definition 2.1.5. *A topological space X is a* cell complex *if*

- *X is Hausdorff;*
- *$X = \coprod_{i\in I} e_i$, where e_i are cells with attaching maps $f_i = f_{e_i}$;*
- *for all $i \in I$, $\operatorname{Im}(f_i) \subseteq \coprod_j e_j$ with $\dim e_j < \dim e_i$.*

Definition 2.1.6. *A cell complex X is called a* CW complex *is*

(C) for any cell e in X its closure $\bar{e} \subseteq X$ is in a finite cell subcomplex of X, i.e. it is a disjoint union of finitely many cells;

(W) a subset $U \subseteq X$ is open in X iff *the intersection $U \cap \bar{e}$ is open in $\bar{e}$, for all cells e.*

If X is a CW complex, define its *skeleton decomposition* (filtration)

$$\mathrm{sk}_0(X) \subseteq \mathrm{sk}_1(X) \subseteq \mathrm{sk}_2(X) \subseteq \cdots \subseteq \mathrm{sk}_n(X) \subseteq \cdots \subseteq X$$

where $\mathrm{sk}_i(X) \subseteq X$ is the union of all cells of dimension $\leqslant i$.

Remark 2.1.7. We will see later that CW complexes have an intrinsic characterization within the category **Top** (equipped with the standard Quillen model structure). Whatever these words mean for now, the main point is that CW complexes play a special role in the category **Top** (or **Spaces**), and this role is similar to the role complexes of finitely generated projective modules play in the category $\mathbf{Com}(\mathbf{R})$ of all complexes of modules over a ring R. There is a precise meaning of that "similarity of roles" in terms of cofibrations, but for now we leave this remark as it is, as a very informal intuition.

2.2 Definition of geometric realization

Definition 2.2.1. *For a simplicial set $X_* \in$ **sSets** we define its* geometric realization *to be the space*

$$|X_*| = \left(\bigsqcup_{n \geqslant 0} X_n \times \Delta^n \right) \Big/ \sim$$

where the equivalence relation is given by $(f^*(x), t) \sim (x, f_*(t))$, $x \in X_n$, $s \in \Delta^m$ for every morphism $f\colon [m] \to [n]$ in $\mathbf{\Delta}$.

More formally, $|X_*|$ can be defined as the coequalizer

$$|X_*| := \operatorname{colim} \left[\bigsqcup_{f\colon [n] \to [m]} X_m \times \Delta^n \underset{f_*}{\overset{f^*}{\rightrightarrows}} \bigsqcup_n X_n \times \Delta^n \right]$$

Remark 2.2.2. Since $\mathbf{\Delta}$ is generated by morphisms d^i, s^j, it is enough only to require the identifications $(d_i(x), t) \sim (x, d^i(t))$ and $(s_j(x), t) \sim (x, s^j(t))$ for all i, j.

Remark 2.2.3. Note also that the set $s_j(x) \times \Delta^{n+1}$ is identified with $\{x\} \times \Delta^n \simeq \Delta^n$, so the degenerate simplices $s_j(x) \in X_{n+1}$ make no contribution to $|X_*|$. This explains why the simplices of the form $s_j(x)$ are called *degenerate*.

It is obvious that the geometric realization defines a functor $|-|\colon$ **sSets** $\to$ **Top**. As we will see later, the image of $|-|$ lies in the subcategory **CW** $\subset$ **Spaces** of CW-complexes.

Theorem 2.2.4. *The functor $|-|$ is left adjoint to the singular set functor, i.e.*

$$|-| : \mathbf{sSets} \rightleftarrows \mathbf{Spaces} : S_*(-) \tag{1.6}$$

where $S_*(-)$ is defined by $S_n(M) = \operatorname{Hom}_{\mathbf{Top}}(\Delta^n, M)$ (see example 1.3.4).

Proof. This will follow from a general adjunction theorem, to be discussed later. □

Lemma 2.2.5. *Geometric realization of the standard simplicial simplex is the usual standard simplex: $|\Delta[n]_*| = \Delta^n$.*

Proof. This follows from a more general result of proposition 2.6.1. For the proof of the particular isomorphism $|\Delta[n]_*| = \Delta^n$ see example 2.6.5. □

Corollary 2.2.6. *If $\mathbb{S}^n_*$ is the simplicial sphere, then its geometric realization $|\mathbb{S}^n_*|$ is just the usual (topological) n-sphere $\mathbb{S}^n$.*

Sketch of proof. By definition of the simplicial sphere it was defined by the push-out diagram

$$\begin{tikzcd} \partial\Delta[n]_* \arrow[r, hook] \arrow[d] & \Delta[n]_* \arrow[d] \\ * \arrow[r, hook] & \mathbb{S}^n_* \end{tikzcd}$$

Push-outs are by definition colimits. Since $|-|$ is left adjoint, it commutes with colimits. Therefore, applying functor $|-|$ to the above push-out diagram, we get another push-out diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, hook] \arrow[d] & \Delta^n \arrow[d] \\
* \arrow[r, hook] & |\mathbb{S}^n_*|
\end{tikzcd}$$

But it is obvious that the push-out of the resulting diagram is just the n-sphere $\mathbb{S}^n$. Therefore, $|\mathbb{S}^n_*|$ is isomorphic to $\mathbb{S}^n$. ☐

Theorem 2.2.7. *For any simplicial set X_* its geometric realization $|X_*|$ is a CW complex.*

Proof. If $X_* \in \mathbf{sSets}$, define its *n-th skeleton* by $\mathrm{sk}_n(X_*) = \langle X_k \mid k \leqslant n\rangle$, the simplicial subset of X_* generated by X_k with $k \leqslant n$. Then we have skeleton filtration

$$\mathrm{sk}_0(X_*) \subseteq \mathrm{sk}_1(X_*) \subseteq \cdots \subseteq \mathrm{sk}_n(X_*) \subseteq \cdots \subseteq X_*$$

and

$$X_* = \bigcup_{n\geqslant 0} \mathrm{sk}_n(X_*)$$

Recall that we defined the boundary of $\Delta[n]_*$ by

$$\partial\Delta[n]_* = \langle \Delta[n]_k \mid k < n\rangle.$$

We have coCartesian (push-out) squares

$$\begin{tikzcd}
\coprod \partial\Delta[n]_* \arrow[r, "\sqcup f_x|_{\partial\Delta[n]}"] \arrow[d, hook] & \mathrm{sk}_{n-1}(X)_* \arrow[d, hook] \\
\coprod \Delta[n]_* \arrow[r, "\sqcup f_x"] & \mathrm{sk}_n(X)_*
\end{tikzcd}$$

where the disjoint unions are taken over all non-degenerate simplices $x \in X_n$, and f_x are the representing maps for such $x \in X_n$, i.e. $f_x \colon \Delta[n]_* \to X_*$ is the map corresponding to $x \in X_n$ under the isomorphism (see example 1.3.5)

$$\mathrm{Hom}_{\mathbf{sSets}}(\Delta[n]_*, X) \simeq X_n$$

By Theorem 2.2.4, the functor $|-|$ commutes with colimits, and hence we get a push-out diagram

$$\begin{tikzcd}
\coprod |\partial\Delta[n]_*| \arrow[r, "\sqcup|f_x||_{\partial\Delta[n]}"] \arrow[d, hook] & |\,\mathrm{sk}_{n-1}(X)_*| \arrow[d, hook] \\
\coprod |\Delta[n]_*| \arrow[r, "\sqcup|f_x|"] & |\,\mathrm{sk}_n(X)_*|
\end{tikzcd}$$

This last diagram is a push-out diagram, so we construct $|X_*|$ inductively, by attaching cells one by one. Therefore $|X_*|$ is a cell complex. **Check it is actually a CW complex.** ☐

Corollary 2.2.8. *The n-cells of $|X_*|$ are in one-to-one correspondence with* non-degenerate *simplices in X_n.*

2.3 Two generalizations of geometric realization

There are two important generalizations of the geometric realization construction. Let us briefly discuss them here.

Fat realization

Consider the subcategory $\mathbf{\Delta}_{\text{inj}}$ of $\mathbf{\Delta}$ with $\text{Ob}(\mathbf{\Delta}_{\text{inj}}) = \text{Ob}(\mathbf{\Delta})$, and the morphisms of $\mathbf{\Delta}_{\text{inj}}$ being the *injective* morphisms in $\mathbf{\Delta}$. It is easy to see that $\mathbf{\Delta}_{\text{inj}}$ is generated by the maps d^i, so we just "forget" the degeneracy maps s^j.

Definition 2.3.1. *A* pre-simplicial *(or* semi-simplicial*) object in a category* $\mathbf{C}$ *is a functor* $\mathbf{\Delta}_{inj}^{op} \to \mathbf{C}$.

Notation 2.3.2. We denote the category of pre-simplicial objects in $\mathbf{C}$ by $\mathbf{presC}$.

There is an obvious forgetful functor $\mathbf{sSets} \to \mathbf{presSets}$ restricting simplicial sets from $\mathbf{\Delta}^{op}$ to $\mathbf{\Delta}_{\text{inj}}^{op}$.

Definition 2.3.3. *Let X_* be a pre-simplicial set. Its* fat realization $\|X_*\|$ *is defined by*

$$\|X_*\| = \left(\coprod_{n \geqslant 0} X_n \times \Delta^n \right) \Big/ \sim$$

where the equivalence relation is given by $(d_i(x), t) \sim (x, d^i(t))$ for $x \in X_n$, $t \in \Delta^{n-1}$.

Since the equivalence relation defining the fat realization is weaker than the equivalence relation for the geometric realization, the space $\|X_*\|$ is "bigger" than $|X_*|$ (that's why $\|X_*\|$ is called "fat"). In fact, $\|X_*\|$ is always infinite-dimensional, even if X_* is the simplicial set consisting of one point in each degree (i.e. when X_* is just the point viewed as a discrete simplicial set). Nevertheless, there is a canonical projection $\|X_*\| \to |X_*|$, which under a suitable condition is a homotopy equivalence.

One advantage of $\|X_*\|$ over $|X_*|$ is that the fat realization of simplicial spaces preserves weak equivalences under much weaker hypotheses. If $f \colon X_* \to Y_*$ is a level-wise weak equivalence of simplicial spaces which are cofibrant in each dimension, then the induced map $\|X_*\| \to \|Y_*\|$ is a weak equivalence.

For more details and applications to simplicial manifolds, see [Dup01].

Example 2.3.4. Let $X_* = X$ be a discrete simplicial set. Then $\|X\| = X \times \|*\| = X \times \bigcup_{n \geqslant 0} \Delta^n$, while $|X| = X$.

Geometric realization of simplicial spaces

Any set can be viewed as a discrete topological space: this gives a natural inclusion **Sets** $\hookrightarrow$ **Spaces** which in turn extends to **sSets** $\hookrightarrow$ **sSpaces**. We can further extend the functor $|-|$ to all simplicial spaces **sSpaces** by the same formula as in the Definition 2.2.1.

$$\begin{tikzcd} \mathbf{sSets} \arrow[r, "|-|"] \arrow[d, hook] & \mathsf{Spaces} \\ \mathbf{sSpaces} \arrow[ur, dashed, "|-|"'] & \end{tikzcd}$$

The difference is that for a simplicial *space* X_*, the topology on each $X_n \times \Delta^n$ is the product topology (in the category **Spaces**) coming from both X_n and Δ^n, while for a simplicial *set* X_*, the topology on $X_n \times \Delta^n = \bigsqcup_{x \in X_n} \Delta^n$ has the union topology coming only from Δ^n.

Working with simplicial spaces, we have to impose additional conditions to avoid certain pathologies. For example, it may happen that for $f\colon X_* \to Y_*$ the map $f\colon X_n \to Y_n$ is a weak homotopy equivalence on each level, but the induced map $|f|\colon |X_*| \to |Y_*|$ is no longer a homotopy equivalence. There are other pathologies as well. There is a class of "good" spaces (called *Reedy cofibrant*) for which geometric realization works well. This class was first introduced in [Seg74].

For us it would be enough to know the following two examples of "good" simplicial spaces. First of all, suppose X_{**} is a bisimplicial set (i.e. a simplicial object in the category **sSets**). Then, for all p, q, the simplicial spaces $|X_{*q}|$ and $|X_{p*}|$ are good.

The second example of "good" simplicial spaces is given by the so-called *ANR spaces*, or absolute neighborhood retracts. By definition, a topological space X is ANR space if for any continuous map $f\colon A \to X$, where $A \subseteq Y$ is a closed subset of a normal space Y, there is Y_0 an open neighborhood $A \subseteq Y_0 \subseteq Y$ of A such f extends to a map $\tilde{f}\colon Y_0 \to X$.

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "f"'] & Y_0 \arrow[r, hook] \arrow[dl, dashed, "\tilde{f}"] & Y \\ X & & \end{tikzcd}$$

(We remind that a normal space is a space with the property that any two disjoint closed subsets have disjoint open neighborhoods.) Then if each component X_n os a simplicial space X_* is an ANR space, then X_* is good.

Example 2.3.5. Any C^∞-manifold is an ANR space. This follows from Titze's Extension theorem. Hence, simplicial C^∞-manifolds are good simplicial spaces.

2.4 Kan extensions

Consider two functors $F\colon \mathbf{C} \to \mathbf{E}$, $G\colon \mathbf{C} \to \mathbf{D}$. Suppose we would like to find a functor $S\colon \mathbf{E} \to \mathbf{D}$ such that $F \simeq S \circ G$. In other words, we would like to complete F and G to a commutative triangle:

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "F"] \arrow[d, "G"'] & \mathbf{D} \\
\mathbf{E} \arrow[ur, dashed, "S"'] &
\end{tikzcd} \tag{1.7}$$

In general, this is not possible. For example, if for some morphisms $f_1, f_2 \in \mathrm{Mor}(\mathbf{C})$ we have $F(f_1) \neq F(f_2)$ but $G(f_1) = G(f_2)$, then, of course, the required S doesn't exist. So it natural to relax the condition $F \simeq S \circ G$, and only ask for "best possible approximation" for the commutativity of (1.7). To be precise, we look for universal natural transformations $F \Rightarrow S \circ G$ or $F \Leftarrow S \circ G$ (whichever one exists, if any).

$$\begin{tikzcd}
\mathbf{C} \arrow[rr, "F"] \arrow[dr, "G"'] & \arrow[d, phantom, "\Downarrow"] & \mathbf{D} \\
& \mathbf{E} \arrow[ur, "S"'] &
\end{tikzcd}$$

We will make things precise in a moment, but first let us fix some notation. If $\varphi\colon S \Rightarrow S'$ is a natural transformation between two functors $S, S'\colon \mathbf{E} \to \mathbf{D}$, then by $\varphi \circ G\colon S \circ G \Rightarrow S' \circ G$ we denote the natural transformation given by

$$(\varphi \circ G)_X = \varphi_{G(X)}\colon S(G(X)) \to S'(G(X))$$

for all $X \in \mathrm{Ob}(\mathbf{C})$.

If $\eta\colon F \Rightarrow S \circ G$ and $\varphi\colon S \Rightarrow S'$ are natural transformations, we get another natural transformation $\eta'\colon F \overset{\eta}{\Rightarrow} S \circ G \overset{\varphi \circ G}{\Rightarrow} S' \circ G$. We say η' is *induced* from η by φ.

Definition 2.4.1. *A* left Kan extension *of F along G is a functor $\mathrm{Lan}_G(F)\colon \mathbf{E} \to \mathbf{D}$ given together with a universal natural transformation $\eta_{un}\colon F \Rightarrow \mathrm{Lan}_G(F) \circ G$ from which any other $\eta\colon F \Rightarrow S \circ G$ is induced by a unique $\varphi\colon \mathrm{Lan}_G(F) \to S$:*

$$\begin{tikzcd}
F \arrow[rr, "\eta"] \arrow[dr, "\eta_{un}"'] & & S \circ G \\
& \mathrm{Lan}_G(F) \circ G \arrow[ur] &
\end{tikzcd}$$

If $\mathrm{Lan}_G(F)$ exists for all possible $F\colon \mathbf{C} \to \mathbf{D}$, then we actually get a pair of adjoint functors

$$\mathrm{Lan}_G(-) :\ \mathbf{Fun}(\mathbf{C}, \mathbf{D}) \rightleftarrows \mathbf{Fun}(\mathbf{E}, \mathbf{D}) : G_*$$

where $G_* = (-) \circ G$. So there is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Fun}(\mathbf{E},\mathbf{D})}(\mathrm{Lan}_G(F), S) \simeq \mathrm{Hom}_{\mathbf{Fun}(\mathbf{C},\mathbf{D})}(F, S \circ G)$$

Dually, one can define a *right Kan extension* as a functor $\mathrm{Ran}_G(F)\colon \mathbf{E} \to \mathbf{D}$ together with a universal natural transformation $\varepsilon_{un}\colon \mathrm{Ran}_G(F) \circ G \Rightarrow F$. Again, if it exists for all F, then one has an adjoint pair

$$G_* : \mathbf{Fun}(\mathbf{E}, \mathbf{D}) \leftrightarrows \mathbf{Fun}(\mathbf{C}, \mathbf{D}) : \mathrm{Ran}_G(-)$$

A good picture to keep in mind when thinking about Kan extensions is the following

$$\begin{tikzcd}
\mathbf{Fun}(\mathbf{E}, \mathbf{D}) \arrow[d, "G_*"] \\
\mathbf{Fun}(\mathbf{C}, \mathbf{D}) \arrow[u, bend left=60, "\mathrm{Lan}_G(-)"] \arrow[u, bend right=60, "\mathrm{Ran}_G(-)"']
\end{tikzcd}$$

Example 2.4.2 (Representations of groups)**.** Let G be a discrete group, and $\underline{G}$ is G viewed as a groupoid with one object $*$. Then functors $\underline{G} \to \mathbf{Vect}_k$ from $\underline{G}$ to vector spaces over a field k are exactly representations of the group G, i.e. $\mathbf{Fun}(\underline{G}, \mathbf{Vect}_k) = \mathbf{Rep}_k(G)$.

If $H \subseteq G$ is a subgroup of G, we have the inclusion $\iota\colon \underline{H} \hookrightarrow \underline{G}$, which induces the restriction functor $\mathrm{Res}^G_H\colon \mathbf{Rep}_k(G) \to \mathbf{Rep}_k(H)$, $\mathrm{Res}^G_H = \iota_*$. Functor Res^G_H sends a G-module V to itself but viewed as an H-module.

In this case, both left and right Kan extensions $\mathrm{Lan}_\iota(-), \mathrm{Ran}_\iota(-)\colon \mathbf{Rep}_k(H) \to \mathbf{Rep}_k(G)$ exist. Namely, $\mathrm{Lan}_\iota = \mathrm{Ind}^G_H$ is the induction functor, sending an H-module V to G-module $\mathrm{Ind}^G_H(V) := k[G] \otimes_{k[H]} V$. The right Kan extension $\mathrm{Ran}_\iota = \mathrm{Coind}^G_H$ is the coinduction functor, sending an H-module V to $\mathrm{Coind}^G_H(V) := \mathrm{Hom}_H(k[G], V)$.

$$\begin{tikzcd}
\mathbf{Rep}_k(G) \arrow[d, "\mathrm{Res}^G_H"] \\
\mathbf{Rep}_k(H) \arrow[u, bend left=60, "\mathrm{Ind}^G_H"] \arrow[u, bend right=60, "\mathrm{Coind}^G_H"']
\end{tikzcd}$$

Example 2.4.3 (Geometric realization)**.** Recall from Example 1.3.2 that assigning to $[n] \in \mathbf{\Delta}$ the geometric n-simplex $\Delta^n \in \mathbf{Top}$ defines a cosimplicial space $\Delta^* \in \mathbf{csTop}$. Let $Y : \mathbf{\Delta} \hookrightarrow \mathbf{sSets}$ denote the Yoneda embedding. Then the left Kan extension $\mathrm{Lan}_Y(\Delta^*)\colon \mathbf{sSets} \to \mathbf{Top}$ exists, and is given by the geometric realization functor $|-|$:

$$\begin{tikzcd}
\mathbf{\Delta} \arrow[r, "\Delta^*"] \arrow[d, hook, "Y"'] & \mathbf{Top} \\
\mathbf{sSets} \arrow[ur, dashed, "\mathrm{Lan}_Y(\Delta^*)=|-|"']
\end{tikzcd}$$

Example 2.4.4 (Derived functors)**.** Let $\mathcal{A}, \mathcal{B}$ be two abelian categories, $F\colon \mathcal{A} \to \mathcal{B}$ an additive functor. F can be extended term-wise to a functor on the category of complexes $F\colon \mathbf{Com}(\mathcal{A}) \to \mathbf{Com}(\mathcal{B})$. Denote by $\mathbf{D}(\mathcal{A})$ and $\mathbf{D}(\mathcal{B})$ the derived categories of $\mathcal{A}$ and $\mathcal{B}$, obtained by localizing $\mathbf{Com}(\mathcal{A})$ and $\mathbf{Com}(\mathcal{B})$ on the class of quasi-isomorphisms. Denote by $\gamma_\mathcal{A}$ and $\gamma_\mathcal{B}$ the corresponding localization functors. Then we have the following diagram

$$\begin{tikzcd}
\mathbf{Com}(\mathcal{A}) \arrow[r, "F"] \arrow[d, "\gamma_\mathcal{A}"'] & \mathbf{Com}(\mathcal{B}) \arrow[r, "\gamma_\mathcal{B}"] & \mathbf{D}(\mathcal{B}) \\
\mathbf{D}(\mathcal{A}) \arrow[urr, dashed] & &
\end{tikzcd}$$

The *total left defived functor* $\mathbb{L}F$ of the functor F is by definition the *right* Kan extension $\mathrm{Ran}_{\gamma_\mathcal{A}}(\gamma_\mathcal{B} \circ F)$, and the *total right defived functor* $\mathbb{R}F$ of F is the *left* Kan extension $\mathrm{Lan}_{\gamma_\mathcal{A}}(\gamma_\mathcal{B} \circ F)$.

2.5 Comma categories

Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor and $d \in \mathrm{Ob}(\mathbf{D})$ be a fixed object of $\mathbf{D}$.

Definition 2.5.1. *The* left comma category F/d *is a category having the objects*

$$\mathrm{Ob}(F/d) = \{(c, f) \mid c \in \mathrm{Ob}(\mathbf{C}), f \in \mathrm{Hom}_\mathbf{D}(F(c), d)\}$$

and morphisms

$$\mathrm{Hom}_{F/d}((c, f), (c', f')) = \{h \in \mathrm{Hom}_\mathbf{C}(c, c') \mid f = f' \circ F(h)\}$$

In other words, the set of morphisms in F/d from (c, f) to (c', f') consists of all morphisms $h\colon c \to c'$ in $\mathbf{C}$ making the following diagram commute:

$$\begin{tikzcd}
F(c) \arrow[rr, "F(h)"] \arrow[dr, "f"'] & & F(c') \arrow[dl, "f'"] \\
& d &
\end{tikzcd}$$

We will sometimes denote objects of F/d by arrows $F(c) \xrightarrow{f} d$. There exists a natural forgetful functor $j\colon F/d \to \mathbf{C}$ sending $j\colon (c, f) \mapsto c$. Let $\mathrm{const}_d\colon F/d \to \mathbf{D}$ be the constant functor sending every $(c, f) \in \mathrm{Ob}(F/d)$ to $d \in \mathbf{D}$ and any morphism $h \in \mathrm{Mor}(F/d)$ to id_d. Then there is a natural transformation $\eta\colon F \circ j \Rightarrow \mathrm{const}_d$ given by $\eta = \{\eta_{(c,f)}\colon F(c) \xrightarrow{f} d\}$. A convenient way to organize this data is the following diagram:

$$\begin{tikzcd}
F/d \arrow[r, "j"] \arrow[d] \arrow[dr, "\mathrm{const}_d"' sloped] & \mathbf{C} \arrow[d, "F"] \arrow[dl, phantom, "\Swarrow\eta"] \\
* \arrow[r] & \mathbf{D}
\end{tikzcd}$$

Here $*$ denotes the category with one object $*$, and only one morphism id_*. We think of the diagram above as a "fibration of categories". We will make it precise later in the course.

Dually, one defines the *right comma category* $d\backslash F$. Its objects are $\mathrm{Ob}(d\backslash F) = \{(c, f) \mid f\colon d \to F(c)\}$ and morphisms $\mathrm{Hom}_{d\backslash F}((c, f), (c', f'))$ are maps $h\colon c \to c'$ in $\mathbf{C}$ making the following diagram commute

$$\begin{tikzcd} F(c) \arrow[rr, "F(h)"] & & F(c') \\ & d \arrow[ul, "f"] \arrow[ur, "f'"'] & \end{tikzcd}$$

A right comma category $d\backslash F$ comes together with a forgetful functor $j\colon d\backslash F \to \mathbf{C}$ and a natural transformation $\eta\colon \mathrm{const}_d \Rightarrow F \circ j$ which can be organized into the following diagram:

$$\begin{tikzcd} d\backslash F \arrow[r, "j"] \arrow[d] \arrow[dr, "\mathrm{const}_d"' sloped] & \mathbf{C} \arrow[d, "F"] \\ * \arrow[r] & \mathbf{D} \arrow[from=1-1, to=1-2, phantom, shift right=4, "\Uparrow \eta"] \end{tikzcd}$$

The (left) comma categories F/d can be combined together for different $d \in \mathrm{Ob}(\mathbf{D})$ to define the *global comma category* $F/\mathbf{D}$. It's the category with objects

$$\mathrm{Ob}(F/\mathbf{D}) = \{(c, d, f) \mid c \in \mathrm{Ob}(\mathbf{C}),\ d \in \mathrm{Ob}(\mathbf{D}),\ f\colon F(c) \to d\}$$

and morphisms $\mathrm{Hom}_{F/\mathbf{D}}((c, d, f), (c', d', f'))$ being the pairs (h, g), $h \in \mathrm{Hom}_{\mathbf{C}}(c, c')$, $g \in \mathrm{Hom}_{\mathbf{D}}(d, d')$ making the following diagram commute

$$\begin{tikzcd} F(c) \arrow[r, "F(h)"] \arrow[d, "f"'] & F(c') \arrow[d, "f'"] \\ d \arrow[r, "g"] & d' \end{tikzcd}$$

There is a forgetful functor $j\colon F/\mathbf{D} \to \mathbf{C} \times \mathbf{D}$ defined by $(c, d, f) \mapsto (c, d)$.

Example 2.5.2. Let $\mathbf{C} = \mathbf{D}$ and $F = \mathrm{id}_{\mathbf{C}}$. Then the category F/d is usually denoted by $\mathbf{C}/d$ and is called the category *over* d. Similarly, $d\backslash F$ is denoted $d\backslash\mathbf{C}$ and is called the category *under* d. In this case, the global comma category $F/\mathbf{C} = \mathbf{Mor}(\mathbf{C})$ is just the *category* of morphisms in $\mathbf{C}$.

Example 2.5.3 (The simplex category of a small category)**.** Recall (see end of Section 1.5) that to every $[n] \in \mathrm{Ob}(\mathbf{\Delta})$ we can associate a small category $\overrightarrow{[n]}$. This gives a fully faithful functor from $\mathbf{\Delta}$ to the category of all small categories $\mathbf{Cats}$:

$$F\colon \mathbf{\Delta} \hookrightarrow \mathbf{Cats},\ F([n]) = \overrightarrow{[n]}$$

Definition 2.5.4. *For* $\mathbf{C} \in \mathbf{Cats}$*, we define the* simplex category $\mathbf{\Delta}/\mathbf{C}$ *to be the left comma category* $F/\mathbf{C}$*.*

By definition, the objects of $\mathbf{\Delta}/\mathcal{C}$ can be interpreted as elements of the simplicial nerve of $\mathbf{C}$:

$$\mathrm{Ob}(\mathbf{\Delta}/\mathbf{C}) = \{([n], f) \mid [n] \in \mathrm{Ob}(\mathbf{\Delta}), f\colon \overrightarrow{[n]} \to \mathbf{C}\} = \bigcup_{n \geqslant 0} B_n\mathbf{C}$$

Morphisms $\mathrm{Mor}(\mathbf{\Delta}/\mathbf{C})$ are the morphisms in $\mathbf{\Delta}$ inducing morphisms of diagrams $\overrightarrow{[n]} \to \mathbf{C}$ in $\mathbf{C}$.

2.6 Kan extensions using comma categories

Let $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{C} \to \mathbf{E}$ be two functors. We will give an "explicit" formula for computing the left Kan extension $\mathrm{Lan}_G(F)$. We will use notation from the sections 2.4 and 2.5.

Fix $e \in \mathrm{Ob}(\mathbf{E})$, and consider the composition $G/e \xrightarrow{j} \mathbf{C} \xrightarrow{F} \mathbf{D}$ sending $(c, f) \mapsto F(c)$. Assuming that the colimit exists $\mathrm{colim}(F \circ j)$ exists, define

$$L_G(F)(e) := \mathrm{colim}\{G/e \xrightarrow{j} \mathbf{C} \xrightarrow{F} \mathbf{D}\} = \mathrm{colim}(F \circ j)$$

Note that any morphism $\varphi\colon e \to e'$ in $\mathbf{E}$ induces a morphism $\varphi_*\colon G/e \to G/e'$ mapping $[G(c) \to e] \mapsto [G(c) \to e \xrightarrow{\varphi} e']$ for each object $[G(c) \to e] \in \mathrm{Ob}(G/e)$. Hence, we get a morphism of diagrams

$$\begin{tikzcd}
G/e \arrow[r, "j"] \arrow[d, "\varphi_*"'] & \mathbf{C} \arrow[r, "F"] \arrow[d, equal] & \mathbf{D} \arrow[d, equal] \\
G/e' \arrow[r, "j'"] & \mathbf{C} \arrow[r, "F"] & \mathbf{D}
\end{tikzcd}$$

By the universal property of colimits, this induces a unique morphism of colimits $L_G(F)(e) \to L_G(F)(e')$ (again assuming that $L_G(F)(e')$ exists) which we call $L_G(F)(\varphi)$:

$$\begin{tikzcd}
F \circ j \arrow[r, "\varphi_*"] \arrow[dr] & F \circ j' \arrow[r] & \mathrm{colim}(F \circ j') \\
& \mathrm{colim}(F \circ j) \arrow[ur, dashed, "\exists!\, L_G(F)(\varphi)"'] &
\end{tikzcd}$$

Proposition 2.6.1. *If* $\mathrm{colim}\{G/e \to \mathbf{C} \to \mathbf{D}\}$ *exists for all* $e \in \mathrm{Ob}(\mathbf{E})$*, then there is an ismorphism of functors*

$$L_G(F) \simeq \mathrm{Lan}_G(F) \tag{1.8}$$

Proof. It is easy to check that $L_G(F)$ satisfies the universal property of a left Kan extension. $\square$

Remark 2.6.2. The isomorphism (1.8) characterizes the so-called *pointwise* (left) Kan extensions, i.e. left Kan extensions commuting with corepresentable functors $h^X = \mathrm{Hom}_{\mathbf{D}}(X, -)$ for all $X \in \mathrm{Ob}(\mathbf{D})$. If $\mathbf{D}$ has all small limits, then every left Kan extension with codomain $\mathbf{D}$ is automatically pointwise.

Proposition 2.6.1 has two useful consequences.

Corollary 2.6.3. *If* $\mathbf{C}$ *is small and* $\mathbf{D}$ *is cocomplete (i.e. it has all small colimits) then* any *functor* $F\colon \mathbf{C} \to \mathbf{D}$ *has a left Kan extension along* any *functor* G.

Corollary 2.6.4. *Assume that the functor* G *in Proposition 2.6.1 is* fully faithful. *Then the universal natural transformation* $\eta_{un}\colon F \Rightarrow \mathrm{Lan}_G(F) \circ G$ *is an isomorphism of functors.*

Proof. For every $c \in \mathrm{Ob}(\mathbf{C})$ consider G/Gc. The fact that G is fully faithful implies that the category G/Gc *has a terminal object.* By definition, the category G/Gc has objects

$$\mathrm{Ob}(G/Gc) = \{(c', f') \mid c' \in \mathrm{Ob}(\mathbf{C}),\ f'\colon G(c') \to G(c)\}$$

The claim is that the object (c, id_c) is the terminal object in G/Gc. Indeed, for any $(c', f') \in \mathrm{Ob}(G/Gc)$, we have

$$\mathrm{Hom}_{G/Gc}((c', f'), (c, \mathrm{id}_c)) = \{h \in \mathrm{Hom}_{\mathbf{C}}(c', c) \mid f' = \mathrm{id}_c \circ G(h) \equiv G(h)\} = \{G^{-1}(f)\}$$

Being fully faithful, functor G induces isomorphisms on all Hom-sets, so the set $\{G^{-1}(f)\}$ consist of exactly one element. Therefore, (c, id_c) is indeed the terminal object in G/Gc.

By the universal property of colimits, if a category $\mathbf{I}$ has a terminal object $*$, then for any functor $F\colon \mathbf{I} \to \mathbf{D}$ the colimit $\operatorname{colim} F$ exists and is naturally isomorphic to $F(*)$.

Thus, in our case, for every $c \in \mathrm{Ob}(\mathbf{C})$

$$\mathrm{Lan}_G(F)e = \operatorname{colim}\{G/G(c) \xrightarrow{j} \mathbf{C} \xrightarrow{F} \mathbf{D}\} \simeq (F \circ j)(c, \mathrm{id}_c) = F(c)\,.$$

So the natural transformation $\eta_{un}\colon F \Rightarrow \mathrm{Lan}_G(F) \circ G$ is naturally isomorphic to id_F. $\square$

Example 2.6.5. Let $|-|\colon \mathbf{sSets} \to \mathbf{Spaces}$ be the geometric realization functor, $Y\colon \mathbf{\Delta} \hookrightarrow \mathbf{sSets}$ the Yoneda embedding, and $\Delta^*\colon \mathbf{\Delta} \to \mathbf{Spaces}$ the geometric simplex functor, sending $[n]$ to the geometric n-simplex simplex Δ^n. By Example 2.4.3, the functor $|-|$ is isomorphic to $\mathrm{Lan}_Y(\Delta^*)$.

Since the Yoneda embedding Y is a fully faithful functor, we can apply corollary 2.6.4 to it. Therefore, the natural transformation $\eta_{un}\colon \Delta^* \Rightarrow |-| \circ Y$ is an isomorphism, and so in particular $|\Delta[n]_*| \simeq \Delta^n$. In other words, geometric realization of the standard simplex $\Delta[n]_*$ is isomorphic to the geometric simplex Δ^n.

3 Homotopy theory of categories

3.1 The classifying space of a small category

Definition 3.1.1. *The* classifying space *of a small category* $\mathbf{C}$ *is* $B\mathbf{C} := |B_*\mathbf{C}|$*, the geometric realization of the nerve* $B_*\mathbf{C}$ *(see Section 1.5).*

Terminology 3.1.2. We say that a category $\mathbf{C}$ is *connected*, *contractible*, etc. if so is the space $B\mathbf{C}$. We say that a functor $F\colon \mathbf{C} \to \mathbf{D}$ is a *covering*, *homotopy equivalence*, *fibration*, etc. if the corresponding map of topological spaces $BF\colon B\mathbf{C} \to B\mathbf{D}$ is a covering, homotopy equivalence, etc.

The space $B\mathbf{C}$ can be characterized uniquely up to homeomorphism by the following axioms.

(BC1) "Naturality:" assignment $\mathbf{C} \mapsto B\mathbf{C}$ extends to a functor $B\colon \mathbf{Cats} \to \mathbf{Spaces}$;

(BC2) "Normalization:" The composition of functors

$$\begin{array}{rcccl} B|_F\colon \boldsymbol{\Delta} & \xhookrightarrow{F} & \mathbf{Cats} & \xrightarrow{B} & \mathbf{Spaces} \\ [n] & \longmapsto & \overrightarrow{[n]} & \longmapsto & B\overrightarrow{[n]} \end{array}$$

coincides with the geometric simplex functor Δ^*. In other words, $B\overrightarrow{[n]} = \Delta^n$ for each $n \geqslant 0$ and for each $f \in \mathrm{Hom}_{\boldsymbol{\Delta}}([n],[m])$ the corresponding map $Bf\colon \Delta^n \to \Delta^m$ sends the j-th vertex of Δ^n to the $f(j)$-th vertex of Δ^m.

(BC3) "Gluing:" For every (small) category $\mathbf{C} \in \mathbf{Cats}$, there is a natural isomorphism

$$B\mathbf{C} \simeq \mathrm{colim}_{([n],f)\in \boldsymbol{\Delta}/\mathbf{C}}\, B\overrightarrow{[n]}$$

In other words, $B\mathbf{C}$ is obtained as the left Kan extension

$$\begin{tikzcd} \boldsymbol{\Delta} \arrow[r, "B|_F"] \arrow[d, hook, "F"'] & \mathbf{Spaces} \\ \mathbf{Cats} \arrow[ur, dashed, "\mathrm{Lan}_F(B|_F)"'] & \end{tikzcd}$$

Corollary 3.1.3. *The classifying space functor* B *satisfies the following.*

(BC4) *If* $\mathbf{C} \subseteq \mathbf{D}$ *is a subcategory, then* $B\mathbf{C} \subseteq B\mathbf{D}$ *is a subcomplex.*

(BC5) *If* $\mathbf{C} = \coprod_\alpha \mathbf{C}_\alpha$ *is the disjoint union of categories (coproduct in the category* $\mathbf{Cats}$*), then* $B\mathbf{C} = \coprod_\alpha B\mathbf{C}_\alpha$ *of* $\mathbf{C}$ *is the disjoint union of classifying spaces of* $\mathbf{C}_\alpha$*.*

(BC6) *For any two categories* $\mathbf{C}, \mathbf{D}$ *there is an isomorphism* $B(\mathbf{C} \times \mathbf{D}) \simeq B\mathbf{C} \times B\mathbf{D}$*.*

Remark 3.1.4. Note that $B\mathbf{C} \times B\mathbf{D}$ is the product in the category **Spaces**, which is different from the product in the category **Top** of all topological spaces (see Remark 2.1.3). However, if either $B\mathbf{C}$ or $B\mathbf{D}$ is a *finite* CW complex, the product $B\mathbf{C} \times B\mathbf{D}$ coincides with the usual product of topological spaces, see [Mil57].

Remark 3.1.5. Informally, $B\mathbf{C}$ is built from geometric simplices $B\overrightarrow{[n]}$ realized as

- $B\overrightarrow{[0]} \simeq \Delta^0 \simeq \mathrm{pt}$: the objects $\mathrm{Ob}(\mathbf{C})$;

- $B\overrightarrow{[1]} \simeq \Delta^1 \simeq [0,1]$ the morphisms $0 \xrightarrow{f} 1$ in $\mathrm{Mor}(\mathbf{C})$;

- $B\overrightarrow{[2]} \simeq \Delta^2$ composable pairs

$$\begin{tikzcd} & 1 \arrow[dr, "f_1"] & \\ 0 \arrow[ur, "f_0"] \arrow[rr, "f_1\circ f_0"'] & & 2 \end{tikzcd}$$

- $B\overrightarrow{[3]} \simeq \Delta^3$ composable triples

$$\begin{tikzcd} & 1 \arrow[drr, "f_2\circ f_1"] \arrow[ddr, "f_1"] & & \\ 0 \arrow[ur, "f_0"] \arrow[rrr, dashed] \arrow[drr, "f_1\circ f_0"'] & & & 3 \\ & & 2 \arrow[ur, "f_2"'] & \end{tikzcd}$$

Thus we have an explicit CW decomposition with the set of 0-cells being $\mathrm{sk}_0(B\mathbf{C}) = \mathrm{Ob}(\mathbf{C})$, the set of 1-cells

$$\overline{\mathrm{sk}}_1(B\mathbf{C}) = \mathrm{sk}_1(B\mathbf{C}) \setminus \mathrm{sk}_0(B\mathbf{C}) = \mathrm{Mor}(\mathbf{C}) \setminus \{\mathrm{id}_c \mid c \in \mathrm{Ob}(\mathbf{C})\},$$

the set of 2-cells $\overline{\mathrm{sk}}_2(B\mathbf{C}) = \mathrm{sk}_2(B\mathbf{C}) \setminus \mathrm{sk}_1(B\mathbf{C})$ being the set of pairs of composable morphisms (f_0, f_1) excluding pairs of the form (f_0, id) and (id, f_1), and so on.

Example 3.1.6. Let $\mathbf{C} = \mathbb{Z}/2\mathbb{Z}$ be the category with one object $*$ and $\mathrm{Mor}(\mathbf{C}) = \{\mathrm{id}, \sigma\}$, with $\sigma^2 = \mathrm{id}$. Then for each $n \geqslant 0$ there exist exactly one n-cell corresponding to the chain of morphisms $[* \xrightarrow{\sigma} * \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} *]$ of length n. Then $\mathrm{sk}_0(B\mathbf{C}) = *$, $\mathrm{sk}_1(B\mathbf{C}) = \mathbb{RP}^1$, etc. The n-th skeleton is $\mathrm{sk}_n(B\mathbf{C}) = \mathbb{RP}^n$. Therefore, the classifying space is $B\mathbf{C} \simeq \mathbb{RP}^\infty$, the infinite real projective space.

3.2 Homotopy-theoretic properties of the classifying spaces

Let $F_0, F_1 \colon \mathbf{C} \to \mathbf{D}$ be two functors between (small) categories.

Lemma 3.2.1. *A natural transformation $h \colon F_0 \Rightarrow F_1$ induces a homotopy $B\mathbf{C} \times [0,1] \to B\mathbf{D}$ between BF_0 and BF_1.*

Proof. Define a functor $H \colon \mathbf{C} \times \overrightarrow{[1]} \to \mathbf{D}$ as follows. On objects, $H(c, 0) := F_0(c)$ and $H(c', 1) := F_1(c')$ for $c, c' \in \mathrm{Ob}(\mathbf{C})$. To define H on morphisms, we first recall that the natural transformation h is given by the family of morphisms

$$h = \{h_c \colon F_0(c) \to F_1(c)\}_{c \in \mathrm{Ob}(\mathbf{C})}$$

such that for all $f \colon c \to c'$, $h_{c'} \circ F_0(f) = F_1(f) \circ h_c$. Then given a morphism $(f, 0 \to 1) \in \mathrm{Mor}(\mathbf{C} \times \overrightarrow{[1]})$, $f \in \mathrm{Hom}_{\mathbf{C}}(c, c')$, define $H(f, 0 \to 1) \in \mathrm{Mor}(\mathbf{D})$ to be the morphism

$$h_{c'} \circ F_0(f) = F_1(f) \circ h_c \colon F_0(c) \to F_1(c').$$

It is easy to check that this defines a functor $H \colon \mathbf{C} \times \overrightarrow{[1]} \to \mathbf{D}$. Applying the classifying space functor B to H gives $BH \colon B(\mathbf{C} \times \overrightarrow{[1]}) \to B\mathbf{D}$. By Property (BC4), $B(\mathbf{C} \times \overrightarrow{[1]}) \simeq B\mathbf{C} \times B\overrightarrow{[1]} \simeq B\mathbf{C} \times [0,1]$, and so BH induces a map $BH \colon B\mathbf{C} \times [0,1] \to B\mathbf{D}$ satisfying $BH|_{B\mathbf{C} \times \{0\}} = BF_0$ and $BH|_{B\mathbf{C} \times \{1\}} = BF_1$. Therefore, BH gives a homotopy between BF_0 and BF_1. □

Proposition 3.2.2. *Given a pair of adjoint functors*

$$L \colon \mathbf{C} \rightleftarrows \mathbf{D} : R$$

then $B\mathbf{C}$ and $B\mathbf{D}$ are homotopy equivalent, with mutually inverse homotopy equivalences given by the maps BL and BR.

Proof. A pair of adjoint functors (L, R) comes with two natural transformations $\eta \colon \mathrm{id}_{\mathbf{C}} \Rightarrow RL$ and $\varepsilon \colon LR \Rightarrow \mathrm{id}_{\mathbf{D}}$, called the unit and counit of the adjunction. Then Lemma 3.2.1 implies

$$B(\mathrm{id}_{\mathbf{C}}) = \mathrm{id}_{B\mathbf{C}} \sim BR \circ BL \quad \text{and} \quad BL \circ BR \sim \mathrm{id}_{B\mathbf{D}} = B(\mathrm{id}_{\mathbf{D}}),$$

where "$\sim$" denotes homotopy equivalence. □

Corollary 3.2.3. *If $F \colon \mathbf{C} \to \mathbf{D}$ is an* equivalence *of categories, then $BF \colon B\mathbf{C} \to B\mathbf{D}$ is a homotopy equivalence of topological spaces.*

Remark 3.2.4. In practice, it often happens that a category $\mathbf{C}$ is not small, but *skeletally small* (i.e. it is equivalent to a small category $\mathbf{C}_0$). The categories $\mathbf{mod}(A)$ of finitely generated modules and $\mathbb{P}(A)$ of finitely generated projective modules over a ring A are two examples of skeletally small categories which are not small.

Then, even though functor B as constructed in Section 3.1 can not be directly applied to such $\mathbf{C}$, we can still define $B\mathbf{C}$ to be $B\mathbf{C}_0$. Then Corollary 3.2.3 insures that $B\mathbf{C}$ is well defined up to homotopy.

Corollary 3.2.5. *If $\mathbf{C}$ has initial or terminal object, then $B\mathbf{C}$ is contractible.*

Proof. If $e \in \mathrm{Ob}(\mathbf{C})$ is initial, consider the constant functor $p\colon \mathbf{C} \to *$ from $\mathbf{C}$ to the category $*$ with one object $*$ and one morphism id_*. Functor p has a left adjoint functor $i\colon * \hookrightarrow \mathbf{C}$ sending $*$ to e. Proposition 3.2.2 implies that $B\mathbf{C} \sim B* \simeq \mathrm{pt}$. □

Example 3.2.6. For any category $\mathbf{C}$ and $d \in \mathrm{Ob}(\mathbf{C})$, both left and right comma categories $\mathbf{C}/d$ and $d\backslash\mathbf{C}$ are contractible, because they have terminal object d/d and initial object $d\backslash d$ respectively.

Example 3.2.7. Any additive category $\mathbf{C}$ is contractible since by definition, $\mathbf{C}$ contains the zero object, which is both initial and terminal in $\mathbf{C}$. In particular, the classifying space of the category $\mathbb{P}(A)$ is contractible, $B\mathbb{P}(A) \sim \mathrm{pt}$.

Example 3.2.8. If $(\mathbf{C}, \otimes)$ is a symmetric monoidal category, then it often happens that the unit object $\mathbb{I}$ is also initial. So in this case once again $B\mathbf{C}$ is contractible. Examples of such categories include $\mathbb{P}(A)$ with direct sum $\oplus$ as monoidal structure, or the category **Fin** of finite sets with monoidal structure given by the disjoint union $\sqcup$ of sets.

One reason why we distinguish symmetric monoidal categories is that the monoidal structure $\otimes$ induces on the space $B\mathbf{C}$ a multiplication $B\mathbf{C} \times B\mathbf{C} \to B\mathbf{C}$, which makes it an H-space. For the precise definition of H-space and more details on the subject see, for example, [Sta63].

Unfortunately many interesting categories turn out to be contractible, and so not that interesting from the homotopy theory point of view. A standard way to remedy this problem is to replace $\mathbf{C}$ by the groupoid $\mathbf{Iso}(\mathbf{C})$ consisting of isomorphisms of $\mathbf{C}$. More precisely, $\mathrm{Ob}(\mathbf{Iso}(\mathbf{C})) = \mathrm{Ob}(\mathbf{C})$, and $\mathrm{Mor}(\mathbf{Iso}(\mathbf{C})) = \{\text{isomorhisms in } \mathbf{C}\}$. Then by (BC5) we get

$$B\mathbf{Iso}(\mathbf{C}) \simeq \bigsqcup_{\text{isoclasses}} B\,\mathrm{Aut}_{\mathbf{C}}(p),$$

where p denotes a representative of the corresponding isoclass, and $B\,\mathrm{Aut}_{\mathbf{C}}(p)$ is the classifying space of the group of automorphisms $\mathrm{Aut}_{\mathbf{C}}(p)$ of $p \in \mathrm{Ob}(\mathbf{C})$. In particular, for the skeletally small category **Fin** the classifying space of groupoid $\mathbf{Iso}(\mathbf{Fin})$ is

$$B\mathbf{Iso}(\mathbf{Fin}) \simeq \bigsqcup_{n \geq 0} BS_n\,,$$

with S_n denoting the symmetric group of n elements.

Remark 3.2.9. Note that from axioms (BC1)—(BC3) it follows that $B\mathbf{C} \simeq B\mathbf{C}^{op}$, but usually there is no functor $\mathbf{C} \to \mathbf{C}^{op}$ inducing this homeomorphism.

3.3 Connected components

Let X be a CW complex. Denote by $\pi_0(X)$ the set of its connected components. It can also be defined as $\pi_0(X) = \mathrm{sk}_0(X)/\sim$, where two 0-cells x_0, x_1 are equivalent, $x_0 \sim x_1$, if and only if there is a 1-cell e adjacent to both x_0 and x_1. This motivates the following definition and implies the following lemma.

Definition 3.3.1. *For a (small) category* $\mathbf{C}$*, define* $\pi_0(\mathbf{C}) := \pi_0(B\mathbf{C})$.

Lemma 3.3.2. *Let "$\sim$" be the equivalence relation on* $\mathrm{Ob}(\mathbf{C})$ *generated by* $c \sim c'$ iff $\exists f\colon c \to c'$ *or* $\exists f\colon c' \to c$*. Then* $\pi_0(\mathbf{C}) \simeq \mathrm{Ob}(\mathbf{C})/\sim$.

Example 3.3.3. Let G be a discrete group, and X a (left) G-set. Consider the action groupoid $G \ltimes X$ (cf. Example 1.6.4). Then $\pi_0(G \ltimes X) = X/G$ is the set of G-orbits in X.

Note that a G-set X can be viewed as a functor $\underline{G} \to \mathbf{Sets}$ sending $* \mapsto X$. This example motivates the following construction due to Grothendieck.

Example 3.3.4 (Translation category)**.** Let $X\colon \mathbf{C} \to \mathbf{Sets}$ be a functor from a small category $\mathbf{C}$ to $\mathbf{Sets}$. Define the *translation category* $\mathbf{C} \ltimes X$ as a category with $\mathrm{Ob}(\mathbf{C} \ltimes X) = \{(i,x) \mid i \in \mathbf{C}, x \in X(i)\}$, and morphisms defined by

$$\mathrm{Hom}_{\mathbf{C}\ltimes X}\big((i,x),(i',x')\big) = \{f \in \mathrm{Hom}_{\mathbf{C}}(i,i') \mid X(f)x = x'\}.$$

Lemma 3.3.5.

$$\pi_0(\mathbf{C} \ltimes X) \simeq \mathrm{colim}(X)$$

Proof. Indeed, for each $i \in \mathrm{Ob}(\mathbf{C})$ denote the universal maps $X(i) \to \mathrm{colim}(X)$ by φ_i. Define $\varphi\colon \mathrm{Ob}(\mathbf{C} \ltimes X) \to \mathrm{colim}(X)$ by $(i,x) \mapsto \varphi_i(x)$. If $(i,x_i) \sim (j,x_j)$ then by definition of the equivalence relation $\exists f_{ij}\colon i \to j$ (or $j \to i$) s.t. $X(f_{ij})x_i = x_j$, and so there is $X(f_{ij})\colon X(i) \to X(j)$ (or $X(j) \to X(i)$). By definition of the universal maps φ_i, $\varphi_j(x_j) = \varphi_i(x_i)$. Hence, by Lemma 3.3.2 we get a well-defined map $\pi_0(\mathbf{C} \ltimes X) \to \mathrm{colim}(X)$. The inverse map $\mathrm{colim}(X) \to \pi_0(\mathbf{C} \ltimes X)$ is induced by the universal property of colim. $\square$

3.4 Coverings of categories

For a short review of the theory of coverings of topological spaces, as well as for notations, see Section 2 of Appendix A.

Let $\mathbf{C}$ be a small category, and let $p\colon E \to B\mathbf{C}$ be a covering of $B\mathbf{C}$. Define the fiber transition functor $E\colon \mathbf{C} \to \mathbf{Sets}$ as follows. Any object $c \in \mathrm{Ob}(\mathbf{C})$ gives a 0-cell in $B\mathbf{C}$, so we define $E(c) := E_c := p^{-1}(c)$. Ant morphism $f\colon c \to c'$ in $\mathbf{C}$ gives an element of $B_1\mathbf{C}$. By Lemma 1.3.6, $B_1\mathbf{C}$ is naturally isomorphic to $\mathrm{Hom}_{\mathbf{sSets}}(\Delta[1]_*, B_*\mathbf{C})$. Thus the morphism $f\colon c \to c'$ corresponds to a morphism of simplicial sets $f\colon \Delta[1]_* \to B_*\mathbf{C}$. Applying the geometric realization functor $|-|$, one gets then a map of spaces $|f|\colon \Delta^1 \to B\mathbf{C}$, i.e. a path in $B\mathbf{C}$. This allows us to define the functor E on morphisms. Namely, for a morphism $f\colon c \to c'$ the map $E(f)\colon E_c \to E_{c'}$ is sending a point $e \in E_c$ to the target $e' = \widetilde{f}(e) \in E_{c'}$

of the unique lifting $\widetilde{f}\colon \Delta^1 \to E$ of the path f satisfying $\widetilde{f}(0) = e$ (cf. Theorem 2.0.3 in Appendix A).

The functor $E\colon \mathbf{C} \to \mathbf{Sets}$ we have defined is actually *morphism-invertible*, i.e. it maps all morphisms in $\mathbf{C}$ into isomorphisms in $\mathbf{Sets}$.

Conversely, given a functor $X\colon \mathbf{C} \to \mathbf{Sets}$, consider the translation category $\mathbf{C} \ltimes X$ (see Example 3.3.4). This category is equipped with the forgetful functor $F\colon \mathbf{C} \ltimes X \to \mathbf{C}$ sending $(c, x) \mapsto c$. Applying the classifying space functor B, we obtain a map of spaces

$$BF\colon B(\mathbf{C} \ltimes X) \to B\mathbf{C}.$$

Lemma 3.4.1. *If a functor $X\colon \mathbf{C} \to \mathbf{Sets}$ is morphism-inverting, then $BF\colon B(\mathbf{C} \ltimes X) \to B\mathbf{C}$ is a covering of $B\mathbf{C}$.*

These two constructions are inverse to each other, and

$$\mathbf{Cov}(B) \xrightarrow{\sim} \mathbf{Fun}^{\mathrm{inv}}(\mathbf{C}, \mathbf{Sets})$$

is an equivalence of categories $\mathbf{Cov}(B)$ of coverings of $B\mathbf{C}$ and the category $\mathbf{Fun}^{\mathrm{inv}}(\mathbf{C}, \mathbf{Sets})$ of morphism-inverting functors (see Theorem 2.0.7 in Appendix A).

Definition 3.4.2. *The fundamental groupoid $\mathbf{\Pi}(\mathbf{C})$ of a category $\mathbf{C}$ is the localization*

$$\mathbf{\Pi}(\mathbf{C}) := \mathbf{C}[\mathrm{Mor}(\mathbf{C})^{-1}]$$

of $\mathbf{C}$ along all morphisms $\mathrm{Mor}(\mathbf{C})$.

Theorem 3.4.3. *The localization functor $\gamma\colon \mathbf{C} \to \mathbf{\Pi}(\mathbf{C})$ induces a natural isomorphism of categories*

$$\mathbf{Fun}(\mathbf{\Pi}(\mathbf{C}), \mathbf{Sets}) \xrightarrow{\sim} \mathbf{Fun}^{\mathrm{inv}}(\mathbf{C}, \mathbf{Sets}) \simeq \mathbf{Cov}(\mathbf{C})$$

Remark 3.4.4. In Example 1.6.6 to every functor $F\colon \mathbf{C} \to \mathbf{Sets}$ we associated a simplicial set $\mathrm{hocolim}(F)$ which we called Bousfield-Kan construction, or a homotopy colimit. Comparing the two definitions, we see that $|\,\mathrm{hocolim}(F)| = B(\mathbf{C} \ltimes F)$. In other words, coverings are geometric realizations of homotopy colimits of fiber functors.

In many cases, covering of a particular category $\mathbf{C}$ can be described explicitly.

Example 3.4.5 (Discrete groupoids)**.** For the details of this example see P.May's book [May99, Chapter 2].

Let $\mathbf{C}$ be a small connected groupoid. In this case, a covering of $\mathbf{C}$ is a functor $p\colon \mathbf{E} \to \mathbf{C}$, where $\mathbf{E}$ is also a small connected groupoid, and the functor p should satisfy the following two conditions:

(1) "surjectivity:" the map $p\colon\ \mathrm{Ob}(\mathbf{E}) \to \mathrm{Ob}(\mathbf{C})$ is surjective;

(2) "local triviality:" for every $e \in \mathrm{Ob}(\mathbf{E})$ the map $p|_{\mathrm{St}(e)}\colon\ \mathrm{St}(e) \xrightarrow{\sim} \mathrm{St}(p(e))$ is a bijection. Here $\mathrm{St}(e)$ is by definition the set of objects $\mathrm{Ob}(e\backslash\mathbf{E})$.

Note that $p^{-1}(\mathrm{St}(c)) = \bigsqcup_{e \in E_c} \mathrm{St}(e)$, where E_c denotes the set $E_c = p^{-1}(c) \subset \mathrm{Ob}(\mathbf{E})$.

The corresponding translation functor $E\colon \mathbf{C} \to \mathbf{Sets}$ is defined by $c \mapsto E_c = p^{-1}(c)$ on objects $c \in \mathrm{Ob}(\mathbf{C})$. On morphisms, for any $f\colon c \to c'$ the corresponding morphism $E(f)\colon E_c \to E_{c'}$ is given by

$$e \mapsto e' := \text{target of } \left(p|_{\mathrm{St}(e)}\right)^{-1}(f)$$

Theorem 3.4.6. *If $p\colon E \to B$ is a covering of spaces, then $\mathbf{\Pi}(p)\colon \mathbf{\Pi}(E) \to \mathbf{\Pi}(B)$ is a covering of groupoids in the above sense. In fact, the functor $\mathbf{\Pi}(-)$ is an equivalence of categories*

$$\mathbf{\Pi}\colon \mathbf{Cov}(B) \xrightarrow{\sim} \mathbf{Cov}\left(\mathbf{\Pi}(B)\right)$$

3.5 Explicit presentations of fundamental groups of categories

Let $\mathbf{C}$ be a small category, and $c \in \mathrm{Ob}(\mathbf{C})$ be a fixed object.

Definition 3.5.1. *The* fundamental group $\pi_1(\mathbf{C}, c)$ *is the group*

$$\pi_1(\mathbf{C}, c) := \pi_1(B\mathbf{C}, c) \simeq \mathrm{Aut}_{\mathbf{\Pi}(\mathbf{C})}(c).$$

For a subset $T \subset \mathrm{Mor}(\mathbf{C})$ of morphisms of $\mathbf{C}$, define its *graph* to be the 1-dimensional CW-subcomplex of $B\mathbf{C}$ with edges being (non-identity) morphisms in T, and vertices being all sources and targets of morphisms from T. We call the set of morphisms T a *tree* if its graph is contractible. If $\mathbf{C}$ is connected, we can choose a *maximal tree* T which has all objects of $\mathbf{C}$ as its vertices (the existence of such a maximal tree is an easy application of Zorn's lemma).

Proposition 3.5.2. *Let $\mathbf{C}$ be a small connected category, and T a maximal tree in* $\mathrm{Mor}(\mathbf{C})$. *Fix an object $c_0 \in \mathrm{Ob}(\mathbf{C})$. Then $\pi_1(\mathbf{C}, c_0)$ has the following presentation:*

$$\pi_1(\mathbf{C}, c_0) \simeq \langle \overline{f} \mid f \in \mathrm{Mor}(\mathbf{C}) \rangle / R$$

where R is the normal subgroup of the free group $\langle \overline{f} \mid f \in \mathrm{Mor}(\mathbf{C}) \rangle$, generated by relations

- *$\overline{f} = 1$ if either $f = \mathrm{id}_c$ for some $c \in \mathrm{Ob}(\mathbf{C})$ or $f \in T$;*
- *$\overline{f} \cdot \overline{g} = \overline{f \circ g}$ for all composable morphisms $f, g \in \mathrm{Mor}(\mathbf{C})$.*

Sketch. Let $f\colon c_1 \to c_2$ be a morphism in $\mathbf{C}$. We can assume it's not id and it doesn't belong to the tree T. Since the tree T is maximal, there exist unique (non-oriented) paths γ_1 from c_0 to c_1 and γ_2' from c_2 to c_0 inside the tree T. Together with f, these two paths give a loop in $B\mathbf{C}$ based at the point c_0 by first going along γ_1, then along f and finally along γ_2'. Denote the resulting loop by γ_f. This allows us to define a homomorphism

$$\langle \overline{f} \mid f \in \mathrm{Mor}(\mathbf{C}) \rangle \to \pi_1(\mathbf{C}, c_0)$$

sending $\overline{f}$ to the homotopy class $[\gamma_f]$ of the loop γ_f. This homomorphism induses the required isomorphism $\langle \overline{f} \mid f \in \mathrm{Mor}(\mathbf{C}) \rangle / R \simeq \pi_1(\mathbf{C}, c_0)$. $\square$

Example 3.5.3. Let G be a discrete group, and let $\mathbf{C} = \underline{G}$ be the corresponding groupoid with one object. Proposition 3.5.2 implies that the tree $T = *$ is the trivial tree, and $\pi_1(BG) \simeq G$. In fact, we will see later that higher homotopy groups vanish, i.e. $\pi_i(BG) = 0$, $\forall i \geqslant 2$. Thus $BG = \mathrm{K}(G, 1)$ is the Eilenberg-Maclane space, see [May99, Chapter 4].

3.6 Homology of small categories

Motivation

If G is a discrete group, we can define its homology (and similarly cohomology) as the homology of Eilenberg-Maclane complex $C_\bullet(G)$ with $C_n(G) = \mathbb{Z}[G]^{\otimes n}$ and the differential $\partial\colon C_n(G) \to C_{n-1}(G)$ given by

$$\partial(g_0, \dots, g_n) = (g_1, \dots, g_n) + \sum_{i=0}^{n-1} (-1)^{i+1} (g_0, \dots, g_i g_{i+1}, \dots, g_n) + (-1)^{n+1} (g_0, \dots, g_{n-1})$$

Then the homology of G is defined by $\mathrm{H}_\bullet(G) := \mathrm{H}_\bullet[C(G)]$. Notice that as an abelian group, $C_n(G) \simeq \mathbb{Z}[B_n G]$ is the free group generated by the n-simplices of the nerve $B_* G$ of the group G (see Example 1.6.2), and moreover, the differential ∂ on $C_\bullet(G)$ is exactly the alternating sum $\partial = \sum (-1)^i d_i$ of the face maps of simplicial set $B_* G$. Thus, the complex $C_\bullet(G)$ is isomorphic to the standard cellular complex $C_\bullet(BG)$ of the CW complex BG. Therefore, there is a natural isomorphism

$$\mathrm{H}_\bullet(G) \simeq \mathrm{H}_\bullet(BG, \mathbb{Z}).$$

Homology of a category

We now generalize this to an arbitrary category.

Definition 3.6.1. *For a small category $\mathbf{C}$ define the complex $C_\bullet(\mathbf{C})$ to have n-th component $\mathbf{C}_n(\mathbf{C}) = \mathbb{Z}[B_n \mathbf{C}]$ and differential $\partial\colon \mathbf{C}_n(\mathbf{C}) \to \mathbf{C}_n(\mathbf{C})$ given by*

$$\partial = \sum_{i=0}^{n} (-1)^i d_i$$

where the maps d_i are the face maps of the simplicial set $B_* \mathbf{C}$ (see Section 1.5).

Then the *homology groups* of $\mathbf{C}$ are defined as the homology groups of the complex $C_\bullet(\mathbf{C})$, i.e.

$$\mathrm{H}_\bullet(\mathbf{C}) := \mathrm{H}_\bullet(C(\mathbf{C})) \simeq \mathrm{H}_\bullet(B\mathbf{C}, \mathbb{Z})$$

An important consequence of Theorem 3.4.3 is the following.

Corollary 3.6.2. *Local systems (see [BMP14, Sec.7, Ch.1]) on $B\mathbf{C}$ are in one-to-one correspondence with morphisms-inverting functors $A\colon \mathbf{C} \to \mathbf{Ab}$, i.e. with functors $\Pi(\mathbf{C}) \to \mathbf{Ab}$ from the fundamental groupoid of $\mathbf{C}$.*

In the case of a group G, coefficients are G-modules, which are just functors $\underline{G} \to \mathbf{Ab}$.

This allows to introduce *homology of* $\mathbf{C}$ *with coefficients in a local system.* Namely, let $A\colon \mathbf{C} \to \mathbf{Ab}$ be a morphism-inverting functor. Define the complex $C_\bullet(\mathbf{C}, A)$ as follows, where $\coprod$ denotes the coproduct in the category $\mathbf{Ab}$, i.e. $\coprod$ is the direct sum $\oplus$ of abelian groups.

$$\begin{aligned} C_0(\mathbf{C}, A) &= \coprod_{c \in \mathrm{Ob}(\mathbf{C})} A(c) \\ C_1(\mathbf{C}, A) &= \coprod_{f\colon c_0 \to c_1} A(c_0) \\ C_n(\mathbf{C}, A) &= \coprod_{c_0 \to \cdots \to c_n} A(c_0) \end{aligned}$$

The differential on $C_\bullet(\mathbf{C}, A)$ is given by

$$\partial_n^A = \sum_{i=0}^{n} (-1)^i A(d_i) \,:\, C_n(\mathbf{C}, A) \to C_{n-1}(\mathbf{C}, A)$$

For example, $\partial_1^A\colon \bigsqcup_{f\colon c_0 \to c_1} A(c_0) \to \bigsqcup_{c} A(c)$ restricted to the component indexed by $f\colon c_0 \to c_1$ is given by the map

$$\partial_1^A|_{f\colon c_0 \to c_1}\colon A(c_0) \to A(c_0) \oplus A(c_1)$$

sending $x \mapsto (x, A(f)x)$. So $\mathrm{H}_0(\mathbf{C}, A) = \mathrm{Coker}(\partial_1^A)$.

Lemma 3.6.3. *For any local system $A\colon \mathbf{C} \to \mathbf{Ab}$ there is a natural isomorphism*

$$\mathrm{H}_0(\mathbf{C}, A) \simeq \mathrm{colim}(A).$$

Proof. Consider the translation category $\mathbf{C} \ltimes A$ associated to A (see Example 3.3.4). Then

$$\pi_0(\mathbf{C} \ltimes A) = \mathrm{Ob}(\mathbf{C} \ltimes A)\Big/ \sim \;\; = \bigsqcup_{c \in \mathrm{Ob}(\mathbf{C})} A(c)\Big/ \sim$$

where

$$\begin{aligned} (c_0, x) \sim (c_1, x') &\Leftrightarrow \exists f\colon c_0 \to c_1 \text{ s.t. } A(f)x = x' \\ &\Leftrightarrow \exists f\colon c_0 \to c_1 \text{ s.t. } A(f)x - x' = 0 \end{aligned}$$

Thus $\pi_0(\mathbf{C} \ltimes A) = \bigsqcup_{c \in \mathrm{Ob}(\mathbf{C})} A(c)\Big/\langle (x, A(f)x) \mid x \in A(c_0),\, f\colon c_0 \to c_1 \rangle = \mathrm{Coker}(\partial_1^A)$. The only thing left is to use Lemma 3.3.5 saying $\pi_0(\mathbf{C} \ltimes A) \simeq \mathrm{colim}(A)$. $\square$

Since **Ab** is an abelian category, the category $\mathbf{Fun}(\mathbf{C}, \mathbf{Ab})$ is also abelian, having enough projectives and injectives since **Ab** does. So $\operatorname{colim}\colon \mathbf{Fun}(\mathbf{C}, \mathbf{Ab}) \to \mathbf{Ab}$ is an additive right exact functor. Therefore it has (classical) left derived functors $\operatorname{colim}_n := \mathbb{L}_n \operatorname{colim}$.

Theorem 3.6.4 (Quillen)**.** *Fo any $A\colon \mathbf{C} \to \mathbf{Ab}$, there is an isomorphism*

$$\mathrm{H}_n(\mathbf{C}, A) \simeq \operatorname{colim}_n(A),\ \forall n \geqslant 0$$

Sketch of proof. The theorem follows from Grothendieck's δ-functor criterion (see [BMP14, Section 3.4]). Namely, one checks that the sequence of functors $\{A \mapsto \mathrm{H}_n(\mathbf{C}, A)\}_{n\geqslant 0}$ is an exact δ-functor which is effaceable in degree $p > 0$. The sequence of derived functors $\{\operatorname{colim}_n\}_{n\geqslant 0}$ satisfies these properties automatically. Since we have a functorial in A isomorphism

$$\mathrm{H}_0(\mathbf{C}, A) \simeq \operatorname{colim}_0(A),$$

the universal property of δ-functors implies isomorphisms

$$\mathrm{H}_n(\mathbf{C}, A) \simeq \operatorname{colim}_n(A),$$

for all $n \geqslant 0$. $\square$

Theorem 3.6.5 (Quillen)**.** *For any $A\colon \mathbf{C} \to \mathbf{Ab}$, denote by $\underline{A}$ the corresponding local system on $B\mathbf{C}$ (see Corollary 3.6.2). Then there is a natural isomorphism*

$$\mathrm{H}_\bullet(B\mathbf{C}, \underline{A}) \simeq \mathrm{H}_\bullet(\mathbf{C}, A)$$

Sketch of proof. Filter the CW complex $B\mathbf{C}$ by its skeleta:

$$\mathrm{sk}_0(B\mathbf{C}) \subseteq \mathrm{sk}_1(B\mathbf{C}) \subseteq \mathrm{sk}_2(B\mathbf{C}) \subseteq \dots$$

This filtration induces a filtration

$$C_\bullet(\mathrm{sk}_0(B\mathbf{C})) \subseteq C_\bullet(\mathrm{sk}_1(B\mathbf{C})) \subseteq \dots$$

on the cellular complex $C_\bullet(B\mathbf{C}, A)$. The associated spectral sequence (E^r_{pq}, d^r) with

$$E^1_{pq} = \mathrm{H}_{p+q}\left[C_\bullet(\mathrm{sk}_p(B\mathbf{C}))/C_\bullet(\mathrm{sk}_{p-1}(B\mathbf{C}))\right] \simeq \mathrm{H}_{p+q}\left[C_\bullet(\mathrm{sk}_p / \mathrm{sk}_{p-1})\right]$$

converging to $H_{p+q}(\mathbf{C}, A)$ has $E^1_{pq} = 0$ if $q \neq 0$. Indeed, the quotient $\mathrm{sk}_p / \mathrm{sk}_{p-1}$ is homeomorphic to a bouquet of p-spheres $\mathbb{S}^p$ indexed by the p-cells of $B\mathbf{C}$, and so $\mathrm{H}_{p+q}(\mathrm{sk}_p / \mathrm{sk}_{p-1})$ vanishes for $q \geqslant 1$. Therefore, the spectral sequence degenerates at the first page, i.e. $E^1 = E^\infty$. The complex $E^1_{\bullet 0}$ is the (normalized) chain complex $C_\bullet(\mathbf{C}, A)$. Hence the spectral sequence degenerating on the first page gives the desired isomorphism. $\square$

3.7 Quillen's Theorem A

Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor between two categories, and let $BF\colon B\mathbf{C} \to B\mathbf{D}$ be the induced map of classifying spaces. Let's recall (see Section 2.5) that left and right comma categories F/d and $d\backslash F$ come equipped with forgetful functors to $\mathbf{C}$, and also with a natural transformations $F \circ j \Rightarrow \mathrm{const}_d$ and $\mathrm{const}_d \Rightarrow F \circ j$ respectively. This data fits into two dual commutative diagrams

$$
\begin{tikzcd}
F/d \arrow[r, "j"] \arrow[d] \arrow[dr, "\mathrm{const}_d"'] & \mathbf{C} \arrow[d, "F"] \arrow[dl, phantom, "\Leftarrow \eta"] \\
* \arrow[r] & \mathbf{D}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
d\backslash F \arrow[r, "j"] \arrow[d] \arrow[dr, "\mathrm{const}_d"'] & \mathbf{C} \arrow[d, "F"] \arrow[dl, phantom, "\Rightarrow \eta"] \\
* \arrow[r] & \mathbf{D}
\end{tikzcd}
$$

Applying the classifying space functor B to these diagrams gives two sequences

$$B(d\backslash F) \overset{Bj}{\to} B\mathbf{C} \overset{f}{\to} B\mathbf{D}$$
$$B(F/d) \overset{Bj}{\to} B\mathbf{C} \overset{f}{\to} B\mathbf{D}$$

where we denoted $f = BF$. Because the compositions are constant, we have natural maps to the homotopy fiber (see diagram (A.4))

$$B(d\backslash F) \to Ff \leftarrow B(F/d)$$

Theorem 3.7.1 (Quillen's Theorem A)**.** *If $F\colon \mathbf{C} \to \mathbf{D}$ is a functor such that every comma category $d\backslash F$ is contractible for every $d \in \mathbf{D}$. Then $BF\colon B\mathbf{C} \to B\mathbf{D}$ is a homotopy equivalence.*

Remark 3.7.2. Since $B\mathbf{D} \simeq B\mathbf{D}^{\mathrm{op}}$, in Theorem 3.7.1 the assumption of right comma categories $d\backslash F$ being contractible can be replaced with the assumption that left comma categories F/d are contractible, and the theorem will still be true.

Example 3.7.3. Let's assume that $F\colon \mathbf{C} \to \mathbf{D}$ has a left adjoint functor $L\colon \mathbf{D} \to \mathbf{C}$. Then L induces for every $d \in \mathbf{D}$ an equivalence of categories $d\backslash F \overset{\sim}{\to} L(d)\backslash\mathbf{C}$. Indeed, unit and counit of adjunction, $\eta\colon \mathrm{id}_{\mathbf{D}} \Rightarrow FL$ and $\varepsilon\colon LF \Rightarrow \mathrm{id}_{\mathbf{C}}$, induce the *isomorphisms* $FLF \overset{F\circ\varepsilon}{\Rightarrow} F$ and $LFL \overset{\eta\circ L}{\Rightarrow} L$. Then the functors $d\backslash F \to L(d)\backslash\mathbf{C}$ sending $[d \to Fc] \mapsto [Ld \to LFc]$ and $L(d)\backslash\mathbf{C} \to d\backslash F$ sending $[Ld \to c] \mapsto [FLd \to Fc]$ are mutually inverse, giving the required equivalence of categories.

Since $L(d)\backslash\mathbf{C}$ has initial object, it is contractible (see Corollary 3.2.5), and so Theorem 3.7.1 implies that $BF\colon B\mathbf{C} \overset{\sim}{\to} B\mathbf{D}$. We have already proved that adjunctions between functors induce homotopy equivalences of the classifying spaces, see Proposition 3.2.2. So Quillen's theorem A agrees with this result.

Example 3.7.4. Let $i\colon \underline{\mathbb{N}} \hookrightarrow \underline{\mathbb{Z}}$ be the inclusion of $\mathbb{N}$ into $\mathbb{Z}$ viewed as an inclusion of categories with one object $*$. We leave it as an exercise to check that $*\backslash i$ is contractible. So there is a homotopy equivalence

$$B\underline{\mathbb{N}} \simeq B\underline{\mathbb{Z}} \simeq \mathbb{S}^1$$

Let $X_{**} = \{X_{pq}\}$ be a bisimplicial set. We denote horizontal face and degeneracy maps by d_i^h, s_j^h, and vertical — by d_i^v, s_j^v.

Definition 3.7.5. *The* geomentric realization *of a bisimplicial set X_{**} is*

$$BX = \coprod_{p,q\geqslant 0} X_{pq} \times \Delta^p \times \Delta^q \Big/ \sim$$

where the equivalence relation $\sim$ is given by the same formulas as in Definition 2.2.1 of the geometric realization of a simplicial set, but separate relations for horizontal and vertical directions.

There exists a *diagonalization functor* $\mathbf{ssSets} \overset{\mathrm{diag}}{\to} \mathbf{sSets}$ from bisimplicial sets to simplicial sets, sending a bisimplicial set $\{X_{pq}\}$ to the simplicial set $\{X_{nn}\}$ with face maps $d_i = d_i^h \circ d_i^v = d_i^v \circ d_i^h$ and degeneracy maps $s_j = s_j^h \circ s_j^v = s_j^h \circ s_j^v$.

Lemma 3.7.6. *For any bisimplicial set $X = \{X_{pq}\}$ the functor* diag *induces a homotopy equivalence*

$$BX \overset{\sim}{\to} B\,\mathrm{diag}(X)$$

Let $f = f_{**}\colon X_{**} \to Y_{**}$ be a map of bisimplicial sets. Every object $c \in \mathrm{Ob}(\mathbf{C}) = B_0\mathbf{C} \simeq \mathrm{Hom}_{\mathbf{sSets}}(\Delta[0]_*, B_*\mathbf{C})$ gives an element $s_0^p(c) \in B_p\mathbf{C}$ for $p \geqslant 0$. Define the *fiber* of f to be the bisimplicial subset $f^{-1}(c) \subseteq X_{**}$ given by

$$f^{-1}(c) = \{f_{pq}^{-1}(s_0^p(c)) \subseteq X_{pq}\}$$

Any morphism $\alpha\colon c \to c'$ yields a map of bisimplicial sets $f^{-1}(c) \overset{\alpha_*}{\to}$.

Lemma 3.7.7. *Let $f\colon X_{**} \to Y_{**}$ be as above.*

(1) *If for every $p \geqslant 0$ the maps $f_{p*}\colon X_{p*} \to Y_{p*}$ are homotopy equivalences, then $Bf\colon BX \to BY$ is a homotopy equivalence.*

(1) *If for every morphism $\alpha\colon c \to c'$ in $\mathbf{C}$ the associated map $Bf^{-1}(c) \to Bf^{-1}(c')$ is a homotopy equivalence, then for every $c \in \mathrm{Ob}(\mathbf{C})$ the inclusion $f^{-1}(c) \hookrightarrow X_{**}$ fits into a homotopy fibration sequence (see Definition 3.0.10)*

$$Bf^{-1}(c) \to BX \to B\mathbf{C}$$

Proposition 3.7.8 (Quillen's lemma)**.** *For any functor $F\colon \mathbf{C} \to \mathbf{D}$, the canonical forgetful functor $j\colon \mathbf{D}\backslash F \to \mathbf{C}$ from the global right comma category is a homotopy equivalence.*

Proof. Recall (see Section 2.5) that $\mathbf{D}\backslash F$ has objects

$$\mathrm{Ob}(\mathbf{D}\backslash F) = \{(c, d, f) \mid c \in \mathrm{Ob}(\mathbf{C}), d \in \mathrm{Ob}(\mathbf{D}),\, d \xrightarrow{f} F(c)\}$$

and morphisms being pairs (φ, ψ), $\varphi\colon c \to c'$ and $\psi\colon d \to d'$ making the obvious diagram commute.

Let $X = \{X_{pq}\}$ be a bisimplicial set defined by

$$X = \{d_p \to \cdots \to d_0 \to F(c_0),\ c_0 \to \cdots \to c_p\}_{p,q} \tag{1.9}$$

with horizontal and vertical face and degeneracy maps coming from the nerves $B_*\mathbf{C}$ and $B_*\mathbf{D}$, respectively.

Note that $B_*(\mathbf{D}\backslash F) = \mathrm{diag}(X)$, and therefore $B(\mathbf{D}\backslash F)$ is homotopy equivalent to BX by Lemma 3.7.6.

Consider the natural projection $f\colon X_{**} \to B_*\mathbf{C}$. Then the fiber $f^{-1}(c_0)$ has elements

$$\{d_q \to \cdots \to d_0 \to F(c_0)\} = \begin{tikzcd}[column sep=small] d_q \arrow[r] \arrow[dr] & \dots \arrow[r] & d_0 \arrow[dl] \\ & F(c_0) & \end{tikzcd} = B_q(\mathbf{D}/F(c_0))$$

For the first equality, we downwards arrow in the left hand side are just compositions of the corresponding arrows in $\{d_q \to \cdots \to d_0 \to F(c_0)\}$. So $f^{-1}(c_0) \simeq B_*(\mathbf{D}/F(c_0))$. Since the category $\mathbf{D}/F(c_0)$ has a terminal object, the corresponding space $B(\mathbf{D}/F(c_0))$ is contractible for any c_0. By Lemma 3.7.7 we conclude that $Bf\colon BX \to B\mathbf{C}$ is a homotopy equivalence.

Combining this with the homotopy equivalence $B(\mathbf{D}\backslash F) \simeq BX$ obtained above, we get $B(\mathbf{D}\backslash F) \simeq B\mathbf{C}$. $\square$

We are now ready to prove Theorem 3.7.1.

Proof of Quillen's theorem A. Consider the category $\mathbf{D}\backslash F$ together with two forgetful functors

$$\mathbf{C} \xleftarrow{j_{\mathbf{C}}} \mathbf{D}\backslash F \xrightarrow{j_{\mathbf{D}}} \mathbf{D}^{\mathrm{op}}$$

where $j_{\mathbf{C}}\colon (c, d, f) \mapsto c$ and $j_{\mathbf{D}}\colon (c, d, f) \mapsto d$. By Quillen's lemma 3.7.8, functor $j_{\mathbf{C}}$ is a homotopy equivalence. Consider the following commutative diagram

$$\begin{tikzcd} \mathbf{C} \arrow[d, "F"'] & \mathbf{D}\backslash F \arrow[l, "\sim"'] \arrow[r] \arrow[d, "j_{\mathrm{Mor}}"] & \mathbf{D}^{\mathrm{op}} \arrow[d, equal] \\ \mathbf{D} & \mathrm{Mor}(\mathbf{D}) \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathbf{D}^{\mathrm{op}} \end{tikzcd} \tag{1.10}$$

In the diagram (1.10) above, $\mathrm{Mor}(\mathbf{D})$ denotes the set of morphisms $\mathrm{Mor}(\mathbf{D})$, but viewed as a category. The functors $t\colon \mathrm{Mor}(\mathbf{D}) \to \mathbf{D}$ and $s\colon \mathrm{Mor}(\mathbf{D}) \to \mathbf{D}^{\mathrm{op}}$ are the projection functors $[d \to d'] \mapsto d'$ and $[d \to d'] \mapsto d$ respectively. The functor $j_{\mathrm{Mor}}\colon \mathbf{D}\backslash F \to \mathrm{Mor}(\mathbf{D})$ is the projection functor $(c, d, f) \mapsto f$.

Note that $\mathrm{Mor}(\mathbf{D}) = \mathbf{D}\backslash \mathrm{id}_{\mathbf{D}}$. Thus, by Quillen's lemma functors s, t are homotopy equivalences. Therefore it suffices to prove that $B(\mathbf{D}\backslash F) \xrightarrow{\sim} B\mathbf{D}^{\mathrm{op}}$ is a homotopy equivalence.

The last map factors as $B(\mathbf{D}\backslash F) \xrightarrow{\sim} BX \xrightarrow{\pi} B\mathbf{D}^{\mathrm{op}}$, where X is the bisimplicial set defined by the formula (1.9) in the proof of Quillen's lemma. The map $\pi\colon BX \to B\mathbf{D}^{\mathrm{op}}$ is defined by the formula

$$\pi\colon \{d_q \to \cdots \to d_0 \to F(c_0), c_0 \to \cdots \to c_p\} \mapsto \{d_q \to \cdots \to d_0\}$$

As in the proof of Quillen's lemma, the fiber $\pi^{-1}(d_0) \simeq B_*(d_0\backslash F)$, and the latter simplicial set is contractible by the assumption. Therefore, Lemma 3.7.7 implies that $B\pi$ is a homotopy equivalence. Thus, $B(\mathbf{D}\backslash F) \xrightarrow{\sim} BX \xrightarrow{\pi} B\mathbf{D}^{\mathrm{op}}$ is also a homotopy equivalence, and so from the diagram (1.10) we conclude that BF is also a homotopy equivalence. □

We refer the reader to the original paper [Qui73] by D. Quillen for all the details.

3.8 Fibred and cofibred functors

For the materials in the section please see [SGA03, Exposé VI].

Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor, and let $d \in \mathrm{Ob}(\mathbf{D})$ be a fixed object.

Definition 3.8.1. *The fibre of F over d is the category $F^{-1}(d)$ defined by*

$$\begin{aligned} \mathrm{Ob}(F^{-1}(d)) &= \{c \in \mathrm{Ob}(\mathbf{C}) \mid F(c) = d\} \\ \mathrm{Mor}(F^{-1}(d)) &= \{f \in \mathrm{Mor}(\mathbf{C}) \mid F(f) = \mathrm{id}_d\} \end{aligned}$$

There are two natural functors relating the fibre category $F^{-1}(d)$ with left and right comma categories F/d and $d\backslash F$. Precisely, we have functors

$$\begin{aligned} i_*\colon F^{-1}(d) &\hookrightarrow d\backslash F \\ c &\mapsto (d \xrightarrow{=} Fc, c) = (\mathrm{id}_d, c) \end{aligned}$$

and

$$\begin{aligned} j_*\colon F^{-1}(d) &\hookrightarrow F/d \\ c &\mapsto (Fc \xrightarrow{=} d, c) = (\mathrm{id}_d, c) \end{aligned}$$

In general, these functors are not homotopy equivalences.

Definition 3.8.2. *A functor $F\colon \mathbf{C} \to \mathbf{D}$ is* pre-fibred *if for any $d \in \mathrm{Ob}(\mathbf{D})$ the induced functor i_* has a right adjoint. We denote this right adjoint functor by $i^!\colon d\backslash F \to F^{-1}(d)$.*

Definition 3.8.3. *Dually, we say that a functor $F\colon \mathbf{C} \to \mathbf{D}$ is* pre-cofibred *if the functor $j_*\colon F^{-1}(d) \hookrightarrow F/d$ has a left adjoint, which we will denote $j^*\colon F/d \to F^{-1}(d)$.*

Note that if a functor $F\colon \mathbf{C} \to \mathbf{D}$ is pre-fibred, Proposition 3.2.2 implies that the functors i_* induces a homotopy equivalence $BF^{-1}(d) \xrightarrow{\sim} B(d\backslash F)$. Similarly, for a pre-cofibred functor F, the functor j_* induces a homotopy equivalence $BF^{-1}(d) \xrightarrow{\sim} B(F/d)$.

Let now $F\colon \mathbf{C} \to \mathbf{D}$ be a pre-fibred functor, and let $f\colon d \to d'$ be a fixed morphism in $\mathbf{D}$. Define the *base change functor* $f^*\colon F^{-1}(d') \to F^{-1}(d)$ as the composition

$$F^{-1}(d') \xrightarrow{i'_*} d'\backslash F \xrightarrow{f} d\backslash F \xrightarrow{i^!} F^{-1}(d)$$

The functor $f\colon d'\backslash F \to d\backslash F$ is defined by the pre-composing with f, i.e.

$$f\colon [d' \to Fc] \mapsto [d \xrightarrow{f} d' \to Fc].$$

If $f\colon d \to d'$ and $g\colon d' \to d''$ is a composable pair of morphisms in $\mathbf{D}$, there exists a natural transformation of functors $\alpha\colon f^*g^* \Rightarrow (gf)^*$ induced by the counit $\varepsilon\colon i'_* \circ (i')^! \Rightarrow \mathrm{id}_{d\backslash F}$ of the adjunction $(i'_*, (i')^!)$:

$$\begin{tikzcd}
F^{-1}(d'') \arrow[r, hook, "i''_*"] & d''\backslash F \arrow[r, "g"] & d'\backslash F \arrow[r, "(i')^!"] \arrow[dr, "\mathrm{id}"'] & F^{-1}(d') \arrow[d, "i'_*"] \arrow[dl, phantom, "\varepsilon"] & & \\
& & & d'\backslash F \arrow[r, "f"] & d\backslash F \arrow[r, "i^!"] & F^{-1}(d)
\end{tikzcd} \tag{1.11}$$

Definition 3.8.4. *A pre-fibred functor $F\colon \mathbf{C} \to \mathbf{D}$ is called* fibred *if for all composable pairs f, g in* $\mathrm{Mor}(\mathbf{D})$ *the natural transformation $\alpha\colon f^*g^* \Rightarrow (gf)^*$ defined above is an isomorphism.*

Definition 3.8.5. *Dually, a functor $F\colon \mathbf{C} \to \mathbf{D}$ is called* cofibred *if for all composable pairs f, g in* $\mathrm{Mor}(\mathbf{D})$*, $f\colon d \to d'$ and $g\colon d' \to d''$, the natural transformation $\beta\colon g_*f_* \Rightarrow (gf)_*$ induced by the unit $\eta\colon \mathrm{id} \Rightarrow j'_* \circ (j')^*$ of the adjunction $((j')^*, j'_*)$ is an isomorphism. Here for any morphism f in $\mathbf{D}$ the induced functor f_* is defined as a composition*

$$F^{-1}(d) \xrightarrow{j_*} F/d \xrightarrow{f} F/d' \xrightarrow{j^*} F^{-1}(d')$$

similar to the definition for a fibred functor above. The natural transformation β is defined by a diagram similar to the diagram (1.11)*.*

Corollary 3.8.6 (of Theorem 3.7.1)**.** *Assume a functor $F\colon \mathbf{C} \to \mathbf{D}$ is either fibered or cofibred, and assume $F^{-1}(d)$ is contractible for all $d \in \mathrm{Ob}(\mathbf{D})$. Then $BF\colon B\mathbf{C} \to \mathbf{D}$ is a homotopy equivalence.*

Example 3.8.7 (Grothendieck)**.** There is a one-to-one correspondence

$$\{\text{cofibered functors } F\colon \mathbf{C} \to \mathbf{D}\} \leftrightarrow \{\text{functors } \mathbf{D} \to \mathbf{Cats}\}$$

The correspondence is as follows. Given a cofibred functor $F\colon \mathbf{C} \to \mathbf{D}$, we associate to is the functor $F^{-1}\colon \mathbf{D} \to \mathbf{Cats}$ sending $d \mapsto F^{-1}(d) \in \mathrm{Ob}(\mathbf{Cats})$, $d \in \mathrm{Ob}(\mathbf{D})$. The definition 3.8.4 of a fibred functor ensures that F^{-1} is indeed a well-defined functor.

On the other hand, having a functor $X\colon \mathbf{D} \to \mathbf{Cats}$, we associate to it a cofibred functor $\mathbf{D} \ltimes X \xrightarrow{j} \mathbf{D}$.

3.9 Quillen's Theorem B

Theorem 3.9.1 (Quillen's theorem B)**.** *Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor such that for each $f \in \mathrm{Mor}(\mathbf{D})$, $f\colon d \to d'$, the base change functor $f\colon d'\backslash F \to d\backslash F$ is a homotopy equivalence. Then for every $d \in \mathrm{Ob}(\mathbf{D})$, the induced sequence of maps $B(d\backslash F) \overset{Bj}{\to} B\mathbf{C} \overset{BF}{\to} B\mathbf{D}$ of classifying spaces is a homotopy fibration sequence (see Definition 3.0.10).*

Corollary 3.9.2. *For a functor F as in the theorem, there exists a long exact sequence of homotopy groups*

$$\cdots \to \pi_{n+1}(B\mathbf{D}) \to \pi_n(B(d\backslash F)) \to \pi_n(B\mathbf{C}) \to \pi_n(B\mathbf{D}) \to \dots$$

Proof of Theorem 3.9.1. As in the proof of Quillen's lemma 3.7.8 and Quillen's theorem A, consider the bisimplcial set

$$X = \{d_p \to \cdots \to d_0 \to F(c_0),\ c_0 \to \cdots \to c_p\}_{p,q}$$

Let $\pi\colon X \to B_*\mathbf{D}^{\mathrm{op}}$ be the obvious projection. Again, as in Proposition 3.7.8, $\pi^{-1}(d) \simeq B_*(d\backslash F)$.

By the assumption, $d'\backslash F \overset{\sim}{\to} d\backslash F$ is a homotopy equivalence. By Lemma 3.7.7, the sequence $B(d\backslash F) \to BX \to B\mathbf{D}^{\mathrm{op}}$ is a homotopy fibration sequence.

On the other hand, we can factor Bj as follows:

$$Bj\colon B(d\backslash F) \to BX \simeq B\,\mathrm{diag}(X) = B(\mathbf{D}\backslash F) \overset{\sim}{\to} B\mathbf{C} \tag{1.12}$$

where the last homotopy equivalence follows from Quillen's lemma 3.7.8. Summarizing, we have the following commutative diagram

$$\begin{tikzcd}
B(d\backslash F) \arrow[r] \arrow[d, equal] & BX \arrow[r, "\sim"] \arrow[d, "\sim"] & B\mathbf{D}^{\mathrm{op}} \arrow[d, "\simeq"] \\
B(d\backslash F) \arrow[r, "Bj"] & B\mathbf{C} \arrow[r, "BF"] & B\mathbf{D}
\end{tikzcd}$$

In the diagram above, the vertical map $BX \to B\mathbf{C}$ is the homotopy equivalence given by the composition $BX \simeq B\,\mathrm{diag}(X) = B(\mathbf{D}\backslash F) \overset{\sim}{\to} B\mathbf{C}$ from (1.12). Since all vertical arrow of the diagram are homotopy equivalences, and the top row is a homotopy fibration sequence, then so is the bottom row, which finishes the proof. □

Chapter 2

Algebraic K-theory

1 Introduction and overview

The original application of the ideas we discussed in Section 3 is to higher *algebraic K-theory*, see [Qui73].

Higher algebraic K-theory assigns to any ring A a sequence of abelian groups $K_i(A)$, $i \geq 0$, in a functorial way. The groups $K_i(A)$ are defined as the homotopy groups

$$K_i(A) := \pi_i[K(A)],$$

of a certain topological space $K(A)$, which is usually constructed in two steps.

The first step associates to A a small category $\mathbf{K}(A)$ depending functorially on A. Then, the space $K(A)$ is taken to be $B\mathbf{K}(A)$, the geometric realization of $\mathbf{K}(A)$.

In fact, there are several different ways to construct the category $\mathbf{K}(A)$. We will only discuss the two original ones, called "plus" and "Q" constructions, both due to Quillen.

In each of the constructions, one starts with some obvious category $\widetilde{\mathbf{K}}(A)$ related to the ring A, which, however, does not have interesting homotopy theory. Then, to get $\mathbf{K}(A)$ out of $\widetilde{\mathbf{K}}(A)$ one performs a certain natural, but highly non-obvious (a priori) modification, which changes drastically the homotopy properties of $B\widetilde{\mathbf{K}}(A)$.

The comparison between the two constructions suggests that it is natural to define K-theory *not* as a functor $A \mapsto B\mathbf{K}(A) =: K(A)$ from **Rings** to **Spaces**, but as a functor from **Rings** to the category of *(topological) spectra*.

Recall that a *spectrum* is a sequence of (based) topological spaces $\{X_0, X_1, X_2, \dots\}$ related by the so-called *bonding maps* $\sigma_i \colon X_i \xrightarrow{\sim} \Omega X_{i+1}$, which are assumed to be homeomorphisms, often even identities.

To extend the K-theory functor $K \colon \mathbf{Rings} \to \mathbf{Spaces}$ to a functor

$$K \colon \mathbf{Rings} \to \mathbf{Spectra}, \quad A \mapsto (X_i)_{i \geqslant 0}$$

one needs to construct X_1 such that $\Omega X_1 \simeq K(A)$, and then X_2 such that $\Omega X_2 = X_1$, etc. This process is called *delooping* of the K-theory of a ring, and there many different constructions depending on the model for $K(A)$. Later we will briefly review a higher dimensional generalization of Quillen's original Q-construction which gives such deloopings for all n. For some more details, see Section 7.2.

Remark 1.0.1. There is another highly interesting (and apparently less known) generalization of Q-construction to a non-additive setting, due to Loday and Fiedorowicz [FL91].

2 Classical K-theory

We assume that the reader have already seen at least the very basics of K-theory.

Fix an associative unital ring A, and let $\mathbb{P}(A)$ be the category of finitely generated (right) projective A-modules.

2.1 The group $K_0(A)$

Denote by $[\mathbb{P}(A)]$ the set of isomorphism classes of objects in $\mathbb{P}(A)$. The set $[\mathbb{P}(A)]$ has a structure of a unital abelian monoid, with multiplication given by $\oplus$, and the class of the zero module 0 as the unit.

For any unital abelian monoid M there exists unique abelian group $M^{-1}M$ called *the group completion* of M together with a morphism $\gamma\colon M \to M^{-1}M$ of monoids, characterized uniquely by the following property. For any abelian group G and a map $\varphi\colon M \to G$ of monoids, there exists unique group homomorphism $\overline{\varphi}\colon M^{-1}M \to G$ making the following diagram commute:

$$\begin{tikzcd} M \arrow[d, "\gamma"'] \arrow[dr, "\varphi"] & \\ M^{-1}M \arrow[r, dashed, "\overline{\varphi}"'] & G \end{tikzcd}$$

In other words, the functor **AbMon** $\to$ **Groups** assigning $M \mapsto M^{-1}M$ is left adjoint to the forgetful functor **Groups** $\to$ **AbMon**.

Definition 2.1.1. *The group $K_0(A)$ is defined to be $K_0(A) = [\mathbb{P}(A)]^{-1}[\mathbb{P}(A)]$.*

Exercise 1. This definition agrees with the usual one.

2.2 The group $K_1(A)$

Consider the group $GL_n(A)$ of invertible $n \times n$ matrices with coefficients in A. There is a natural inclusion $GL_n(A) \hookrightarrow GL_{n+1}(A)$. Define $GL(A) = \operatorname{colim}_n GL_n(A)$.

Definition 2.2.1. *Define the group $K_1(A)$ to be $K_1(A) = GL(A)/[GL(A), GL(A)]$.*

Consider the matrices $e_{ij}(a)$, for $1 \leqslant i \neq j \leqslant n$ and $a \in A$, having 1 along the diagonal and the element a on the position (i, j). These matrices are called elementary. Let $E_n(A)$ be the subgroup of $GL_n(A)$ generated by the elementary matrices, $E_n(A) = \langle e_{ij}(a) \rangle$.

Exercise 2. For any $a, b \in A$ and $i \neq j$ the elementary matrices satisfy the following relations.

1. $e_{ij}(a)^{-1} = e_{ij}(-a)$
2. $e_{ij}(a) \cdot e_{ij}(b) = e_{ij}(a+b)$
3. $[e_{ij}(a), e_{kl}(b)] = \begin{cases} 1, & \text{if } j \neq k \text{ and } i \neq l \\ e_{il}(ab), & \text{if } j = k \text{ and } i \neq l \\ e_{kj}(-ba), & \text{if } j \neq k \text{ and } i = l \end{cases}$

Corollary 2.2.2. *If $n \geqslant 3$, then $E_n(A)$ is perfect.*

Proof. If $n \geqslant 3$ take j, k, l to be distinct so that $e_{ij}(a) = [e_{ik}(a), e_{kj}(1)]$, for all $1 \geqslant i \neq j \leqslant n$. □

Notice that the inclusion $GL_n(A) \hookrightarrow GL_{n+1}(A)$ restricts to the inclusion $E_n(A) \hookrightarrow E_{n+1}(A)$. Thus we can define $E(A) = \operatorname{colim}_n E_n(A)$.

Lemma 2.2.3 (Whitehead)**.** $E(A) = [GL(A), GL(A)] = [E(A), E(A)]$.

Sketch of proof. First check that for any $\alpha, \beta \in GL_n(A)$,

$$\begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \in E_{2n}(A) \text{ and } \begin{bmatrix} [\alpha, \beta] & 0 \\ 0 & I \end{bmatrix} \in E_{2n}(A)$$

Hence $E_n = [E_n, E_n] \subseteq [GL_n, GL_n] \subseteq E_{2n}$. Taking colimit over n we get the equality $E(A) = [E(A), E(A)] = [GL(A), GL(A)]$. □

2.3 The group $K_2(A)$

For the details of the material of this section see the book [Mil71] by John Milnor.

Definition 2.3.1. *Assume $n \geqslant 3$ and define the* Steinberg group $St_n(A)$ *as the group generated by symbols $x_{ij}(a)$ for $1 \leqslant i \neq j \leqslant n$ and $a \in A$ modulo the following two relations:*

- $x_{ij}(a) \cdot x_{ij}(b) = x_{ij}(a+b)$
- $[x_{ij}(a), x_{kl}(b)] = \begin{cases} 1, & \textit{if } j \neq k \textit{ and } i \neq l \\ x_{il}(ab), & \textit{if } j = k \textit{ and } i \neq l \\ x_{kj}(-ba), & \textit{if } j \neq k \textit{ and } i = l \end{cases}$

Note that these relations are a part of the relations from the Exercise 2. Thus the exists a natural projection $\varphi_n \colon St_n(A) \twoheadrightarrow E_n(A)$. We will think of φ_n as a map to $GL_n(A)$ by composing it with the inclusion $E_n(A) \hookrightarrow GL_n(A)$. The homomorphisms φ_n are compatible with the inclusions $GL_n(A) \hookrightarrow GL_{n+1}(A)$, we can define $St(A) = \operatorname{colim} St_n(A)$ equipped with a homomorphism $St(A) \to GL(A)$.

Note that by the definition, $K_1(A) = \operatorname{Coker} \varphi$.

Definition 2.3.2. *Define the second K-group to be $K_2(A) = \operatorname{Ker}\varphi$.*

Thus we have an exact sequence of groups

$$1 \to K_2(A) \to St(A) \to GL(A) \to K_1(A) \to K_1(A) \to 1$$

Theorem 2.3.3 (Steinberg)**.** *The group $K_2(A)$ is abelian. In fact, $K_2(A)$ is the center of $St(A)$*

$$K_2(A) = Z(St(A))$$

Proof. If $x \in Z(St(A))$, then $\varphi(x) \in Z(E(A))$. But $Z(E(A)) = \{1\}$ (check!), and so $\varphi(x) = 1$. This shows the inclusion $Z(St(A)) \subseteq K_2(A)$.

Need to add proof of the opposite direction!! □

Exercise 3. Show that $K_2(\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$.

2.4 Universal central extensions

Let G be a group, and A be an abelian group.

Definition 2.4.1. *A* central extension *of G by A is a short exact sequence*

$$0 \to A \xrightarrow{i} \widetilde{G} \xrightarrow{p} G \to 1$$

such that $i(A) \subseteq Z(\widetilde{G})$.

Two such extensions $(\widetilde{G}_1, i_1, p_1)$ and $(\widetilde{G}_2, i_2, p_2)$ are equivalent if there exists a homomorphism $f\colon \widetilde{G}_1 \to \widetilde{G}_2$ making the following diagram commute:

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r, "i_1"] \arrow[d, equal] & \widetilde{G}_1 \arrow[r, "p_1"] \arrow[d, "f"'] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & A \arrow[r, "i_2"] & \widetilde{G}_2 \arrow[r, "p_2"] & G \arrow[r] & 1
\end{tikzcd}$$

Lemma 2.4.2 (see [BMP14])**.** *There is a natural bijection*

$$\{\textit{equiv. classes of extensions of } G \textit{ by } A\} \longleftrightarrow \mathrm{H}^2(G, A)$$

Fix a group G, and define the category $\mathbf{Ext}_C(G)$ of central extensions of G with the set of objects given by

$$\operatorname{Ob}(\mathbf{Ext}_C(G)) = \{p\colon \widetilde{G} \twoheadrightarrow G \mid \operatorname{Ker}(p) \subseteq Z(\widetilde{G})\}$$

and morphisms between two central extensions $p_1\colon \widetilde{G}_1 \twoheadrightarrow G$ and $p_1\colon \widetilde{G}_2 \twoheadrightarrow G$ given by

$$\operatorname{Hom}_{\mathbf{Ext}_C(G)}(\widetilde{G}_1, \widetilde{G}_2) = \{f\colon \widetilde{G}_1 \to \widetilde{G}_2 \mid p_2 \circ f = p_1\}$$

Definition 2.4.3. *The* universal central extension *of G is the initial object in $\mathbf{Ext}_C(G)$.*

Theorem 2.4.4. *The universal central extension of G exists iff G is perfect.*

Proof. First let's assume that the universal extension $p\colon \widetilde{G} \twoheadrightarrow G$ exists. Since p is surjective, it suffices to show that $\widetilde{G}$ is perfect.

Assume $\widetilde{G}$ is not perfect. Then the quotient $B = \widetilde{G}/[\widetilde{G}, \widetilde{G}] \neq \{1\}$ is non-trivial. Consider the trivial extension $B \times G \twoheadrightarrow G$ of G by B, and let $\pi\colon \widetilde{G} \to B$ be the canonical projection. Then there are two non-equal morphisms

$$\begin{tikzcd} \widetilde{G} \arrow[rr, shift left, "{(0,p)}"] \arrow[rr, shift right, "{(\pi,p)}"'] \arrow[dr] & & B \times G \arrow[dl] \\ & G & \end{tikzcd}$$

which contradicts the the fact that $\widetilde{G}$ is the initial object in $\mathbf{Ext}_C(G)$.

Assume now that G is perfect. We will need the following construction due to Hopf. Let G be any group for a moment. Consider a presentation $G \simeq F/R$ of the group G. Then there are two natural central extensions

1. $1 \to \frac{R}{[R,F]} \to \frac{F}{[R,F]} \to G \to 1$
2. $1 \to \frac{R\cap[F,F]}{[R,F]} \to \frac{[F,F]}{[R,F]} \to [G,G] \to 1$

Theorem 2.4.5 (Hopf)**.** *s* $\mathrm{H}_2(G,\mathbb{Z}) \simeq \frac{R\cap[F,F]}{[R,F]}$

We claim that if G is perfect, the second sequence is the universal extension of $G = [G,G]$. To see this, we first prove the following

Lemma 2.4.6. *Let $\widetilde{G}, \widetilde{G}'$ be two central extensions of G, with $\widetilde{G}$ being perfect. Then there is* at most *one morphism $\widetilde{G} \to \widetilde{G}'$ in $\mathbf{Ext}_C(G)$.*

Proof of lemma. Let $f_1, f_2\colon \widetilde{G} \to \widetilde{G}'$ be two such morphisms. Then for any $g, h \in \widetilde{G}$ there exist central elements $c, c' \in Z(\widetilde{G}')$ such that

$$f_2(g) = f_1(g)c$$
$$f_1(h) = f_2(h)c'$$

But then $f_1(ghg^{-1}h^{-1}) = f_2(ghg^{-1}h^{-1})$, i.e. f_1 and f_2 agree on $[G,G]$. Since we assumed G is perfect, $f_1 = f_2$. □

Need to complete the proof!!! □

Proposition 2.4.7 (Recognition criterion)**.** *For a central extension $\widetilde{G} \to G$ the following are equivalent:*

1. $\widetilde{G}$ *is universal;*

2. $\widetilde{G}$ *is perfect and every central extension of* $\widetilde{G}$ *splits;*

3. $\mathrm{H}_1(\widetilde{G}, \mathbb{Z}) = \mathrm{H}_2(\widetilde{G}, \mathbb{Z}) = 0$.

Theorem 2.4.8 (Kervaire,Steinberg)**.** *For an associative unital ring A, the natural exact sequence*

$$1 \to K_2(A) \to St(A) \to E(A) \to 1 \tag{2.1}$$

is the universal extension of $E(A)$. Hence, by Hopf construction, $K_2(A) \simeq \mathrm{H}_2(E(A), \mathbb{Z})$.

The key lemma in the proof of the above theorem is the following.

Lemma 2.4.9. *For $n \geqslant 5$, every central extension of $St_n(A)$ splits. Therefore, $St_n(A)$ is the universal central extension of $E_n(A)$ for $\forall n \geqslant 5$.*

Remark 2.4.10. If $n = 2$, $E_2(A)$ is not perfect. For example, $E_2(\mathbb{Z}/2\mathbb{Z})$ is solvable.

Remark 2.4.11. Lemma 2.4.9 fails for $n \leqslant 4$.

Definition 2.4.12. *Define $K_3(A) = \mathrm{H}_3(St(A), \mathbb{Z})$.*

Remark 2.4.13. For $n \geqslant 4$ it is not known how to express $K_n(A)$ in "elementary terms."

3 Higher K-theory via "plus"-construction

3.1 Acyclic spaces and maps

To define Quillen's "plus"-construction we first need the notion of an acyclic space. In our exposition we will follow the paper [HH79]. In this section we will assume all our spaces to be based CW-complexes.

Definition 3.1.1. *A topological space X is called* acyclic *if it has homology of the point. In other words, X is acyclic if its reduced homology groups vanish*

$$\widetilde{\mathrm{H}}_i(X, \mathbb{Z}) = 0, \ \forall i \in \mathbb{Z}$$

Lemma 3.1.2. *Let X be acyclic. Then the following holds.*

(1) *The space X is connected.*

(2) *The fundamental group $G := \pi_1(X, *)$ is a perfect group, i.e. $G = [G, G]$.*

(3) $\mathrm{H}_2(G, \mathbb{Z}) = 0$.

Proof. The first statement is obvious.

The second statement follows immediately from the standard fact that $\mathrm{H}_1(X,\mathbb{Z}) \simeq \pi_1(X)/[\pi_1(X),\pi_1(X)]$ and vanishing of $\mathrm{H}_1(X,\mathbb{Z})$ because of the acyclicity of X.

Recall that for any CW complex X we have its *Postnikov decomposition*

$$\begin{tikzcd} \mathrm{cosk}_1(X) & \mathrm{cosk}_1(X) \arrow[l, two heads] & \ldots \arrow[l, two heads] & \mathrm{cosk}_n(X) \arrow[l, two heads] & \ldots \arrow[l, two heads] \\ & & X \arrow[ull, "f_1"] \arrow[ul, "f_2"'] \arrow[ur, "f_n"'] & & \end{tikzcd}$$

where $\mathrm{cosk}_1(X) = K(\pi_1(X),1) = BG$ is the corresponding Eilenberg-Maclane space, constructed from X by adjoining cells of dimensions $3,4,5,\ldots$ to kill all the higher homotopy groups. The the map $f_1\colon X \to BG$ is the natural inclusion.

Let's now compute $\mathrm{H}_2(G,\mathbb{Z})$. Applying the above construction, we get a map $f = f_1\colon X \to BG$. Let Ff be the homotopy fiber of f. So we get a homotopy fibration $Ff \to X \xrightarrow{f} BG$ (see Section 3 for definitions and details). It induces a long exact sequence of homotopy groups

$$\begin{tikzcd} \ldots \arrow[r] & \pi_2(BG) \arrow[r] \arrow[d, equal] & \pi_1(Ff) \arrow[r] \arrow[d, equal] & \pi_1(X) \arrow[r, "\sim"] \arrow[d, equal] & \pi_1(BG) \arrow[r] \arrow[d, equal] & \pi_0(Ff) \arrow[r] \arrow[d, equal] & \pi_0(X) \arrow[d, equal] \\ & 0 & \{1\} & G & G & \{1\} & \{1\} \end{tikzcd}$$

In this sequence, the map $\pi_1(X) \to \pi_1(BG) = G$ is a natural isomoprhism. Therefore, both groups $\pi_1(Ff) \simeq \{1\}$ and $\pi_1(Ff) \simeq \{1\}$ are trivial, and so Ff is connected simply connected. Therefore, the space Ff is homotopy equivalent to the universal covering $\widetilde{X}$ of X.

Consider the homological Leray–Serre spectral sequence associated with the fibration $\widetilde{X} \to X \to BG$. Its second page is $E^2_{pq} = \mathrm{H}_p(BG, \mathrm{H}_q(\widetilde{X},\mathbb{Z}))$, and the sequence converges to $\mathrm{H}_{p+q}(X,\mathbb{Z})$. Notice that basically by definition, there is an isomorphism $\mathrm{H}_p(BG,\mathrm{H}_q(\widetilde{X},\mathbb{Z})) \simeq \mathrm{H}_p(G,\mathrm{H}_q(\widetilde{X},\mathbb{Z}))$.

Lemma 3.1.3. *Given any first quadrant (homological) spectral sequence $E^2_{p,q} \Rightarrow \mathrm{H}_{p+q}$, there exists a canonical five term exact sequence*

$$\mathrm{H}_2 \to E^2_{2,0} \xrightarrow{d^2_{2,0}} E^2_{0,1} \to \mathrm{H}_1 \to E^2_{1,0} \to 0$$

Proof of Lemma. This is a standard fact. See, for example, Exercise 5.1.2 in [Wei94]. □

Applying Lemma 3.1.3 in our case give an exact sequence

$$\begin{tikzcd} \mathrm{H}_2(X,\mathbb{Z}) \arrow[r] \arrow[d, equal] & \mathrm{H}_2(G,\mathrm{H}_0(\widetilde{X},\mathbb{Z})) \arrow[r] \arrow[d, equal] & \mathrm{H}_0(G,\mathrm{H}_1(\widetilde{X},\mathbb{Z})) \arrow[r] \arrow[d, equal] & \mathrm{H}_1(X,\mathbb{Z}) \arrow[r] \arrow[d, equal] & \mathrm{H}_1(G,\mathrm{H}_0(\widetilde{X},\mathbb{Z})) \arrow[d, equal] \\ 0 & \mathrm{H}_2(G,\mathbb{Z}) & 0 & 0 & 0 \end{tikzcd}$$

In the diagram above, $\mathrm{H}_2(X,\mathbb{Z})=0$ and $\mathrm{H}_1(X,\mathbb{Z})=0$ because of the acyclicity of X. Since X is connected, and $\widetilde{X}$ is its double cover, it is also connected, and hence $\mathrm{H}_0(\widetilde{X},\mathbb{Z})=\mathbb{Z}$. Since $\widetilde{X}$ is simply connected by the definition of the universal cover, the group $\mathrm{H}_1(\widetilde{X},\mathbb{Z})$ vanishes. Thus the exact sequence implies $\mathrm{H}_2(G,\mathbb{Z})$, which finishes the proof. □

Definition 3.1.4. *Let X, Y be two based connected CW complexes. A map $f\colon X\to Y$ is* acyclic *if its homotopy fiber Ff is acyclic.*

Remark 3.1.5. Note that a space X is acyclic if and only if the constant map $X\to *$ is acyclic.

Corollary 3.1.6. *If $f\colon X\to Y$ is acyclic, then*

(1) *The induced homomorphism $f_*\colon \pi_1(X)\to\pi_1(Y)$ is surjective;*

(2) $\mathrm{Ker}(f_*)$ *is a perfect normal subgroup of $\pi_1(X)$.*

Proof. Consider the long exact sequence associated to the fibration sequence $Ff\to X\to Y$

$$\cdots\to\pi_1(Ff)\to\pi_1(X)\to\pi_1(Y)\to\pi_0(Ff)\to\dots$$

Since f is acyclic, its homotopy fiber Ff is connected, and so $\pi_0(Ff)=\{1\}$. Thus the map $f_*\colon\pi_1(X)\to\pi_1(Y)$ is surjective.

Lemma 3.1.2 implies that $\pi_1(Ff)$ is perfect. Then $\mathrm{Ker}\,F_*$ is also perfect since it is an image of a perfect group. □

Exercise 1. Show that is $f\colon X\to Y$ is acyclic and s.t. $f_*\colon\pi_1(X)\to\pi_1(Y)$ is an isomorphism, then f is a homotopy equivalence.

A useful characterization of acyclic maps (due to Quillen) is provided by the following

Proposition 3.1.7. *Let X, Y be based connected CW complexes. Then $f\colon X\to Y$ is acyclic iff the induced map*

$$f_*\colon\ \mathrm{H}_\bullet(X,f^*\mathcal{L})\overset{\sim}{\to}\mathrm{H}_\bullet(Y,\mathcal{L})\tag{2.2}$$

is an isomorphism for any local system (locally constant sheaf, or equivalently, a $\pi_1(Y)$-module) $\mathcal{L}$ on Y.

Proof. To simplify notation, we will write $\mathcal{L}$ instead of $f^*\mathcal{L}$ for the pull-back of $\mathcal{L}$ to X. Assume first that f is acyclic, with acyclic homotopy fiber Ff. Since $\pi_1(Ff)\to\pi_1(X)\to\pi_1(Y)$ is exact, the image of $\pi_1(Ff)$ in $\pi_1(Y)$ is trivial, and so $\pi_1(Ff)$ acts trivially on $\mathcal{L}$. Since f is acyclic, $\mathrm{H}_i(Ff,\mathbb{Z})=0$ for all $i\geqslant 1$ and $\mathrm{H}_0(Ff,\mathbb{Z})=\mathbb{Z}$. Hence, by the Universal Coefficients theorem, $\mathrm{H}_i(Ff,\mathcal{L})=0$ for all $i\geqslant 1$ and $\mathrm{H}_0(Ff,\mathcal{L})\simeq\mathcal{L}$. Consider the Leray-Serre spectral sequence for the fibration $Ff\to X\to Y$, i.e.

$$E^2_{pq}=\mathrm{H}_p(Y,\mathrm{H}_q(Ff,\mathcal{L}))\Rightarrow\mathrm{H}_{p+q}(X,\mathcal{L})$$

This sequence degenerates on the second page, because $E^2_{pq} = 0$ for $q \neq 0$ as we discussed above. Hence, using the isomorphism $\mathrm{H}_0(Ff, \mathcal{L}) = \mathcal{L}$, we get the isomorhisms

$$\mathrm{H}_p(Y, \mathcal{L}) \simeq \mathrm{H}_p(X, \mathcal{L}),\ \forall p \geqslant 0$$

Let's prove the opposite direction. Assume the isomorphism (2.2) holds for any local system $\mathcal{L}$ on Y. The proof of acyclicity of f will be in two steps.

First, assume that Y is simply connected, i.e. $\pi_1(Y) = \{1\}$. Then it suffices to only consider $\mathcal{L} = \mathbb{Z}$, the constant sheaf, as $\pi_1(Y)$ acts trivially on any $\mathcal{L}$. Assume $\mathrm{H}_\bullet(X, \mathbb{Z}) \simeq \mathrm{H}_\bullet(Y, \mathbb{Z})$.

Lemma 3.1.8 (Comparison theorem)**.** *Let $E^r_{pq} \Rightarrow \mathrm{H}_n$ and $E'^r_{pq} \Rightarrow \mathrm{H}'_n$ be two spectral sequences. Let $h\colon \mathrm{H}_\bullet$ to $\mathrm{H}'_\bullet$ be a homomorphism compatible with a map $f\colon E \to E'$ of the spectral sequences. If for some r the map $f^r\colon E^r_{pq} \to E'^r_{pq}$ is an isomorphism for all p, q, then $h\colon \mathrm{H}_\bullet \to \mathrm{H}'_\bullet$ is an isomorphism.*

Applying the Comparison theorem to Leray-Serre spectral sequences associated to the fibrations $Ff \to X \xrightarrow{f} Y$ and $* \to Y \xrightarrow{\mathrm{id}} Y$, we conclude $\widetilde{\mathrm{H}}_\bullet(Ff, \mathbb{Z}) = 0$, which means that both Ff and f are acyclic.

The case of non-simply connected Y can be reduced to the previous one by the following *trick*. Let $\widetilde{Y}$ be the universal covering of Y. Put $\widetilde{X} = X \times_Y \widetilde{Y}$ which is a covering of X. We denote the natural map $\widetilde{X} \twoheadrightarrow \widetilde{Y}$ by $\widetilde{f}$.

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d] & \widetilde{Y} \arrow[d] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Then there are natural isomorphisms

$$\mathrm{H}_\bullet(\widetilde{Y}, \mathbb{Z}) \simeq \mathrm{H}_\bullet(Y, \mathcal{L})$$
$$\mathrm{H}_\bullet(\widetilde{X}, \mathbb{Z}) \simeq \mathrm{H}_\bullet(X, \mathcal{L})$$

where $\mathcal{L} = \mathbb{Z}[\pi_1(Y)]$. The assumption $\mathrm{H}_\bullet(X, \mathcal{L}) \simeq \mathrm{H}_\bullet(Y, \mathcal{L})$ implies that the map $\widetilde{f}$ induces an isomorphism $\mathrm{H}_\bullet(\widetilde{X}, \mathbb{Z}) \simeq \mathrm{H}_\bullet(\widetilde{Y}, \mathbb{Z})$. Since $\widetilde{Y}$ is simply-connected, we can apply the previous case, and conclude that $F\widetilde{f}$ is an acyclic space. But $\widetilde{X} \to X$ and $\widetilde{Y} \to Y$ are both coverings, and so by the path lifting property of coverings

$$\begin{tikzcd}
F\widetilde{f} \arrow[r] \arrow[d, shift right, "\simeq"'] & \widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d] & \widetilde{Y} \arrow[d] \\
Ff \arrow[r] \arrow[u, shift right] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

Hence $F\widetilde{f} \simeq Ff$, and so Ff is acyclic, which means that the map f is acyclic. $\square$

3.2 Plus construction

Definition 3.2.1. *Let X be a based connected CW complex. Let $N \trianglelefteq \pi_1(X)$ be a perfect normal subgroup of $\pi_1(X)$. A* +-construction *on X relative to N is an acyclic map $f\colon X \to X_N^+$, depending on N, such that*

$$\operatorname{Ker}\left[f_*\colon \pi_1(X) \to \pi_1(X_N^+)\right] = N$$

In other words, the induced map $f_*\colon \pi_1(X) \to \pi_1(X_N^+)$ is isomorphic to the projection $\pi_1(X) \twoheadrightarrow \pi_1(X)/N$.

Theorem 3.2.2 (Quillen)**.** *Let X, N be as in the Definition 3.2.1. Then*

(1) *The +-construction $f\colon X \to X_N^+$ exists.*

(2) *The map $f\colon X \to X_N^+$ is universal (initial) in the homotopy category among all maps $g\colon X \to Y$, where Y is based connected, such that $g_*(N) = \{1\}$ in $\pi_1(Y)$. In particular, X_N^+ is unique up to homotopy.*

Recall that any group G has unique largest perfect subgroup, which is automatically normal. We denote this subgroup by $P(G)$ and call it the *perfect radical* of G.

Definition 3.2.3. *A +-construction on X with respect to the perfect radical $P\left[\pi_1(X)\right]$ is called* the plus construction*, and is denoted by $f\colon X \to X^+$.*

3.3 Higher K-groups via plus construction

Let A be a unital associative ring. Consider the group $GL(A)$ (see Section 2.2 for the notation). Consider the classifying space $BGL(A)$. By the very definition,

$$\begin{aligned} \mathrm{H}_\bullet(BGL(A), \mathbb{Z}) &= \mathrm{H}_\bullet(GL(A), \mathbb{Z}) \\ \pi_i(BGL(A), \mathbb{Z}) &= \begin{cases} GL(A), & \text{if } i = 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Definition 3.3.1. *For any $i \geqslant 1$ define $K_i(A)$ to be $K_i(A) = \pi_i(BGL(A)^+)$.*

Corollary 3.3.2. *The following is an immediate consequence of the definition.*

1. *For $i = 1$, we have*

$$\begin{aligned} K_1(A) &= \pi_1(BGL(A)^+) \\ &= \pi_1(BGL(A)^+)/P(\pi_1(BGL(A)^+)) \\ &= GL(A)/[GL(A), GL(A)] \end{aligned}$$

and so Quillen's definition of $K_1(A)$ coincides with the classical definition, see Section 2.2. The last equality follows from the Whitehead lemma 2.2.3.

2. $K_i\colon$ **Rings** $\to$ **Ab** *is a functor from the category of unital associative rings to the category of abelian groups.*

3. *The space $BGL(A)^+$ is uniquely up to homotopy characterized by the properties that $\pi_1(BGL(A)^+) = K_1(A)$ and $\mathrm{H}_\bullet(BGL(A)^+, \mathcal{L}) = \mathrm{H}_\bullet(GL(A), \mathcal{L})$ for any $K_1(A)$-module $\mathcal{L}$.*

4. *$BGL(A)^+$ is a homotopy commutative H-space. Indeed, there is a natural inclusion $GL_n(A) \times GL_m(A) \to GL_{n+m}(A)$, for any $\alpha \in GL_n(A)$ and $\beta \in GL_m(A)$ given by*

$$(\alpha, \beta) \mapsto \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

Composing this map with the inclusion $GL_{n+m}(A) \hookrightarrow GL(A)$ and applying the classifying space functor, we get

$$\begin{tikzcd} BGL_n(A) \times BGL_m(A) \arrow[r] \arrow[dr] & BGL(A) \arrow[r] & BGL(A)^+ \\ & BGL(A)^+ \wedge BGL(A)^+ \arrow[ur, dashed] & \end{tikzcd}$$

Theorem 3.3.3 (Quillen's Recognition Criterion)**.** *The map $f\colon BGL(A) \to BGL(A)^+$ is universal among all maps to homotopy commutative H-spaces: given any map $g\colon BGL(A) \to H$ of H-spaces, there exists a map $h\colon BGL(A)^+ \to H$ making the following diagram commute:*

$$\begin{tikzcd} BGL(A) \arrow[rr, "g"] \arrow[dr, "f"'] & & H \\ & BGL(A)^+ \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

The map h is not unique, but the induced map $h_*\colon \pi_i(BGL(A)^+) \to \pi_i(H)$ is independent of the choice of h.

To incorporate $K_0(A)$, we make the following

Definition 3.3.4. *The* K-theory space *of A is the space*

$$K(A) = K_0(A) \times BGL(A)^+ = \coprod_{K_0(A)} BGL(A)^+$$

Then automatically

$$\pi_0(K(A)) = K_0(A)$$
$$\pi_i(K(A)) = K_i(A),\ \forall i \geqslant 1$$

Remark 3.3.5. There are other models for the space $K(A)$. In particular, we will discuss the Q-construction later.

Proposition 3.3.6. *Quillen's definition of $K_2(A)$ agrees with the classical Milnor-Steinberg definition 2.3.2.*

Remark 3.3.7. By Kervaire–Steinberg theorem 2.4.8, the exact sequence

$$1 \to K_2(A) \to St(A) \to E(A) \to 1$$

is the universal central extension of $E(A)$. Classification theorem of central extensions (see Lemma 2.4.2) implies $K_2(A) = \mathrm{H}_2(E(A), \mathbb{Z})$.

Proof. We need to show that $\pi_2(BGL(A)^+) \simeq \mathrm{H}_2(E(A), \mathbb{Z})$.

Lemma 3.3.8. *Let $N \trianglelefteq G$ be a perfect normal subgroup of a group G. Consider the $+$-construction $f\colon BG \to BG_N^+$ corresponding to N, and let Ff be its homotopy fiber. Then*

(1) *$\pi_1(Ff)$ is isomorphic to the universal central extension of N.*

(2) *$\pi_2(BG_N^+) \simeq \mathrm{H}_2(N, \mathbb{Z})$.*

Proof of lemma. Consider the homotopy fibration $Ff \to BG \to BG_N^+$. It yields

$$\begin{tikzcd}
\ldots \arrow[r] & \pi_2(BG_N^+) \arrow[r] \arrow[d, equal] & \pi_1(Ff) \arrow[r] & \pi_1(BG) \arrow[r] \arrow[d, equal] & \pi_1(BG_N^+) \arrow[r] \arrow[d, equal] & \pi_0(F) \arrow[r] \arrow[d, equal] & \ldots \\
& \{1\} & & G & G/N & \{1\} &
\end{tikzcd}$$

Therefore, we have an exact sequence

$$1 \to \pi_2(BG_N^+) \to \pi_1(Ff) \to N \to 1$$

Note, by the general properties of homotopy fibrations **(ref.???)**, $\pi_2(BG_N^+)$ is in the center of $\pi_1(Ff)$. By Lemma 3.1.2, $\pi_1(Ff)$ is a perfect group, and so

$$\mathrm{H}_1(\pi_1(Ff), \mathbb{Z}) = 0$$
$$\mathrm{H}_2(\pi_1(Ff), \mathbb{Z}) = 0$$

Recognition criterion 2.4.7 of universal central extensions implies that $\pi_1(Ff)$ is isomorphic to the universal central extension of N. Proof of Theorem 2.4.4 implies the isomorphism $\pi_2(BG_N^+) \simeq \mathrm{H}_2(N, \mathbb{Z})$, which follows from Hopf theorem 2.4.5. This finishes the proof of the lemma. □

Proposition immediately follows from the lemma applied to $N = E(A) \subseteq GL(A) = G$, which gives isomorphism $K_2(A) = \pi_2(BGL(A)^+) = \mathrm{H}_2(E(A, \mathbb{Z})$. □

Lemma 3.3.9. *Let $N \trianglelefteq G$ be a perfect normal subgroup of a group G and $f\colon BG \to BG_N^+$ be the corresponding $+$-construction. Then the composition $BN \to BG \to BG_N^+$ factors as follows:*

$$\begin{tikzcd} BN \arrow[r] \arrow[dr] & BG \arrow[r] & BG_N^+ \\ & BN^+ \arrow[ur, dashed] & \end{tikzcd}$$

So BN^+ is homotopy equivalent to the universal covering space of BG_N^+. Hence we have isomorphisms

$$\pi_1(BN^+) \simeq \pi_i(BG_N^+),\ \forall i \geq 2$$

Corollary 3.3.10. *For any ring A, $K_i(A) \simeq \pi_i(BE(A)^+)$ for $i \geq 2$.*

Proof. Apply the previous lemma to $N = E(A)$, $G = GL(A)$. □

Let $1 \to A \to \widetilde{G} \to G \to 1$ be the universal central extension of G. Theorem 2.4.4 implies that $G, \widetilde{G}$ are perfect groups. Consider the composite map

$$\begin{tikzcd} BA \arrow[r] & B\widetilde{G} \arrow[r] \arrow[dr] & BG \arrow[r] & BG^+ \\ & & B\widetilde{G}^+ \arrow[ur, dashed] & \end{tikzcd}$$

This gives a sequence $BA \to B\widetilde{G}^+ \to BG^+$.

Lemma 3.3.11. *The sequence $BA \to B\widetilde{G}^+ \to BG^+$ is a homotopy fibration sequence. Hence*

$$\pi_i(B\widetilde{G}^+) = 0,\ i = 0, 1, 2$$
$$\pi_i(B\widetilde{G}^+) = \pi_i(BG^+),\ i \geqslant 3$$

Corollary 3.3.12. *For $i \geqslant 3$ there is an isomorphism $K_i(A) \simeq \pi_i(BSt(A)^+)$. In particular, $K_3(A) \simeq \mathrm{H}_3(St(A), \mathbb{Z})$ (Gersten's theorem).*

Proof. Apply Lemma 3.3.9 to the extension

$$1 \to K_2(A) \to St(A) \to E(A) \to 1$$

Lemma 3.3.9 implies that $\pi_i(BSt(A)^+) \simeq \pi_i(BE(A)^+)$ for $i \geq 3$. Then by Lemma 3.3.8, $\pi_i(BE(A)^+) \simeq K_i(A)$. □

3.4 Milnor K-theory of fields

Let's re-write the definition of Steinberg group $St(A)$, see 2.3.1, as follows. It is a group generated by symbols $x_{ij}(a)$ for $i, j \geqslant 1$, $i \neq j$, $a \in A$ satisfying the following relations:

$$\begin{aligned} x_{ij}(a)x_{ij}(b) &= x_{ij}(a+b) \\ x_{ij}(a)x_{kl}(b) &= x_{ij}(b)x_{ij}(a),\ i \neq l, j \neq k \\ x_{ij}(a)x_{jk}(b) &= x_{jk}(b)x_{ik}(ab)x_{ij}(a) \end{aligned}$$

We also recall from Section 2.3 that there is a natural projection $\varphi\colon St(A) \to E(A)$ sending $x_{ij}(a) \mapsto e_{ij}(a)$ with $\operatorname{Ker}\varphi \simeq K_2(A)$.

Assume that a, b are *commuting* elements in $E(A)$. Choose $\widetilde{a}, \widetilde{b} \in St(A)$ s.t. $\varphi(\widetilde{a}) = a$ and $\varphi(\widetilde{b}) = b$. Define operation

$$a \bullet b := [\widetilde{a}, \widetilde{b}] = \widetilde{a}\widetilde{b}\widetilde{a}^{-1}\widetilde{b}^{-1}$$

Lemma 3.4.1. *For any commuting $a, b \in E(A)$,*

(1) *$a \bullet b$ is well-defined, i.e. it doesn't depend on the choices of $\widetilde{a}, \widetilde{b}$;*

(2) *for any $s \in GL(A)$, $(sas^{-1}) \bullet (sbs^{-1}) = a \bullet b$.*

Proof. The proof is just a direct calculation. □

Definition 3.4.2. *Let $x, y \in A^\times$ be* commuting *units in A. Define their Steinberg symbol by*

$$\{x, y\} = \begin{bmatrix} x & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} y & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & y^{-1} \end{bmatrix} = \begin{bmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^{-1} \end{bmatrix} \bullet \begin{bmatrix} y & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in K_2(A)$$

For any unit $u \in A^\times$ define

$$\begin{aligned} g_{ij}(u) &:= x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u) \\ h_{ij}(u) &:= g_{ij}(u)g_{ij}(-1) \end{aligned}$$

Then by the definition, $\{u, v\} = [h_{ij}(u), h_{ik}(v)]$

Exercise 2. If $u + v = 1$, then $\{u, v\} = h_{12}(uv)h_{12}(u)^{-1}h_{12}(v)^{-1}$.

Lemma 3.4.3. *If both $u, 1-u$ are units in A, then*

$$\begin{aligned} \{u, 1-u\} &= 1 \\ \{u, -u\} &= 1 \end{aligned}$$

Proof. First, notice that $g_{12}(-1) = g_{12}(1) = x_{12}(1)x_{12}(-1)x_{21}(1)$, and moreover,

$$g_{12}(u)g_{21}(1) = x_{12}(-u^2)g_{12}(u)$$
$$x_{21}(1)g_{12}(v) = g_{12}(v)g_{12}(-v^2)$$

Since $u + v = 1$, $\frac{1}{u} + \frac{1}{v} = \frac{1}{uv}$. We have the following sequence of equalities in $St(A)$:

$$g_{12}(u)g_{12}(-1)g_{12}(v) = x_{12}(-u^2)g_{12}(u)x_{12}(-1)g_{12}(v)x_{12}(-v^2) \quad = x_{12}(uv)x_{21}(-u^{-1})x_{12}(0)x_{21}(-v^{-1})x_{12}(v) = x$$

Multiplying by $g_{12}(-1)$ implies that $h_{12}(u)h_{12}(v) = h_{12}(uv)$ if $u + v = 1$. Therefore, $\{u, v\} = 1$ if $u + v = 1$.

Now, write $-u = \frac{1-u}{1-u^{-1}}$ in $A^\times$. Then

$$\{u, -u\} = \{u, 1-u\} \cdot \{u, 1-u^{-1}\}^{-1} \qquad = \{u^{-1}, 1-u^{-1}\} = 1$$

which finishes the proof of the lemma. □

Let now $A = \mathbb{F}$ be a field. Consider $\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$, viewed as a $\mathbb{Z}$-module via $n.v = v^n$ for $n \in \mathbb{Z}$, $v \in \mathbb{F}^\times$. Therefore we can consider the tensor algebra

$$T(\mathbb{F}^\times) = \mathbb{Z} \oplus \mathbb{F}^\times \oplus \left(\mathbb{F}^\times\right)^{\otimes 2} \oplus \cdots \oplus \left(\mathbb{F}^\times\right)^{\otimes n} \oplus \ldots$$

Definition 3.4.4. *Define Milnor K-theory ring $K^M(\mathbb{F})$ by*

$$K^M(\mathbb{F}) = T(\mathbb{F}^\times) \Big/ \langle u \otimes (1-u) \mid u \in \mathbb{F}^\times \setminus \{0, 1\} \rangle$$

Define $K_i^M(\mathbb{F})$ to be the i-th graded component of $K_i^M(\mathbb{F})$. So

$$K^M(\mathbb{F}) = \bigoplus_{i \geqslant 0} K_i^M(\mathbb{F})$$

So by definition, we have isomorphisms between Milnor's and Quillen's K-groups

$$K_0^M(\mathbb{F}) = \mathbb{Z} \simeq K_0(\mathbb{F})$$
$$K_1^M(\mathbb{F}) = \mathbb{F}^\times \simeq K_1(\mathbb{F})$$

where the isomorphism $\mathbb{F}^\times \simeq K_1(\mathbb{F})$ is given by

$$\begin{tikzcd}
\mathbb{F}^\times = GL_1(\mathbb{F}) \arrow[r, hook] \arrow[dr, dashed] & GL(\mathbb{F}) \arrow[d, two heads] \\
& K_1(\mathbb{F})
\end{tikzcd}$$

The homomorphism $\mathbb{F}^\times \to K_1(\mathbb{F})$ has a natural inverse induced by the determinant map

$$\begin{tikzcd}
GL(\mathbb{F}) \arrow[r, "\det"] \arrow[d, two heads] & \mathbb{F}^\times \\
K_1(\mathbb{F}) \arrow[ur] &
\end{tikzcd}$$

Notation 3.4.5. The image of $u_1 \otimes \cdots \otimes u_n$ in $K^M(\mathbb{F})$ will be denoted by $(u_1, \ldots, u_n)$.

Theorem 3.4.6 (Matsumoto)**.** *The canonical map $\mathbb{F}^\times \otimes_{\mathbb{Z}} \mathbb{F}^\times \to K_2(\mathbb{F})$ mapping $u \otimes v \mapsto \{u, v\}$ induces an isomorphism $K_2^M(\mathbb{F}) \xrightarrow{\sim} K_2(\mathbb{F})$.*

Thus the group $K_2(\mathbb{F})$ is given by the following presentation. It has generators $\{u, v\}$ for each $u, v \in \mathbb{F}^\times$ subject to the relations

$$\begin{aligned} \{uu', v\} &= \{u, v\}\{u', v\} \\ \{u, vv'\} &= \{u, v\}\{u, v'\} \\ \{u, 1-u\} &= 1, \ \forall u \neq 0, 1 \end{aligned}$$

Remark 3.4.7. By Lemma 3.4.3, the relation $\{u, 1-u\} = 1$ implies that $\{u, -u\} = 1$. Therefore, we have the skew-symmetricity property

$$\begin{aligned} \{u, v\}\{v, u\} &= \{u, -uv\}\{v, -uv\} \\ &= \{uv, -uv\} \\ &= 1 \end{aligned}$$

Corollary 3.4.8. *If $\mathbb{F}$ is a finite field, then $K_2(\mathbb{F}) = 1$.*

Proof. This follows from the relations above and the fact that for a finite field $\mathbb{F}$, the group $\mathbb{F}^\times$ is cyclic. □

Remark 3.4.9. Quillen showed that for a finite field $\mathbb{F} = \mathbb{F}_q$ with $q = p^m$ for a prime p, $K_{2i}(\mathbb{F}_q) = 1$ and $K_{2i+1}(\mathbb{F}_q) = \mathbb{Z}/(q^i - 1)\mathbb{Z}$.

3.5 Loday product

Consider two associative unital rings A, B. For a ring R we denote by R^n the R-module $R^{\oplus n}$. Then for any $n, m \in \mathbb{N}$ there is a natural (but not unique) isomorphism of $A \otimes B$-modules

$$f_{nm} \colon A^n \otimes B^m \to (A \otimes B)^{nm}$$

which induces maps

$$GL_n(A) \times GL_m(B) \to GL_{nm}(A \otimes B).$$

These maps induce the maps of plus constructions

$$\widetilde{f}_{nm} \colon BGL_n(A)^+ \times BGL_m(B)^+ \to BGL_{nm}(A \otimes B)^+ \to BGL(A \otimes B)^+$$

A different choice of f_{nm} will give a different map f'_{nm} which would be conjugate to the original one, and so the induced maps $\widetilde{f}'_{nm}$ and $\widetilde{f}_{nm}$ on the plus constructions are homotopic to each other. Moreover, the maps $\widetilde{f}_{nm}$ are compatible with stabilizations in n, m (up to homotopy). Since $BGL(A \otimes B)^+$ is a homotopy commutative H-space, we can define the map $g\colon BGL(A)^+ \wedge BGL(B)^+ \to BGL(A \otimes B)^+$ which is induced by the maps

$$\widetilde{g}_{nm}(a,b) = \widetilde{f}_{nm}(a,b) - \widetilde{f}_{nm}(a,*) - \widetilde{f}_{nm}(*,b)$$

where "$-$" denotes the structure of the H-space $BGL(A \otimes B)^+$. The map g is well-defined up to weak homotopy equivalence. Taking homotopy groups, we obtain *Loday's cup product*

$$\cup\colon K_i(A) \otimes K_j(B) \to K_{i+j}(A \otimes B)$$

Theorem 3.5.1 (Loday)**.** *The $\cup$-product is functorial in A, B, and if A is* commutative*, then the multiplication map $m\colon A \otimes A \to A$ is an algebra homomorphism, and so we get maps*

$$K_i(A) \otimes K_j(A) \xrightarrow{\cup} K_{i+j}(A \otimes A) \xrightarrow{K_{i+j}(m)} K_{i+j}(A)$$

making $K(A) = \bigoplus_{i \geqslant 0} K_i(A)$ into a graded commutative ring.

Exercise 3. If $A = \mathbb{F}$ is a field, then Loday's product $\cup\colon K_1(\mathbb{F}) \otimes K_1(\mathbb{F}) \to K_2(\mathbb{F})$ coincides with the Steinberg symbol map φ, see 2.3.1.

In general, for a field $A = \mathbb{F}$ there is a natural graded ring homomorphism $\nu\colon K^M(\mathbb{F}) \to K(\mathbb{F})$ from Milnor K-theory to the Quillen K-theory, given by

$$\nu_i(u_1, \ldots, u_i) = u_1 \cup \cdots \cup u_i$$

In degrees $i = 0, 1$ and 2 the map ν_i is an isomorphism of groups: $K_i^M(\mathbb{F}) \simeq K_i(\mathbb{F})$. For $i = 0, 1$ this is obvious; for $i = 2$ this follows from Matsumoto theorem 3.4.6. For $i \geqslant 3$ the maps ν_i are no longer isomorphisms, and we define groups

$$K_i^{\text{indec}}(\mathbb{F}) := \operatorname{Coker}(\nu_i)$$

called *indecomposable part* of $K_i(\mathbb{F})$. The problem is how to compute it: this is open (or conjectural) in general (for infinite fields).

3.6 Bloch groups

If Γ is an abelian group, define $\Lambda^2\Gamma$ as

$$\Lambda^2\Gamma = \Gamma \otimes_{\mathbb{Z}} \Gamma \Big/ \langle a \otimes b + b \otimes a \rangle$$

We denote the image of $a \otimes b$ in $\Lambda^2\Gamma$ by $a \wedge b$. Note that $2(a \wedge a) = 0$ in $\Lambda^2\Gamma$, but $a \wedge a$ is not necessarily zero.

Definition 3.6.1. *Define the set of* five term relations *$FT(\mathbb{F})$ for a field $\mathbb{F}$ by*

$$FT(\mathbb{F}) = \left\{ (x, y, \frac{y}{x}, \frac{1-x^{-1}}{1-y^{-1}}, \frac{1-x}{1-y}),\ x \neq y,\ x, y \in \mathbb{F} \setminus \{0,1\} \right\} \subset (\mathbb{F} \setminus \{0,1\})^5$$

We will suppress $\mathbb{F}$ and write simply FT is the field $\mathbb{F}$ is understood. From now on, fix a field $\mathbb{F}$. Let $\mathbb{Z}[FT]$ be the free abelian group generated by elements of $FT = FT(\mathbb{F})$. Then we have

$$\mathbb{Z}[FT] \xrightarrow{\rho} \mathbb{Z}[\mathbb{F} \setminus \{0,1\}] \xrightarrow{\nu} \Lambda^2(\mathbb{F}^\times) \tag{2.3}$$

where $\nu(z) = z \wedge (1-z)$ and $\rho(z_0, z_1, z_2, z_3, z_4) = [z_0] - [z_1] + [z_2] - [z_3] + [z_4]$.

Exercise 4. Prove that $\nu \circ \rho = 0$, i.e. that the sequence above is a chain complex. Then Matsumoto's theorem 3.4.6 implies that $\operatorname{Coker} \nu \simeq K_2(\mathbb{F})$.

Definition 3.6.2. *Define the* pre-Bloch group $\mathcal{P}(\mathbb{F})$ *to be*

$$\mathcal{P}(\mathbb{F}) = \operatorname{Coker} \rho = \mathbb{Z}[\mathbb{F} \setminus \{0,1\}] \Big/ \langle FT \rangle$$

The Bloch group $\mathcal{B}(\mathbb{F})$ *is defined as*

$$\mathcal{B}(\mathbb{F}) = \operatorname{Ker} \nu / \operatorname{Im}(\rho) = \operatorname{Ker} \left[\mathcal{P}(\mathbb{F}) \xrightarrow{\nu} \mathbb{F}^\times \wedge \mathbb{F}^\times \right]$$

Theorem 3.6.3 (Suslin)**.** *If $\mathbb{F}$ is any infinite field, there is a short exact sequence*

$$0 \to \widetilde{\mu}_{\mathbb{F}} \to K_3^{\text{indec}}(\mathbb{F}) \to \mathcal{B}(\mathbb{F}) \to 0$$

where $\widetilde{\mu}_{\mathbb{F}}$ denotes the unique non-trivial extension of the group $\mu_{\mathbb{F}}$ of roots of 1 *in $\mathbb{F}$ by $\mathbb{Z}/2\mathbb{Z}$.*

Corollary 3.6.4. *Since $\widetilde{\mu}_{\mathbb{F}}$ is torsion, there is an isomorphism of rationalized abelian groups*

$$\mathcal{B}(\mathbb{F})_{\mathbb{Q}} \simeq K_3^{\text{indec}}(\mathbb{F})_{\mathbb{Q}},$$

where for an abelian group A its rationalization $A_{\mathbb{Q}}$ is the tensor product $A_{\mathbb{Q}} = A \otimes_{\mathbb{Z}} \mathbb{Q}$.

3.7 Homology of Lie groups "made discrete"

$K_3(\mathbb{C})$ and homology

Recall that the groups $K_i(\mathbb{F})$ were defined by $K_i(\mathbb{F}) = \pi_i(BGL(\mathbb{F})^+)$ for $i \geqslant 1$. Using Hurewicz homomorphism $H\colon \pi_i(BGL(\mathbb{F})^+) \to \mathrm{H}_i(BGL(\mathbb{F})^+)$ (see, for example, [May99, Chapter 15.1]) we get

$$\begin{tikzcd}
K_i(\mathbb{F}) \arrow[r, "H"] \arrow[ddr, dashed, "H"'] & \mathrm{H}_i(BGL(\mathbb{F})^+, \mathbb{Z}) \arrow[d, equal] \\
& \mathrm{H}_i(BGL(\mathbb{F}), \mathbb{Z}) \arrow[d, equal] \\
& \mathrm{H}_i(GL(\mathbb{F})^\delta, \mathbb{Z})
\end{tikzcd}$$

where by $GL(\mathbb{F})^\delta$ we mean $GL(\mathbb{F})$ viewed as an abstract group.

Let now $\mathbb{F} = \mathbb{C}$, $i = 3$. The obvious inclusion $SL_2(\mathbb{C}) \hookrightarrow GL(\mathbb{C})$ induces a map on homology (with $\mathbb{Z}$-coefficients)

$$i_*\colon \mathrm{H}_3(SL_2(\mathbb{C})^\delta) \to \mathrm{H}_3(GL(\mathbb{C})^\delta)$$

Remark 3.7.1. (1) The famous Milnor Conjecture states that for a suitable coefficients there should be an isomorphism

$$\mathrm{H}_\bullet(GL(\mathbb{C})^\delta, A) \simeq \mathrm{H}_\bullet(GL(\mathbb{C}), A)$$

(2) There is a stabilization theorem due to Suslin.

Theorem 3.7.2 (Suslin)**.** *For any $i \geqslant n$ there is an isomorphism*

$$\mathrm{H}_i(GL_n(\mathbb{C})) \xrightarrow{\sim} \mathrm{H}_i(GL(\mathbb{C}))$$

Theorem 3.7.3 (Dupont–Sah)**.** *For $\mathbb{F} = \mathbb{C}$ the group $K_3^{\mathrm{indec}}(\mathbb{C})$ is a direct summand of $K_3(\mathbb{C})$, and the image of the restriction of the Hurewicz map $K_3(\mathbb{C}) \xrightarrow{H} \mathrm{H}_3(GL(\mathbb{C}))$ coincides with the image of the map i_*. Thus, the Hurewicz map H induces an isomorphism*

$$K_3^{\mathrm{indec}}(\mathbb{C}) \simeq \mathrm{H}_3(SL_2(\mathbb{C})^\delta)$$

Problem: we would like to identify $\mathrm{H}_3(SL_2(\mathbb{C}))$ in terms of $\mathcal{B}(\mathbb{C})$.

Homology of discrete groups

First we recall the standard method to compute homology of a discrete group. Let G be a discrete group. For any G-module A denote by A_G the module of coinvariants, defined as $A_G = A \otimes_{\mathbb{Z}[G]} \mathbb{Z}$. Then the homology $\mathrm{H}_\bullet(G, A)$ can be computed as the homology $\mathrm{H}[P_\bullet(A)_G]$ of the complex $P_\bullet(A)_G$, where $P_\bullet(A)$ is a projective resolution of the G-module A, and $P_\bullet(A)_G$ is the complex of coinvariants of $P_\bullet(A)$. For $A = \mathbb{Z}$, we simply write P for $P_\bullet(\mathbb{Z})$.

Let X be a space (or a set). In Example 1.6.3 we defined a simplicial set E_*X with $E_nX = X \times \cdots \times X = X^{n+1}$. The maps d_i and s_j send $(x_0, \ldots, x_n)$ to $(x_0, \ldots, \widehat{x_i}, \ldots, x_n)$ and $(x_0. \ldots, x_j, x_j, \ldots, x_n)$ respectively.

Let $C_\bullet(X)$ be the complex of free abelian groups

$$C_n(X) = \mathbb{Z}[E_n(X)], \quad d_n = \sum_{i=0}^{n} d_i$$

It is easy to see that this complex is acyclic.

For $X = G$, the natural diagonal action of G on $C_n(G)$ defined by

$$\begin{gathered} G \times C_n(G) \to C_n(G) \\ g.(g_0, \ldots, g_n) = (gg_0, \ldots, gg_n) \end{gathered}$$

makes $C_\bullet(G)$ into a right dg-module over G, i.e. it is a complex of G-modules where the differential commutes with the G-action. The augmentation map $\varepsilon\colon C_0(G) = \mathbb{Z}[G] \to \mathbb{Z}$ defined by $g_0 \mapsto 1$ makes $C_\bullet(G) \to \mathbb{Z}$ into a resolution of the trivial G-module $\mathbb{Z}$ by free $\mathbb{Z}[G]$-modules, and hence

$$\mathrm{H}_\bullet(G, \mathbb{Z}) \simeq \mathrm{H}_\bullet(C(G)_G)$$

More generally, for any G-space X the complex $C_\bullet(X)$ is a complex of right G-modules (in the same way as for $X = G$).

Consider for each $n \geqslant 0$ the *configuration space of ordered* $(n+1)$-tuples of distinct points in X:

$$Conf_{n+1}(X) = \left\{(x_0, \ldots, x_n) \in X^{n+1} \mid x_i \neq x_j, \, \forall i \neq j\right\}$$

Note that the action of G on X extends diagonally to all configuration spaces. Let

$$C_n^{\neq}(X) = \mathbb{Z}\left[Conf_{n+1}(X)\right] \subseteq C_n(X)$$

Then $C_\bullet^{\neq}(X)$ is a subcomplex of G-modules in $C_\bullet(X)$.

Definition 3.7.4. *We say that G acts on X k-*transitively *if the induced action of G on $Conf_k(X)$ is transitive.*

Remark 3.7.5. Notice that k-transitivity implies $(k-1)$-transitivity. Moreover, 1-transitive action is just the usual transitive action.

Lemma 3.7.6. *Assume G acts on X k-transitively. Then*

(1) $C_i^{\neq}(X)_G \simeq \mathbb{Z}$ *for* $i = 0, 1, \ldots, k-1$.

(2) $C_k^{\neq}(X)_G$ *consists entirely of boundaries, so* $C_k^{\neq}(X)_G \twoheadrightarrow \mathrm{H}_k(C^{\neq}(X)_G)$.

Proof. Because of k-transitivity, G acts transitively on all distinct $(i+1)$-tuples for any $i \leqslant k-1$, so $(x_0, \ldots, x_i)$ can be moved by G to a single fixed one. This proves the first statement. The second statement follows obviously from the first. $\square$

Homology of Lie groups "made discrete"

Let's move back to our situation, where we had $G = SL_2(\mathbb{C})^\delta$, and take $X = \mathbb{C}P^1$. Then G acts on X via Möbius transformations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} .z = \frac{az+b}{cz+d}$$

Then we get $C_\bullet^{\neq}(\mathbb{C}P^1)$, a complex of $SL_2(\mathbb{C})^\delta$-modules. We will suppress δ from now on, always thinking of $SL_2(dC)$ as a discrete group, unless otherwise stated. The following two facts are standard.

Lemma 3.7.7. (1) *The action of $SL_2(\mathbb{C})$ on $\mathbb{C}P^1$ is* 3*-transitive.*

(2) *Any four* distinct *points z_0, z_1, z_2, z_3 are determined, up to $SL_2(\mathbb{C})$-action by the cross-ration*

$$[z_0 : z_1 : z_2 : z_3] := \frac{(z_0 - z_1)(z_1 - z_2)}{(z_0 - z_2)(z_1 - z_3)} \in \mathbb{C} \setminus \{0, 1\}$$

Hence $C_i^{\neq}(\mathbb{C}P^1) = \mathbb{Z}$ for $i = 0, 1, 2$, and by Lemma 3.7.7, $C_3^{\neq}(\mathbb{C}P^1) \simeq \mathbb{Z}[\mathbb{C} \setminus \{0, 1\}]$. Now, it is straightforward to check that the FT relation from the Definition 3.6.2 of Bloch group is equivalent to

$$\sum_{i=0}^{4} (-1)^i [z_0 : \cdots : \widehat{z_i} \cdots : z_4] = 0$$

This means that the kernel of the cross-ratio map

$$\sigma \colon C_3^{\neq}(\mathbb{C}P^1) \to \mathbb{Z}[\mathbb{C} \setminus \{0, 1\}] \twoheadrightarrow \mathcal{P}(\mathbb{C})$$

is precisely the boundaries in $C_3^{\neq}(\mathbb{C}P^1)$. Therefore, by Lemma 3.7.6, σ induces an *isomorphism*

$$\sigma \colon \mathrm{H}_3 \left[C_\bullet^{\neq}(\mathbb{C}P^1) \right] \xrightarrow{\simeq} \mathcal{P}(\mathbb{C})$$

Lemma 3.7.8 (Dupont–Sah)**.** *In $\mathcal{P}(\mathbb{C})$ the following relation holds:*

$$[z] = \left[\frac{1}{1-z}\right] = \left[1 - \frac{1}{z}\right] = -\left[\frac{1}{z}\right] = -\left[\frac{z}{1-z}\right] = -\left[1-z\right]$$

This lemma allows us to extend the cross-ration map σ to the diagonals, i.e. to all tuples z_0, z_1, z_2, z_3 of complex numbers, not necessarily distinct, by simply setting $[z_0 : z_1 : z_2 : z_3] = 0$ if $z_i = z_j$ for some $i \neq j$. In this way, one can extend the cross-ratio isomorphism to a map

$$\begin{tikzcd}
\mathrm{H}_3 \left[C_\bullet(\mathbb{C}P^1)_G \right] \arrow[r, "\widetilde{\sigma}"] & \mathcal{P}(\mathbb{C}) \\
\mathrm{H}_3 \left[C_\bullet^{\neq}(\mathbb{C}P^1)_G \right] \arrow[u] \arrow[ur, "\simeq", "\sigma"'] &
\end{tikzcd}$$

Next, define $\gamma \colon C_3(G) \to C_3(\mathbb{C}P^1)$ by

$$(g_0, g_1, g_2, g_3) \mapsto (g_0 \cdot \infty, g_1 \cdot \infty, g_2 \cdot \infty, g_3 \cdot \infty)$$

Composing it with $\widetilde{\sigma}$ we get a map

$$\begin{tikzcd}
\mathrm{H}_3(G) \arrow[r, "\simeq"] \arrow[drr, "\lambda"] & \mathrm{H}(C(G)_G) \arrow[r, "\gamma"] & \mathrm{H}_3(C(\mathbb{C}P^1)_G) \arrow[d, "\widetilde{\sigma}"] \\
& & \mathcal{P}(\mathbb{C})
\end{tikzcd}$$

Theorem 3.7.9 (Bloch–Wigner, Dupont–Sah)**.** *The image of λ is precisely the Bloch group, and there is an exact sequence of groups*

$$0 \to \mathbb{Q}/\mathbb{Z} \xrightarrow{i} \mathrm{H}_3[SL_2(\mathbb{C})] \xrightarrow{\lambda} \mathcal{B}(\mathbb{C}) \to 0$$

Remark 3.7.10. If we identify (by localization) $\mathbb{Q}/\mathbb{Z} \simeq \operatorname{colim} \mathbb{Z}/n\mathbb{Z}$, the embedding i is induced by $\mathbb{Z}/n\mathbb{Z} \hookrightarrow SL_2(\mathbb{C})$ defined by

$$1 \mapsto \begin{bmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{bmatrix}$$

Then we have

$$\begin{aligned} \mathrm{H}_3(\operatorname{colim} \mathbb{Z}/n\mathbb{Z}) &\simeq \operatorname{colim} \mathrm{H}_3(\mathbb{Z}/n\mathbb{Z}) \\ &\simeq \operatorname{colim} \mathbb{Z}/n\mathbb{Z} \\ &\simeq \mathbb{Q}/\mathbb{Z} \end{aligned}$$

3.8 Relation to polylogarithms, and some conjectures

The classical k-logarithm is an analytic function on the unit disk $|z| < 1$ defined by

$$\mathcal{L}i_k(z) = \sum_{i=1}^{\infty} \frac{z^n}{n^k}$$

Note that $\mathcal{L}i_1(z) = -\log(1-z)$ and

$$z\frac{d}{dz}\mathcal{L}i_k(z) = \mathcal{L}i_{k-1}(z), \ \forall k \tag{2.4}$$

We fix a choice of logarithm branch for $\log(1-z)$ once and for all. Using Equation 2.4 we can write

$$\mathcal{L}i_k(z) = \int_0^z \mathcal{L}i_{k-1}(z)\frac{dz}{z}$$

and continue $\mathcal{L}i_k(z)$ analytically to a multivalued function on $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$.

Definition 3.8.1 (Zagier,Goncharov). *For $k \geqslant 1$ let $\widetilde{\mathcal{L}}_k(z)$ be*

$$\widetilde{\mathcal{L}}_k(z) := \sum_{n=0}^{k-1} \frac{2^n B_n}{n!} \log^n |z| \cdot \mathcal{L}i_{k-n}(z)$$

Here B_n denotes the n-th Bernoulli number. Then the modified k-logarithm $\mathcal{L}_k(z)$ *is defined as*

$$\mathcal{L}_k(z) = \begin{cases} \operatorname{Re}\left[\widetilde{\mathcal{L}}_k(z)\right], & \text{if } k \text{ is odd} \\ \operatorname{Im}\left[\widetilde{\mathcal{L}}_k(z)\right], & \text{if } k \text{ is even} \end{cases}$$

Theorem 3.8.2 (Zagier,Goncharov). *For $k \geqslant 2$, $\mathcal{L}_k(z)$ are single-valued continuous functions on $\mathbb{C}P^1$, and $\mathcal{L}_2(z)$ satisfies the functional equation*

$$\mathcal{L}_2(x) - \mathcal{L}_2(y) + \mathcal{L}\left(\frac{y}{x}\right) - \mathcal{L}_2\left(\frac{1-x^{-1}}{1-y^{-1}}\right) + \mathcal{L}_2\left(\frac{1-x}{1-y}\right) = 0$$

In general, for any infinite field $\mathbb{F}$ one defines abelian subgroups $\mathcal{R}_k(\mathbb{F}) \subseteq \mathbb{Z}\left[\mathbb{F}^\times \setminus \{1\}\right]$ s.t. for $\mathbb{F} = \mathbb{C}$, the map

$$\begin{aligned} \mathbb{Z}\left[\mathbb{C} \setminus \{0,1\}\right] &\to \mathbb{R} \\ [z] &\mapsto \mathcal{L}_k(z) \end{aligned}$$

is well-defined.

Let $\mathcal{G}_k(\mathbb{F}) := \mathbb{Z}\left[\mathbb{F}^\times \setminus \{1\}\right] \Big/ \mathcal{R}_k(\mathbb{F})$ and

$$\mathcal{G}_\bullet(\mathbb{F})_\mathbb{Q} = \bigoplus_{n=1}^{\infty} \mathcal{G}_n(\mathbb{F}) \otimes_\mathbb{Z} \mathbb{Q}$$

One can define a complex, called *Goncharov motivic complex* of the form

$$G_\bullet := \operatorname{Tot}\left[\mathcal{G}_\bullet(\mathbb{F})_\mathbb{Q} \to \Lambda^2_\mathbb{Q}\mathcal{G}_\bullet(\mathbb{F})_\mathbb{Q} \to \Lambda^3_\mathbb{Q}\mathcal{G}_\bullet(\mathbb{F})_\mathbb{Q} \to \dots\right]$$

Conjecture 3.8.3 (Goncharov)**.** $\mathrm{H}_i^{(n)}(G_\bullet) \simeq \mathrm{gr}_n^\gamma\left[K_{2n-i}(\mathbb{F})_\mathbb{Q}\right]$

Above, $\mathrm{H}_i^{(n)}$ denotes the n-th weighted component of H_i, where weights are coming from the grading on $\mathcal{G}_\bullet(\mathbb{F})_\mathbb{Q}$. γ is a certain filtration on $K_{2n-i}(\mathbb{F})_\mathbb{Q}$.

4 Higher K-theory via Q-construction

4.1 Exact categories

Definition 4.1.1. *An* exact category *is an additive category* $\mathbf{C}$ *equipped with a class* $\mathcal{E}$ *of diagrams of the form* $(i,p) := [c' \overset{i}{\hookrightarrow} c \overset{p}{\twoheadrightarrow} c'']$ *for* $c, c', c'' \in \mathrm{Ob}(\mathbf{C})$*. If* $(i,p) \in \mathcal{E}$*, we call* i *an* admissible monic*, and* p *an* admissible epic*. The class* $\mathcal{E}$ *should satisfy the following axioms.*

(E1) $\mathcal{E}$ *is closed under isomorphisms.*

(E2) The set of admissible monics is closed under compositions and push-outs (cobase change) along arbitrary morphisms: if $i\colon c \to c'$ *is an admissible monic and* $f\colon c \to d$ *is an arbitrary morphism,*

$$\begin{tikzcd} c \arrow[r, "i"] \arrow[d, "f"'] & c' \arrow[d, dashed] \\ d \arrow[r, dashed, "\widetilde{i}"] & d \times_c c' \end{tikzcd}$$

then $\widetilde{i}$ *is also an admissible monic.*

(E3) The set of admissible epics is closed under compositions and pull-backs (base change) along arbitrary morphisms: if $p\colon c \to c'$ *is an admissible epic and* $f\colon d \to c'$ *is an*

arbitrary morphism,

$$\begin{tikzcd} d \times^{c'} c \arrow[r, dashed, "\widetilde{p}"] \arrow[d, dashed] & d \arrow[d, "f"] \\ c \arrow[r, "p"] & c' \end{tikzcd}$$

then $\widetilde{p}$ is also an admissible epic.

(E4) Any diagram of the form $[c \hookrightarrow c \oplus d \twoheadrightarrow d]$ in $\mathbf{C}$ necessarily belongs to $\mathcal{E}$.

(E5) If $(i, p) \in \mathcal{E}$, then $i = \mathrm{Ker}(p)$ and $p = \mathrm{Coker}(i)$.

We will denote exact categories as pairs $(\mathbf{C}, \mathcal{E})$, often suppressing $\mathcal{E}$ from the notation when it won't cause any confusion.

Definition 4.1.2. *A functor $F\colon (\mathbf{C}, \mathcal{E}_{\mathbf{C}}) \to (\mathbf{D}, \mathcal{E}_{\mathbf{D}})$ between two exact categories is called* exact *if it sends $\mathcal{E}_{\mathbf{C}}$ to $\mathcal{E}_{\mathbf{D}}$.*

4.2 K-group of an exact category

Definition 4.2.1. *If $\mathbf{C}, \mathcal{E}$ is an exact category, we define its* Grothendieck group $K_0(\mathbb{C}, \mathcal{E})$ *by*

$$K_0(\mathbf{C}, \mathcal{E}) := \mathbb{Z}[\mathrm{Iso}(\mathbf{C})] \Big/ \big([c] = [c'] + [c''] \text{ for } [c' \to c \to c''] \text{ in } \mathcal{E}\big)$$

Example 4.2.2. Let $\mathbf{A}$ be an abelian category. Then $\mathbf{A}$ is an exact category with $\mathcal{E}$ being the class of all short exact sequences. This is called the *canonical exact structure* on $\mathbf{A}$.

Warning: there is another natural exact structure on $\mathbf{A}$, where the class $\mathcal{E}^{\mathrm{split}}$ is taken to be the class of *split exact sequences* $0 \to c \to c \oplus d \to d \to 0$. In general, $K_0(\mathbf{A}, \mathcal{E}) \not\simeq K_0(\mathbf{A}, \mathcal{E}^{\mathrm{split}})$, unless there is a version of Krull-Schmidt theorem in $\mathbf{A}$.

Definition 4.2.3. *An examct category $(\mathbf{C}, \mathcal{E})$ is called* split exact *is $\mathcal{E}$ consists of the diagram of the form $0 \to c \to c \oplus d \to d \to 0$.*

Example 4.2.4. Let A be a unital ring, and let $\mathbf{C} = \mathbb{P}(A)$ be the category of finitely generated projective (right) A-modules. Take $\mathcal{E}_{\mathbb{P}}$ to be the family of all short exact sequences $\{0 \to p' \to p \to p'' \to 0\}$. The category $\mathbf{C}$ is not abelian, but it is exact. Then $K_0(A) = K_0(\mathbb{P}(A), \mathcal{E}_{\mathbb{P}})$.

Example 4.2.5. If X is a quasi-projective scheme, take $\mathbf{C} = \mathbf{Vect}(X)$ to be the category of vector bundles on X. Then by the definition, $K^0(X) = K_0(\mathbf{Vect}(X))$. Here $K^0(X)$ is the Grothendieck ring of X as opposed to $K_0(X)$, which is the Grothendieck group of X defined using *coherent sheaves* instead of *vector bundles*. However, in nice cases (when any coherent sheaf has a finite locally free resolution) the two are isomorphic.

Example 4.2.6. If $\mathbf{C}$ is an exact category, then $\mathbf{C}^{\mathrm{op}}$ is also an exact category in a natural way, where admissible epics in $\mathbf{C}^{\mathrm{op}}$ correspond to admissible monics in $\mathbf{C}$, and vice versa. Then

$$K_0(\mathbf{C}) \simeq K_0(\mathbf{C}^{\mathrm{op}})$$

Example 4.2.7. If $\mathbf{C}_1$ and $\mathbf{C}_2$ are two exact categories, there is a natural structure of an exact category on their product $\mathbf{C}_1 \times \mathbf{C}_2$. We will denote the resulting exact category by $\mathbf{C}_1 \oplus \mathbf{C}_2$. Then there is a natural isomorhism of K-groups

$$K_0(\mathbf{C}_1 \oplus \mathbf{C}_2) \simeq K_0(\mathbf{C}_1) \oplus K_0(\mathbf{C}_2).$$

Example 4.2.8. Let $(\mathbf{C}, \otimes)$ be any skeletally small symmetric monoidal category. Recall from the discussion after Example 3.2.8 that to $\mathbf{C}$ we can associate a groupoid $\mathbf{Iso}(\mathbf{C})$ of isomorphisms in $\mathbf{C}$, and $\mathrm{Iso}(\mathbf{C}) := \pi_0\left[\mathbf{Iso}(\mathbf{C})\right]$ is naturally an abelian monoid. We then define a group $K_0^{\otimes}(\mathbf{C})$ to be the group completion (see Section 2.1) of the monoid $S := \mathrm{Iso}(\mathbf{C})$, i.e.

$$K_0^{\otimes}(\mathbf{C}) = S^{-1}S, \quad \text{where } S = \mathrm{Iso}(\mathbf{C})$$

Now, let $\mathbf{C}$ be an additive category. Then it is also naturally a symmetric monoidal category, with the monoidal structure given by the direct sum $\oplus \colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$. Therefore, we can consider the K-group $K_0^{\oplus}(\mathbf{C})$.

If in addition the category $\mathbf{C}$ is also *exact*, then we have the associated group $K_0(\mathbf{C}, \mathcal{E})$. In general, the two groups $K_0^{\oplus}(\mathbf{C})$ and $K_0(\mathbf{C}, \mathcal{E})$ are not isomorphic, with $K_0(\mathbf{C}, \mathcal{E})$ being naturally a quotient of $K_0^{\oplus}(\mathbf{C})$, depending on the choice of $\mathcal{E}$.

Lemma 4.2.9. *Let* $\mathbf{B}$ *be an exact subcategory of an exact category* $\mathbf{C}$*. Assume that*

1. $\mathbf{B}$ *is closed under extensions, i.e. if* $0 \to b' \to c \to b \to 0$ *is an exact sequence in* $\mathbf{C}$ *with* $b, b' \in \mathrm{Ob}(\mathbf{B})$*, then* c *is also an object in* $\mathbf{B}$*;*

2. $\mathbf{B}$ *is* cofinal *in* $\mathbf{C}$ *in the sense that for any* $c \in \mathrm{Ob}(\mathbf{C})$ *there exists* $c' \in \mathrm{Ob}(\mathbf{C})$ *such that* $c \oplus c' \in \mathrm{Ob}(\mathbf{B})$.

Then $K_0(\mathbf{B})$ *is naturally a subgroup of* $K_0(\mathbf{C})$*.*

Sketch of proof. First one checks that $K_0^{\oplus}(\mathbf{B}) \subseteq K_0^{\oplus}(\mathbf{C})$ is a subgroup (exercise). Then we need to see that this embedding descends to the quotients $K_0(\mathbf{B}) \to K_0(\mathbf{C})$.

If $[c_1 \hookrightarrow c \twoheadrightarrow c_2] \in \mathcal{E}_{\mathbf{C}}$, choose $c_1', c_2' \in \mathrm{Ob}(\mathbf{C})$ such that $b_1 := c_1 \oplus c_1' \in \mathbf{B}$ and $b_2 := c_2 \oplus c_2' \in \mathbf{B}$. Put $b = c_1' \oplus c \oplus c_2'$ in $\mathbf{C}$ so that

$$[b_1 \hookrightarrow b \hookrightarrow b_2] \in \mathcal{E}_{\mathbf{C}}$$

Therefore, since $\mathbf{B}$ is closed under extensions, $b \in \mathrm{Ob}(\mathbf{B})$, and so $[b_1 \hookrightarrow b \hookrightarrow b_2] \in \mathcal{E}_{\mathbf{B}}$.

In $K_0^{\oplus}(\mathbf{C})$ we have $[c] - [c_1] - [c_2] = [b] - [b_1] - [b_2]$, and so

$$\mathrm{Ker}\left[K_0^{\oplus}(\mathbf{C}) \twoheadrightarrow K_0(\mathbf{C})\right] = \mathrm{Ker}\left[K_0^{\oplus}(\mathbf{B}) \twoheadrightarrow K_0(\mathbf{B})\right]$$

Thus $K_0(\mathbf{B})$ is a subgroup of $K_0(\mathbf{C})$. $\square$

Remark 4.2.10. Our argument actually shows that

$$K_0(\mathbf{C}) \Big/ K_0(\mathbf{B}) \simeq K_0^{\oplus}(\mathbf{C}) \Big/ K_0^{\oplus}(\mathbf{B})$$

4.3 Q-construction

Definition 4.3.1. *Let* $(\mathbf{C}, \mathcal{E})$ *be a skeletally small exact category. Then* $Q\mathbf{C}$ *is a category with*

$$\mathrm{Ob}(Q\mathbf{C}) = \mathrm{Ob}(\mathbf{C}),$$

and $\mathrm{Hom}_{Q\mathbf{C}}$*(c,c') is the set of the diagrams, called "roofs," of the form*

$$\begin{tikzcd} & d \arrow[dl, two heads, "p"'] \arrow[dr, hook, "i"] & \\ c & & c' \end{tikzcd}$$

where p *is an admissible epic, and* i *is an admissible monic in* $\mathbf{C}$*, modulo the equivalence relation, where two roofs* $[c \overset{p}{\twoheadleftarrow} d \overset{i}{\hookrightarrow} c']$ *and* $[c \overset{p'}{\twoheadleftarrow} d' \overset{i'}{\hookrightarrow} c']$ *are* equivalent *if there exists an isomorphism* $\varphi \colon d \overset{\sim}{\to} d'$ *in* $\mathbf{C}$ *making the following diagram commute*

$$\begin{tikzcd} c \arrow[d, equal] & d \arrow[l, two heads, "p"'] \arrow[r, hook, "i"] \arrow[d, "\varphi", "\simeq"'] & c' \arrow[d, equal] \\ c & d' \arrow[l, two heads, "p'"'] \arrow[r, hook, "i'"] & c' \end{tikzcd}$$

The composition of two (equivalence classes of) roofs $[c \overset{p}{\twoheadleftarrow} d \overset{i}{\hookrightarrow} c']$ and $[c' \overset{p'}{\twoheadleftarrow} d' \overset{i'}{\hookrightarrow} c'']$ is defined to be the (equivalence class of) the roof

$$\begin{tikzcd} & & d \times_{c'} d' \arrow[dl, two heads, "\widetilde{p'}"'] \arrow[dr, hook, "\widetilde{i'}"] & & \\ & d \arrow[dl, two heads, "p"'] \arrow[dr, hook, "i"] & & d' \arrow[dl, two heads, "p'"'] \arrow[dr, hook, "i'"] & \\ c & & c' & & c'' \end{tikzcd}$$

In this diagram, $d \times_{c'} d'$ is the pull-back of d' along i.

Lemma 4.3.2. *This composition is well-defined.*

4.4 Some remarks on the Q-construction

There are two distinguished types of morphisms in $Q\mathbf{C}$ corresponding to admissible monics and epics in $\mathbf{C}$:

$$\begin{tikzcd} & d \arrow[dl, equal] \arrow[dr, hook, "i"] & \\ d & & c' \end{tikzcd} \qquad \begin{tikzcd} & d \arrow[dl, two heads, "p"'] \arrow[dr, equal] & \\ d & & c' \end{tikzcd}$$

We will denote them respectively by $[d \overset{i}{\hookrightarrow} c']$ and $[c \overset{p}{\twoheadleftarrow} d]$.

Notice that any roof $[c \overset{p}{\twoheadleftarrow} d \overset{i}{\hookrightarrow} c'] \in \mathrm{Mor}(Q\mathbf{C})$ can be decomposed (uniquely up to an isomorphism, in fact) as

$$[c \overset{p}{\twoheadleftarrow} d \overset{i}{\hookrightarrow} c'] = [c \overset{p}{\twoheadleftarrow} d] \circ [d \overset{i}{\hookrightarrow} c']$$

Definition 4.4.1. *Let $\mathbf{C}$ be any category, and let $c \in \mathrm{Ob}(\mathbf{C})$. A* subobject *of c is an equivalence class of monics $d \hookrightarrow c$, two monics being equivalent if they factor through each other.*

Let now $\mathbf{C}$ be an *exact* category. A subobject $d \hookrightarrow c$ of $c \in \mathrm{Ob}(\mathbf{C})$ is called *admissible* if it can be represented by an admissible monic.

By definition of morphisms in $Q\mathbf{C}$, every morphism $[c \twoheadleftarrow d\ intoc']$ defines a unique admissible monic $d \hookrightarrow c'$. If we fix a representative for each subobject in $\mathbf{C}$, thne a morphicm $c \to c'$ in $Q\mathbf{C}$ is given by a pair $d \hookrightarrow c'$, an admissible subobject in c', and an admissible epic $d \twoheadrightarrow c$. Hence,

$$\mathrm{Hom}_{Q\mathbf{C}}(c, c') = \big\{\text{admissible subobjects of } c' \text{ in } \mathbf{C}\big\}$$

Our next observation is the isomorphisms in $Q\mathbf{C}$ are in one-to-one correspondence with isomorphisms in $\mathbf{C}$. Indeed, every isomorphism $i\colon c \xrightarrow{\sim} c'$ in $\mathbf{C}$ gives an isomorphism in $Q\mathbf{C}$ which is represented by either $[c \overset{i}{\hookrightarrow} c']$ or $[c \overset{i^{-1}}{\twoheadleftarrow} c']$. Every isomorphism in $Q\mathbf{C}$ arises this way.

Exercise 1. There is another way to represent morphisms in $Q\mathbf{C}$. Given a pair of admissible subobjects in $\mathbf{C}$, $[c_1 \hookrightarrow c_2 \hookrightarrow c]$, the quotient $c_2/c_1 \in \mathrm{Ob}(\mathbf{C})$ is called an admissible subquotient of $\mathbf{C}$. Then any morphism in $Q\mathbf{C}$ can be represented by an isomorphism $c_2/c_1 \xrightarrow{\sim} c'$ in $\mathbf{C}$.

4.5 The K_0-group via Q-construction

Fix an exact skeletally small category $\mathbf{C}$, and consider $Q\mathbf{C}$, the Q-construction of $\mathbf{C}$, and $BQ\mathbf{C}$, the corresponding classifying space.

Remark 4.5.1. First observe the following.

1. If $\mathbf{C}$ is skeletally small, then $Q\mathbf{C}$ is also skeletally small, and so $BQ\mathbf{C}$ is well-defined.

2. The space $BQ\mathbf{C}$ has a canonical basepoint, given by the zero object $0 \in \mathrm{Ob}(\mathbf{C})$.

3. $BQ\mathbf{C}$ is a connected space, where $[0 \hookrightarrow c]$ gives a path connecting any $c \in \mathrm{Ob}(\mathbf{C})$ with $0 \in \mathrm{Ob}(\mathbf{C})$. Therefore, $\pi_0(Q\mathbf{C}) = \{*\}$ is trivial.

Theorem 4.5.2. *There is an isomorhism $K_0(\mathbf{C}) \simeq \pi_1(BQ\mathbf{C})$. Under this isomorphism, an element of $\pi_1(BQ\mathbf{C})$ corresponding to a class $[c] \in K_0(\mathbf{C})$ is represented by the based loop in $BQ\mathbf{C}$*

$$[o \hookrightarrow c \twoheadrightarrow 0] = [0 \hookrightarrow c] \circ [c \twoheadrightarrow 0]$$

Recall (see Proposition 3.5.2) that for a small (connected) category, if $T \subset \mathrm{Mor}(\mathbf{C})$ is a maximal tree then $\pi_1(B\mathbf{C}, c_0)$ has presentation

$$\pi_1(B\mathbf{C}, c_0) \simeq \langle [\![f]\!] \mid f \in \mathrm{Mor}(\mathbf{C})\rangle \Big/ R,$$

where R is the normal subgroup generated by the relations

$$\begin{aligned}
&[\![t]\!] = 1,\ t \in T \\
&[\![\mathrm{id}_c]\!] = 1,\ \forall c \in \mathrm{Ob}(\mathbf{C}) \\
&[\![f \circ g]\!] = [\![f]\!] \cdot [\![g]\!],\ \forall f, g \in \mathrm{Mor}(\mathbf{C}) \text{ composable}
\end{aligned}$$

A class of $f \colon c_1 \to c_2$ is represented by a unique path

$$c_0 \rightsquigarrow c_1 \xrightarrow{f} c_2 \rightsquigarrow c_0$$

where an arrow $a \rightsquigarrow b$ denotes the unique path in T from the vertex a to the vertex b.

Proof of the theorem. Let T be the family of all morphisms $[0 \hookrightarrow a] \in \mathrm{Mor}(Q\mathbf{C})$. Note that every vertex not equal to 0 occurs exactly once in T. Hence T is a maximal tree, and so $\pi_1(BQ\mathbf{C})$ admits the following presentation:

$$\pi_1(BQ\mathbf{C}) \simeq \langle [\![\mathrm{Mor}(Q\mathbf{C})]\!]\rangle \Big/ S \tag{2.5}$$

where the normal subgroup S of relations is generated by the relations

$$\begin{aligned}
&[\![0 \hookrightarrow a]\!] = 1,\ \forall a \in \mathbf{C} \\
&[\![f]\!] \cdot [\![g]\!] = [\![f \circ g]\!],\ \forall f, g \text{ composable}
\end{aligned}$$

If $[a \to b] \in \mathrm{Mor}(Q\mathbf{C})$, under the presentation (2.5) corresponds to the based loop

$$0 \hookrightarrow a \to b \hookleftarrow 0$$

Now, $[0 \hookrightarrow b' \hookrightarrow b] \in T$ implies that $[\![b' \hookrightarrow b]\!] = 1$ in $\pi_1(BQ\mathbf{C})$ because $[0 \hookrightarrow b' \hookrightarrow b] = [0 \hookrightarrow b'] \circ [b' \hookrightarrow b]$. Hence,

$$[\![a \twoheadleftarrow b' \hookrightarrow b]\!] = [\![a \twoheadleftarrow b']\!] \text{ in } \pi_1(BQ\mathbf{C}) \tag{2.6}$$

Similarly, $[0 \twoheadleftarrow a \twoheadleftarrow b] \in T$ yields the relation

$$[\![0 \twoheadleftarrow a]\!] \cdot [\![a \twoheadleftarrow b]\!] = [\![0 \twoheadleftarrow b]\!]$$

so that

$$[\![a \twoheadleftarrow b]\!] = [\![0 \twoheadleftarrow a]\!]^{-1} \cdot [\![0 \twoheadleftarrow b]\!] \text{ in } \pi_1(BQ\mathbf{C})$$

By the discussion in Section 4.4, we can conclude that $\pi_1(BQ\mathbf{C})$ is generated by the classes of morphisms $[0 \twoheadleftarrow a]$ in $Q\mathbf{C}$.

Now, let $[a \hookrightarrow b \twoheadrightarrow c] \in \mathcal{E}_{\mathbf{C}}$ be an exact sequence in $\mathbf{C}$. Then, in $Q\mathbf{C}$, we have

$$[0 \hookrightarrow c] \circ [c \twoheadleftarrow b] = [0 \twoheadleftarrow a \hookrightarrow b], \tag{2.7}$$

where a is defined by the pull-back diagram defining the composition in $Q\mathbf{C}$

$$\begin{tikzcd}
& & a \arrow[dl, two heads] \arrow[dr, hook] & & \\
& 0 \arrow[dl, equal] \arrow[dr, hook] & & b \arrow[dl, two heads] \arrow[dr, equal] & \\
0 & & c & & b
\end{tikzcd}$$

Hence, in $\pi_1(BQ\mathbf{C})$ we have

$$\begin{aligned}
[\![0 \twoheadleftarrow b]\!] &= [\![0 \twoheadleftarrow c]\!] \cdot [\![c \twoheadleftarrow b]\!] \\
&= [\![0 \twoheadleftarrow c]\!] \cdot [\![0 \twoheadleftarrow a \hookrightarrow b]\!] \\
&= [\![0 \twoheadleftarrow c]\!] \cdot [\![0 \twoheadleftarrow a]\!]
\end{aligned} \tag{2.8}$$

which is the additivity relation. Since every relation $[\![f]\!] \cdot [\![g]\!] = [\![f \circ g]\!]$ can be written in terms of the additivity relations (2.8), $\pi_1(BQ\mathbf{C})$ is generated by $[\![0 \twoheadleftarrow a]\!]$, with (2.8) being the *only* relation. It follows that

$$K_0(\mathbf{C}) \simeq \pi_1(QB\mathbf{C})$$

$\square$

It follows from the proof that the map $\mathrm{Mor}(Q\mathbf{C}) \to K_0(\mathbf{C})$ inducing the isomorphism $\pi_1(BQ\mathbf{C}) \xrightarrow{\sim} K_0(\mathbf{C})$ is given by

$$[a \overset{p}{\twoheadleftarrow} b' \overset{i}{\hookrightarrow} b] \mapsto [\mathrm{Ker}(p)] \in K_0(\mathbf{C})$$

4.6 Higher K-theory

Definition 4.6.1. *For a skeletally small exact category* $\mathbf{C}$ *define its K-space by*

$$K(\mathbf{C}) = \Omega BQ\mathbf{C},$$

where Ω *denotes the based loop space functor. Then the* K-groups $K_i(\mathbf{C})$ *are defined to be*

$$K_i(\mathbf{C}) = \pi_i(\Omega BQ\mathbf{C}) \simeq \pi_{i+1}(\Omega BQ\mathbf{C}), \forall i \geqslant 0.$$

Theorem 4.5.2 implies that $K_0(\mathbf{C})$ defined using Q-construction agrees with the elementary definition.

If $F \colon \mathbf{C} \to \mathbf{D}$ is an exact functor between two exact categories, then F gives a natural functor $QF \colon Q\mathbf{C} \to Q\mathbf{D}$ between the corresponding Q-constructions, which in turn induces a map $BQF \colon BQ\mathbf{C} \to BQ\mathbf{D}$, and hence homomorphisms

$$BQF_i \colon K_i(\mathbf{C}) \to K_i(\mathbf{D})$$

Thus, if **ExCat** denotes the category of all small exact categories, then we have natural functors

$$\begin{aligned} K\colon \mathbf{ExCat} &\to \mathbf{Spaces} \\ \mathbf{C} &\mapsto K(\mathbf{C}) \\ K_i\colon \mathbf{ExCat} &\to \mathbf{Ab} \\ \mathbf{C} &\mapsto K_i(\mathbf{C}) \end{aligned}$$

Example 4.6.2. Let A be an associative unital ring. Define $K_i(A) := K_i(\mathbb{P}(A))$ for all $i \geqslant 0$. For $i = 0$ this agrees with the classical definition.

Theorem 4.6.3. $K_i(A) \simeq \pi_i(K_0(A) \times BGL(A)^+)$, *for all* $i \geqslant 0$. *In other words, for a ring* A, *the two K-theories defined using the plus- and Q-constructions agree.*

Proof. This will follow from the general "$+ = Q$" theorem, see later. $\square$

4.7 Elementary properties

Morita invariance

If A, B are two Morita equivalent rings, then $K_i(A) \simeq K_i(B)$ for $i \geqslant 0$. Recall that two rings are called *Morita invariant* if there exists a projective A-module P such that $B \simeq \mathrm{End}_A(P)$. Hence P is a (B, A)-bimodule, and so $P^* := \mathrm{Hom}_A(P, A)$ is naturally an (A, B)-bimodule.

Then the two categories $\mathbf{Mod}(A)$ and $\mathbf{Mod}(B)$ are equivalent, with the equivalence given by

$$\begin{aligned} \mathbf{Mod}(A) &\to \mathbf{Mod}(B) \\ M &\mapsto M \otimes_A P^* \\ \mathbf{Mod}(A) &\leftarrow \mathbf{Mod}(B) \\ N \otimes_B P &\leftarrow\!\shortmid N \end{aligned}$$

Since the categories $\mathbf{Mod}(A)$ and $\mathbf{Mod}(B)$ are equivalent, the corresponding categories of projectives $\mathbb{P}(A)$ and $\mathbb{P}(B)$ are also equivalent, and hence the K-spaces are also homotopy equivalent $K(A) \simeq K(B)$.

Basic constructions

If $\mathbf{C}^{\mathrm{op}}$ is the opposite category of $\mathbf{C}$, then $Q\mathbf{C}^{\mathrm{op}} \simeq (Q\mathbf{C})^{\mathrm{op}}$. By the properties of the classifying space functor B,

$$B\left(Q\mathbf{C}^{\mathrm{op}}\right) \simeq B\left(Q\mathbf{C}\right)^{\mathrm{op}} \simeq BQ\mathbf{C}$$

Therefore, there is an isomorphism of K-groups $K_i(\mathbf{C}^{\mathrm{op}}) \simeq K_i(\mathbf{C})$, for all $i \geqslant 0$.

If $\mathbf{C}$ and $\mathbf{D}$ are two exact categories, then $\mathbf{C} \oplus \mathcal{D}$ is naturally an exact category. It is easy to see that $Q(\mathbf{C} \oplus \mathbf{D}) \simeq Q\mathbf{C} \times Q\mathbf{D}$, and therefore

$$BQ(\mathbf{C} \oplus \mathbf{D}) \simeq BQ\mathbf{C} \times BQ\mathbf{D}$$

Taking homotopy groups of both sides, we get an isomorphism (cf. Example 4.2.7)

$$K_i(\mathbf{C} \oplus \mathbf{D}) \simeq K_i(\mathbf{C}) \oplus K_i(\mathbf{D}),\ \forall i \geqslant 0$$

In particular, if we apply this result to the categories $\mathbb{P}(A)$ and $\mathbb{P}(B)$ for any two associative unital rings A, B, we get an isomorphism

$$K_i(A \times B) \simeq K_i(A) \oplus K_i(B),\ \forall i \geqslant 0$$

Another useful application is the K-theory of schemes. If X, Y are two quasi-projective schemes, then $\mathbf{Vect}(X \sqcup Y) \simeq \mathbf{Vect}(X) \oplus \mathbf{Vect}(Y)$, and so

$$K_i(X \sqcup Y) \simeq K_i(X) \oplus K_i(Y),\ \forall i \geqslant 0$$

4.8 Quillen–Gersten Theorem

Theorem 4.8.1 (Quillen–Gersten)**.** *Let $\mathbf{C}$ be an exact category, and $\mathbf{B} \subset \mathbf{C}$ be an exact subcategory which is closed under extensions and is cofinal in $\mathbf{C}$ (see Lemma 4.2.9). Then the space $BQ\mathbf{B}$ is homotopy equivalent to the covering space of $BQ\mathbf{C}$ corresponding to the subgroup $K_0(\mathbf{B}) \simeq \pi_1(BQ\mathbf{B}) \subseteq K_0(\mathbf{C}) \simeq \pi_1(BQ\mathbf{C})$. In particular, $K_i(\mathbf{B}) \simeq K_i(\mathbf{C})$ for all $i \geqslant 1$.*

Sketch of proof. We will consider only a special case which often arises in practice. Assume that $\mathbf{C}$ is an exact category equipped with a surjective group homomorphism $\varphi\colon K_0(\mathbf{C}) \twoheadrightarrow G$ for some (necessarily abelian) group G. Define $\mathbf{B}$ to be the full subcategory of $\mathbf{C}$ consisting of all objects $b \in \mathrm{Ob}(\mathbf{C})$ such that $\varphi([b]) = 0 \in G$. Assume in addition that $\mathbf{B}$ is cofinal in $\mathbf{C}$. Then we prove that $K_i(\mathbf{B}) \simeq K_i(\mathbf{C})$ for $i \geqslant 1$. This special case was first proved by Gersten.

Step 1. Consider $\underline{G}$, the group G viewed as a groupoid with one object $\{*\}$. Define a functor $F\colon Q\mathbf{C} \to \underline{G}$ given by

$$F\left([c \overset{p}{\twoheadleftarrow} d \overset{i}{\hookrightarrow} c']\right) = \varphi([d']) \in G = \mathrm{Mor}(\underline{G}),$$

where $d' := \mathrm{Ker}(p)$. Then the induced map $F_*\colon \pi_1(Q\mathbf{C}) \simeq K_0(\mathbf{C}) \to \pi_1(\underline{G}) = G$ is just the original map φ.

Step 2. For any $g \in G = \mathrm{Mor}(\underline{G})$, g is a morphism $g\colon * \to *$, so we can consider the induced functor from the corresponding comma category to itself

$$g\colon F/* \to F/*$$

It easy to see that this functor is a homotopy equivalence, with the homotopy inverse map induced by the functor associated to g^{-1}.

Quillen's theorem B then implies that $B(F/*)$ is the homotopy fiber of the map $BQ\mathbf{C} \to BG$.

Step 3. Let $F^{-1}(*)$ be the fibre category of F over $*$ (see Definition 3.8.1), i.e. $F^{-1}(*)$ is the category with

$$\mathrm{Ob}(F^{-1}(*)) = \mathrm{Ob}(Q\mathbf{C}) = \mathrm{Ob}(\mathbf{C})$$
$$\mathrm{Mor}(F^{-1}(*)) = \{f \in \mathrm{Mor}(Q\mathbf{C}) \mid F(f) = \mathrm{id}_{\underline{G}}\}$$

There is a natural functor $\Phi\colon Q\mathbf{B} \to F^{-1}(*)$. Take any $d \in \mathrm{Ob}(F^{-1}(*))$ and consider the corresponding comma category Φ/d. Then Φ/d is contractible, and so by Quillen's theorem A we have

$$B\Phi\colon BQ\mathbf{B} \overset{\sim}{\to} B\left[F^{-1}(*)\right]$$

is a homotopy equivalence.

Step 4. If $\mathbf{B}$ is cofinal, then the canonical functor $i\colon F^{-1}(*) \to F/*$ (see Section 3.8) has right adjoint. Therefore, Bi is a homotopy equivalence:

$$Bi\colon BF^{-1}(*) \overset{\sim}{\to} B[F/*]$$

Step 5. Combining Steps 2–4, we conclude that

$$BQ\mathbf{B} \xrightarrow[B\Phi]{\simeq} BF^{-1}(*) \xrightarrow[Bi]{\simeq} B(F/*) \hookrightarrow BQ\mathbf{C} \longrightarrow BG$$

is a homotopy fibration sequence. Since $\pi_i(BG) = \begin{cases} G, & \text{if } i = 1 \\ 0, & \text{otherwise} \end{cases}$, from the long exact sequence of homotopy groups we deduce

$$K_i(\mathbf{B}) \simeq K_i(\mathbf{C}),\ \forall i \geqslant 1$$

□

5 The "plus = Q" Theorem

The main goal of this section is the proof of the following famous theorem by Quillen.

Theorem 5.0.1 (Quillen)**.** *For any associative unital ring A, there is a homotopy equivalence of the K-spaces*

$$\Omega BQ\left[\mathbb{P}(A)\right] \simeq K_0(A) \times BGL(A)^+$$

Therefore, the K-groups defined using the plus- and Q-constructions are isomorphic,

$$K_i(A) \simeq K_i(\mathbb{P}(A)),\ \forall i \geq 0$$

The proof of this theorem is not direct. Rather, it proceeds in two steps, stated in the following two propositions. These two steps consist of comparison of both Q- and plus-constructions to the third construction, which we will explain in detail below. For now, let's fix the two statements.

Proposition 5.0.2. *Let* $\mathbf{C}$ *be a split exact category, and let* $\mathbf{S} = \mathbf{Iso}(\mathbf{C})$ *be the category (groupoid) of all isomorphisms in* $\mathbf{C}$*. Then there is a homotopy equivalence of spaces*

$$K(\mathbf{C}) = \Omega BQ\mathbf{C} \simeq B(\mathbf{S}^{-1}\mathbf{S}),$$

where $B(\mathbf{S}^{-1}\mathbf{S})$ *is a "group completion" of the H-space* $B\mathbf{S}$*.*

Proposition 5.0.3. *If* $\mathbf{C} = \mathbb{P}(A)$ *is the category of finitely generated projective modules over a unital associative ring* A*, then*

$$B(\mathbf{S}^{-1}\mathbf{S}) \simeq K_0(A) \times BGL(A)^+$$

5.1 The category $\mathbf{S}^{-1}\mathbf{S}$

Assume that $(\mathbf{S}, \oplus)$ is a symmetric monoidal category, such that

1. $\mathbf{S} = \mathbf{Iso}(\mathbf{S})$, i.e. S is a groupoid;
2. all *translations* in $\mathbf{S}$ are *faithful*. That is, for any $s \in \mathrm{Ob}(\mathbf{S})$, the homomorphism

$$\begin{aligned} s\oplus \colon \ \mathrm{Aut}_{\mathbf{S}}(t) &\to \mathrm{Aut}_{\mathbf{S}}(s \otimes t) \\ u &\mapsto \mathrm{id}_s \oplus u \end{aligned}$$

   is injective.

Definition 5.1.1. *Define a category* $\mathbf{S}^{-1}\mathbf{S}$ *to be the category with objects*

$$\mathrm{Ob}(\mathbf{S}^{-1}\mathbf{S}) := \mathrm{Ob}(\mathbf{S}) \times \mathrm{Ob}(\mathbf{S})$$

and the set of morphisms from (m_1, m_2) *to* (n_1, n_2) *given by the set of compositions in* $\mathbf{S}$

$$(s, f, g) \colon (m_1, m_2)c \xrightarrow{s\oplus} (s \oplus m_1, s \oplus m_2) \xrightarrow{(f,g)} (n_1, n_2) \tag{2.9}$$

with $s \in \mathrm{Ob}(\mathbf{S})$*,* $f \in \mathrm{Hom}_{\mathbf{S}}(s \oplus m_1, n_1)$ *and* $g \in \mathrm{Hom}_{\mathbf{S}}(s \oplus m_2, n_2)$ *modulo the following equivalence relation. Tho triples* (s, f, g) *and* (s', f', g') *are* equivalent *if there exists a morphism* $\alpha \colon s \xrightarrow{\sim} s'$ *in* $\mathbf{S}$ *such that* $\alpha \oplus \mathrm{id}_{m_i}$ *maps* f' *and* g' *to* f *and* g*, respectively.*

Remark 5.1.2. Since we assumed $\mathbf{S}$ to be groupoid, all maps f, g, α in the definition above will be *isomorphisms.*

Remark 5.1.3. The category $\mathbf{S}^{-1}\mathbf{S}$ satisfies the following properties.

1. Any (strict) monoidal functor $\mathbf{S} \to \mathbf{T}$ induces a functor

$$\mathbf{S}^{-1}\mathbf{S} \to \mathbf{T}^{-1}\mathbf{T}$$

2. There are two distinguished types of morphisms in $\mathbf{S}^{-1}\mathbf{S}$. Morphisms of the first type are
$$(f_1, f_2)\colon (m_1, m_2) \to (n_1, n_2)$$
arising from the inclusion $\mathbf{S} \times \mathbf{S} \hookrightarrow \mathbf{S}^{-1}\mathbf{S}$. Morphisms of the second type are formal maps
$$s\oplus\colon (m, n) \to (s \oplus m, s \oplus n)$$
If in $\mathbf{S}$ the transitions are faithful, i.e. the natural homomorphisms $\mathrm{Aut}_{\mathbf{S}}(t) \hookrightarrow \mathrm{Aut}_{\mathbf{S}}(s \oplus t)$ are injective, then any morphism in $\mathbf{S}^{-1}\mathbf{S}$ defines the triple (s, f, g) in 2.9 uniquely up to unique isomorphism.

3. The category $\mathbf{S}^{-1}\mathbf{S}$ is symmetric monoidal, with the monoidal product given by
$$(m_1, m_2) \oplus (n_1, n_2) := (m_1 \oplus n_1, m_2 \oplus n_2),$$
and the natural inclusion $\mathbf{S} \hookrightarrow \mathbf{S}^{-1}\mathbf{S}$ sending $m \mapsto (m, e)$ is a monoidal functor.

4. The induced map of the classifying spaces $B\mathbf{S} \to B(\mathbf{S}^{-1}\mathbf{S})$ is a morphism of H-spaces. The induced map $\pi_0(B\mathbf{S}) \to \pi_0\left[B(\mathbf{S}^{-1}\mathbf{S})\right]$ is a map of abelian monoids, with $\pi_0\left[B(\mathbf{S}^{-1}\mathbf{S})\right]$ being an abelian *group*. Indeed, $[(m,n)]^{-1} = [(n,m)]$ because $\exists \eta \in \mathrm{Mor}(\mathbf{S}^{-1}\mathbf{S})$,
$$\eta\colon e \to (m, n) \oplus (n, m) = (m\oplus, n \oplus m) \simeq (m \oplus n, m \oplus n)$$
given by $\eta = (m \oplus n)\oplus$. Note that η is *not* a natural transformation $0 \Rightarrow \mathrm{id} \oplus \tau$ for 0 the constant functor $\mathbf{S}^{-1}\mathbf{S} \to \mathbf{S}^{-1}\mathbf{S}$ sending all the objects to the unit object e, and $\tau\colon \mathbf{S}^{-1}\mathbf{S} \to \mathbf{S}^{-1}\mathbf{S}$ sending (m, n) to (n, m).

5.2 K-groups of a symmetric monoidal groupoid

Let $(\mathbf{S}, \oplus)$ be a symmetric monoidal category with $\mathbf{S} = \mathbf{Iso}(\mathbf{S})$.

Definition 5.2.1. *For every $i \geqslant 0$ define*
$$K_i^{\oplus}(\mathbf{S}) := \pi_i\left[B(\mathbf{S}^{-1}\mathbf{S})\right]$$

Lemma 5.2.2. *The group $K_0^{\oplus}(\mathbf{S}) := \pi_0(\mathbf{S}^{-1}\mathbf{S})$ is the group completion of the abelian monoid $\pi_0(\mathbf{S})$.*

Proof. Let $\widehat{\pi_0(\mathbf{S})}$ be the group completion of the abelian monoid $\pi_0(\mathbf{S})$. Define a map
$$\begin{aligned} \varphi\colon\ \mathrm{Ob}(\mathbf{S}^{-1}\mathbf{S}) = \mathrm{Ob}(\mathbf{S}) \times \mathrm{Ob}(\mathbf{S}) &\to \widehat{\pi_0(\mathbf{S})} \\ (m, n) &\mapsto [m] - [n] \end{aligned}$$

If $s \in \mathrm{Ob}(\mathbf{S})$ is an object of $\mathbf{S}$ and $f_1, f_2 \in \mathrm{Mor}(\mathbf{S})$ are two morphisms in $\mathbf{S}$, $f_i\colon m_i \to n_i$, giving a morphism $[s, f_1, f_2] \in \mathrm{Mor}(\mathbf{S}^{-1}\mathbf{S})$, then

$$\begin{aligned}\varphi(n_1, n_2) &= [n_1] - [n_2] \\ &= [s \oplus m_1] - [s \oplus m_2] \\ &= [m_1] - [m_2] \\ &= \varphi(m_1, m_2).\end{aligned}$$

Thus φ induces a map $\overline{\varphi}$,

$$\overline{\varphi}\colon \pi_0(\mathbf{S}^{-1}\mathbf{S}) \to \widehat{\pi_0(\mathbf{S})}$$

which is the inverse of the universal map. □

5.3 Some facts about H-spaces

Recall that by an *H-space* (X, μ, e) we mean a topological space X together with a *multiplication map* $\mu\colon X \times X \to X$ and a distinguished point $e \in X$ such that $\mu(x, e) = \mu(e, x) = x, \forall x \in X$. The point e is called the *unit*. We will often suppress e, μ from the notation, and will simply refer to the H-space X.

We say that X is *homotopy associative*, or simply *associative*, if the following diagram commutes *up to homotopy*:

$$\begin{tikzcd} X \times X \times X \arrow[r, "\mu\times\mathrm{id}"] \arrow[d, "\mu\times\mathrm{id}"'] & X \times X \arrow[d, "\mu"] \\ X \times X \arrow[r, "\mu"] & X \end{tikzcd}$$

We say that X is *homotopy commutative*, or simply *commutative*, if the two maps $\mu \circ \tau \sim \mu$ are homotopic, where $\tau\colon X \times X \to X \times X$ is the map $\tau(x, y) = (y, x)$.

Example 5.3.1. Topological groups are H-spaces. They are *strictly associative*.

Example 5.3.2. If $\mathbf{S}$ is a symmetric monoidal category, then the classifying space $B\mathbf{S}$ is a (homotopy) associative and (homotopy) commutative H-space.

Let X be an associative and commutative H-space. Then $\pi_0(X)$ is naturally an abelian monoid. Moreover, $\mathrm{H}_0(X, \mathbf{Z}) = \mathbf{Z}[\pi_0(X)]$ is the monoid ring, and

$$\mathrm{H}_\bullet(X, \mathbb{Z}) = \bigoplus_{i \geqslant 0} \mathrm{H}_i(X, \mathbf{Z})$$

is an associative commutative graded $\mathbf{Z}[\pi_0(X)]$-algebra.

Remark 5.3.3. Let k be a field. Then $\mathrm{H}_\bullet(X, k)$ is a graded bialgebra with the comultiplication Δ induced by the diagonal inclusion $X \overset{\mathrm{diag}}{\to} X \times X$, mulptiplication is induced by the H-space multiplication $\mu\colon X \times X \to X$. The counit of $\mathrm{H}_\bullet(X, k)$ is induced by the projection $X \to \{e\}$, and the unit induced by $\eta\colon \{e\} \hookrightarrow X$.

Definition 5.3.4. *An associative H-space X is called* group-like *if there exists a map* $\mathrm{in}\colon X \to X$ *such that both maps*

$$X \xrightarrow{\mathrm{in} \times \mathrm{id}} X \times X \xrightarrow{\mu} X$$

$$X \xrightarrow{\mathrm{id} \times \mathrm{in}} X \times X \xrightarrow{\mu} X$$

are homotopic to the identity map $\mathrm{id}_X\colon X \to X$*.*

Note that if and H-space X is group-like, then $\pi_0(X)$ is naturally a group. Conversely,

Theorem 5.3.5 ([Whi78][Ch.X,2.2)**.** *] If an H-space X is a CW complex, then $\pi_0(X)$ is a group if and only if X is group-like.*

Example 5.3.6. If $\mathbf{S} = \mathbf{Iso}(\mathbf{S})$ is a symmetric monoidal groupoid, then $B\mathbf{S}$ is group-like if and only if $\pi_0(\mathbf{S})$ is a group.

Definition 5.3.7. *Let X be a homotopy associative and commutative H-space. A* group completion *of X is a morphism $f\colon X \to \widehat{X}$ such that*

1. *the induced map $\pi_0(f)\colon \pi_0(X) \to \pi_0(\widehat{X})$ is a group completion of $\pi_0(X)$;*

2. *for any commutative ring k, the map of graded commutative $\mathbf{Z}[\pi_0(X)]$-algebras $f_\bullet\colon \mathrm{H}_\bullet(X,k) \to \mathrm{H}_\bullet(\widehat{X},k)$ induces an isomorphism*

$$\begin{tikzcd}
\mathrm{H}_\bullet(X,k) \arrow[rr, "f_\bullet"] \arrow[dr] & & \mathrm{H}_\bullet(\widehat{X},k) \\
& \mathrm{H}_\bullet(X,k)\left[\pi_0(X)^{-1}\right] \arrow[ur, dashed, "\simeq"'] &
\end{tikzcd}$$

Remark 5.3.8. If X is a CW complex (such as, for example, $X = B\mathbf{S}$) then we can (and will) assume that $\widehat{X}$ is also a CW complex. Then by Theorem 5.3.5 and by the first axiom of Definition 5.3.7, $\widehat{X}$ is necessarily group-like.

Lemma 5.3.9. *Any group-like H-space X is its own group completion, i.e.* $\mathrm{id}\colon X \to X$ *is a group completion of X. Any group completion of a group completion $f\colon X \to \widehat{X}$ is a homotopy equivalence.*

Proof. The first statement is obvious. The map f is homology equivalence by the second axiom in the Definition 5.3.7. Hence f_* is an isomorphism on π_0 and π_1. Therefore, the map of basepoint components is a "+"-constriction (cf.see Definition 3.2.1) with respect to the trivial subgroup $\{1\} \subset \pi_1(X)$. By the homotopy uniqueness of "+"-constriction (see Theorem 3.2.2), $X \simeq \widehat{X}$. □

5.4 Actions on categories

Definition 5.4.1. *We say that a monoidal category* $(\mathbf{S}, \oplus)$ acts *on a category* $\mathbf{X}$ *by a functor* $\oplus\colon \mathbf{S} \times \mathbf{X} \to \mathbf{X}$ *if for any* $s, t \in \mathrm{Ob}(\mathbf{S})$ *and* $x \in \mathrm{Ob}(\mathbf{X})$ *there exist natural isomorphisms*

$$\begin{aligned} s \oplus (t \oplus x) &\simeq (s \oplus t) \oplus x \\ e \oplus x &\simeq x \end{aligned}$$

satisfying certain (obvious) coherence conditions.

Example 5.4.2. Category $\mathbf{S}$ acts on itself via the monoidal structure functor $\oplus\colon \mathbf{S} \times \mathbf{S} \to \mathbf{S}$.

Example 5.4.3. If $\mathbf{X}$ is a discrete category, then $\mathbf{S}$ acts on $\mathbf{X}$ if and only if $\pi_0(\mathbf{S})$ acts on $\mathrm{Ob}(\mathbf{X})$.

The following is analogous to the construction of the translation category, see Example 3.3.4.

Definition 5.4.4. *Let* $\mathbf{S}$ *and* $\mathbf{X}$ *be two categories as above, and assume* $\mathbf{S}$ *acts on the category* $\mathbf{X}$ *by a functor* $\oplus$*. Define a category* $\mathbf{S} \ltimes \mathbf{X}$ *with objects* $\mathrm{Ob}(\mathbf{S} \ltimes \mathbf{X}) = \mathrm{Ob}(\mathbf{X})$ *and morphisms*

$$\mathrm{Hom}_{\mathbf{S}\ltimes\mathbf{X}}(x, y) = \{(s, \varphi) \mid s \in \mathrm{Ob}(\mathbf{S}), \varphi \in \mathrm{Mor}_{\mathbf{X}}(s \oplus x, y)\}\big/ \sim$$

where $(s, \varphi) \sim (s', \varphi')$ *if there exists* $\alpha\colon s \xrightarrow{\sim} s'$ *an isomorphism in* $\mathbf{S}$ *making the following diagram commute:*

$$\begin{tikzcd} s \oplus x \arrow[rr, "\alpha\oplus\mathrm{id}_x"] \arrow[dr, "\varphi"'] & & s' \oplus x \arrow[dl, "\varphi'"] \\ & y & \end{tikzcd}$$

Definition 5.4.5. *Define* $\mathbf{S}^{-1}\mathbf{X} := \mathbf{S} \ltimes (\mathbf{S} \times \mathbf{X})$*, where* $\mathbf{S}$ *acts on* $\mathbf{S} \times \mathbf{X}$ *diagonally.*

Example 5.4.6. If $\mathbf{X} = \mathbf{S}$, then $\mathbf{S}^{-1}\mathbf{X}$ coincides with the category defined in the Section 5.1.

If $\mathbf{S}$ is a symmetric monoidal category, then we define an action of $\mathbf{S}$ on $\mathbf{S}^{-1}\mathbf{X}$ by the formula

$$\begin{aligned} \mathbf{S} \times \mathbf{S}^{-1}\mathbf{X} &\to \mathbf{S}^{-1}\mathbf{X} \\ s \oplus (t, x) &:= (s \oplus t, x) \end{aligned}$$

Definition 5.4.7. *We say that* $\mathbf{S}$ *acts on* $\mathbf{X}$ homotopy invertibly *if every translation functor* $s\oplus\colon \mathbf{X} \to \mathbf{X}$*,* $x \mapsto s \oplus x$ *is a homotopy equivalence.*

Example 5.4.8. If $\mathbf{S}$ acts on $\mathbf{X}$, then the corresponding action of $\mathbf{S}$ on $\mathbf{S}^{-1}\mathbf{X}$ is homotopy invertible. Indeed, for any $s \in \mathrm{Ob}(\mathbf{S})$ the translation functor $F_s\colon \mathbf{S}^{-1}\mathbf{X} \to \mathbf{S}^{-1}\mathbf{X}$, $F_s(t,x) := (s \oplus t, x)$ has a homotopy inverse functor $G_s\colon \mathbf{S}^{-1}\mathbf{X} \to \mathbf{S}^{-1}\mathbf{X}$ given by

$$G_s(t,x) := (t, s \oplus X)$$

To see that G_s gives a homotopy inverse to F_s, notice that there is a natural transformation $\mathrm{id}_{\mathbf{S}^{-1}\mathbf{X}} \Rightarrow F_s \circ G_s$ given by

$$(t,x) \xrightarrow{(s\oplus)} (s \oplus t, s \oplus x)$$

Having a natural transformation between two functors implies that they are homotopy equivalent, see Lemma 3.2.1. (**So F_s is left inverse to G_s. Why is it also a right inverse??**)

If $\mathbf{S}$ is a symmetric mnoidal category, $\pi_0(\mathbf{S})$ is a multiplicatively cosed subset of

$$\mathrm{H}_0(\mathbf{S}) = \mathrm{H}_0(B\mathbf{S}, \mathbb{Z}) = \mathbb{Z}[\pi_0(\mathbf{S})]$$

If $\mathbf{S}$ acts on $\mathbf{X}$, then $\mathrm{H}_0()\mathbf{S}$ acts on $\mathrm{H}_\bullet(\mathbf{X}) = \mathrm{H}_\bullet(B\mathbf{X}, \mathbb{Z})$ and acts *invertibly* on $\mathrm{H}_\bullet(\mathbf{S}^{-1}\mathbf{X})$. Hence the functor $\mathbf{X} \to \mathbf{S}^{-1}\mathbf{X}$ sending $x \mapsto (e,x)$ induces a map

$$\mathrm{H}_i(\mathbf{X})\left[\pi_0(\mathbf{S})^{-1}\right] \to \mathrm{H}_i(\mathbf{S}^{-1}\mathbf{X}),\ \forall i \geqslant 0 \tag{2.10}$$

Theorem 5.4.9 (Quillen)**.** *Let $\mathbf{S}$ be a symmetric monoidal category satisfying*

1. $\mathbf{S} = \mathbf{Iso}(\mathbf{S})$*;*
2. *all translations in $\mathbf{S}$ are faithful.*

Then the maps 2.10 are isomorphisms for all $\mathbf{X}$ and all $i \geqslant 0$. In particular, $B(\mathbf{S}^{-1}\mathbf{S})$ is a group completion of the H-space $B\mathbf{S}$.

Sketch of proof. We will outline the basic steps of the proof. **Step 1.** Consider the category $\mathbf{S} \ltimes \mathbf{S}$. Recall that this is the category with objects $\mathrm{Ob}(\mathbf{S} \ltimes \mathbf{S}) = \mathrm{Ob}(\mathbf{S})$ and morphisms given by

$$\mathrm{Hom}_{\mathbf{S}\ltimes\mathbf{S}}(x, x') = \{(s,\varphi) \mid s \in \mathrm{Ob}(\mathbf{S}), \varphi \in \mathrm{Hom}_{\mathbf{S}}(s \oplus x, x')\}\Big/\sim$$

where $(s,\varphi) \sim (s',\varphi')$ if there exists $\alpha\colon s \to s'$ a morphism in $\mathbf{S}$ making the following diagram commute

$$\begin{tikzcd} s \oplus x \arrow[rr, "\alpha\oplus\mathrm{id}_x"] \arrow[dr, "\varphi"'] & & s' \oplus x \arrow[dl, "\varphi'"] \\ & x' & \end{tikzcd}$$

Note that if $\mathbf{S} = \mathbf{Iso}(\mathbf{S})$, the category $\mathbf{S}^{-1}\mathbf{S}$ is contractible, because it has initial object $e \in \mathrm{Ob}(\mathbf{S}) = \mathrm{Ob}(\mathbf{S}^{-1}\mathbf{X})$. So $B\mathbf{S} \ltimes \mathbf{S} \simeq \{*\}$.

Step 2. There is a natural projection functor

$$F\colon \mathbf{S}^{-1}\mathbf{X} \to \mathbf{S} \ltimes \mathbf{S}$$
$$(s,x) \mapsto s$$

Lemma 5.4.10. *If* $\mathbf{S}$ *satisfies the two conditions of the theorem, the projection functor* F *is cofibred, with fibre over* $s \in \mathrm{Ob}(\mathbf{S} \ltimes \mathbf{S}) = \mathrm{Ob}(\mathbf{S})$ *being* $\mathbf{X}_s = \mathbf{X}$.

We recall (see Section 3.8) that a functor $F: \mathbf{C} \to \mathbf{D}$ is called cofibred if for any $d \in \mathrm{Ob}(\mathbf{D})$ the inclusion $j_*: F^{-1}(d) \hookrightarrow F/d$ has a left adjoint functor $j^*: F/d \to F^{-1}(d)$ (and so $BF^{-1}(d) \simeq B(F/d)$) and the cobase change $f_*: F^{-1}(d) \to F^{-1}(d')$ respects composition: $(fg)_* = f_*g_*$ for any composable $f: d \to d'$ and $g: d' \to d''$.

Step 3.

Lemma 5.4.11. *If* $F: \mathbf{C} \to \mathbf{D}$ *is a cofibred functor, there is a first quadrant homological spectral sequence*

$$E^2_{pq} = \mathrm{H}_p\left(\mathbf{D}, \mathrm{H}_q F^{-1}\right) \Rightarrow \mathrm{H}_{p+q}(\mathbf{C}, \mathbb{Z}),$$

where the homology of $\mathbf{D}$ *is taken with the coefficients in the functor* $\mathrm{H}_q F^{-1}: \mathbf{D} \to \mathbf{Ab}$

$$d \mapsto \mathrm{H}_q\left(F^{-1}(d), \mathbb{Z}\right)$$

Proof of the Lemma. This spectral sequence is associated with the first quadrant bicomplex $\{M_{pq}\}_{p,q\geqslant 0}$ with

$$M_{pq} = \mathbb{Z}\left\{(d_p \to \cdots \to d_0 \to F(c_0), c_0 \to \cdots \to c_q)\right\}$$

being the free abelian group spanned by all sequences of composable maps in $\mathbf{D}$ and $\mathbf{C}$ (cf. proof of Proposition 3.7.8).

Recall that if $\{M_{pq}\}_{p,q\geqslant 0}$ is a (first quadrant) bicomplex, we have two (convergent) spectral sequences converging to $\mathrm{H}_\bullet\left[\mathrm{Tot}(M)\right]$:

$$\begin{aligned} {}^I E^2_{pq} &= \mathrm{H}'_p\, \mathrm{H}''_{pq}(M) \\ {}^{II} E^2_{pq} &= \mathrm{H}''_q\, \mathrm{H}'_{qp}(M) \end{aligned}$$

In ${}^I E^2_{pq}$, we first take homology of p-th column and then homology of rows. In ${}^{II} E^2_{pq}$, we first take homology of q-th row, and then homology of columns.

Exercise 1. Check that in our case ${}^{II} E^2_{pq}$ degenerates giving

$$\mathrm{H}_q(\mathbf{C}, \mathbb{Z}) = \mathrm{H}_q(B\mathbf{C}, \mathbb{Z}) \simeq \mathrm{H}_q(\mathrm{Tot}(M))$$

The spectral sequence ${}^I E^2_{pq}$ becomes the sequence of the lemma. □

Step 4. Apply Lemma 5.4.11 to the functor of Lemma 5.4.10. This gives a convergent spectral sequence

$$E^2_{pq} = \mathrm{H}_p\left(\mathbf{S} \ltimes \mathbf{S}, \mathrm{H}_q(X_s)\right) \Rightarrow \mathrm{H}_{p+q}(\mathbf{S}^{-1}\mathbf{X}) \tag{2.11}$$

Let $\mathrm{H}_q(X_s)^{\mathrm{loc}} := \mathrm{H}_q(X_s)\left[\pi_0(S)^{-1}\right]$. Since the localization at multiplicatively closed subsets is an exact functor, localizing 2.11 yields

$$\left(E^2_{pq}\right)^{\mathrm{loc}} = \mathrm{H}_p(\mathbf{S} \ltimes \mathbf{S}, \mathrm{H}_q(X_s)^{\mathrm{loc}}) \Rightarrow \mathrm{H}_{p+q}(\mathbf{S}^{-1}\mathbf{X})^{\mathrm{loc}} \simeq \mathrm{H}_{p+q}(\mathbf{S}^{-1}\mathbf{X}) \tag{2.12}$$

where the last isomorphism follows from the fact that $\mathrm{H}_0(\mathbf{S}) = \mathbb{Z}[\pi_0(\mathbf{S})]$ acts on $\mathrm{H}_{p+q}(\mathbf{S}^{-1}\mathbf{X})$ invertibly.

Now, the functor

$$\begin{aligned} \mathbf{S} \ltimes \mathbf{S} &\to \mathbf{Ab} \\ s &\mapsto \mathrm{H}_q(X_s)^{\mathrm{loc}} \end{aligned}$$

is morphism invertible (local system) on the contractible (by Step 1) space $B(\mathbf{S} \ltimes \mathbf{S})$. Hence

$$\mathrm{H}_p(\mathbf{S} \ltimes \mathbf{S}, \mathrm{H}_q(X_s)^{\mathrm{loc}}) \simeq \begin{cases} 0, & p \neq 0 \\ \mathrm{H}_q(X_s)^{\mathrm{loc}}, & p = 0 \end{cases}$$

Thus, the spectral sequence 2.12 degenerates giving the required isomorphism

$$\mathrm{H}_q(\mathbf{X})\left[\pi_0(\mathbf{S})^{-1}\right] \to \mathrm{H}_q(\mathbf{S}^{-1}\mathbf{X}),\ \forall q \geqslant 0$$

□

5.5 Application to "plus"-construction

Let A be an associative unital ring. Consider the full subcategory $\mathbb{F}(A) \subseteq \mathbb{P}(A)$ consisting of finitely generated **free** modules. In what follows, assume that the ring A satisfies **invariant basis property** (IBP), that is, if two modules $A^m \simeq A^n$ are isomorphic, then $m = n$.

Let $\mathbf{Iso}(\mathbb{F}(A))$ be the associated groupoid. Because A satisfies IBP, the category $\mathbf{Iso}(\mathbb{F}(A))$ is equivalent to the category $\mathbf{S}$ with $\mathrm{Ob}(\mathbf{S}) = \{0, A, A^2, \dots\} \simeq \mathbb{N}$, and

$$\mathrm{Hom}_{\mathbf{S}}(n, m) = \begin{cases} \emptyset, & n \neq m \\ GL_n(A), & n = m \end{cases}$$

In other words, $\mathbf{S} = \bigsqcup_{n \geqslant 0} GL_n(A)$, and so

$$B\mathbf{S} = \bigsqcup_{n \geqslant 0} BGL_n(A)$$

Note that $\mathbf{S}$ has a natural monoidal structure given by $A^n \oplus A^m = A^{m+n}$ on objects and

$$\alpha \oplus \beta = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \in GL_{n+m}(A)$$

for $\alpha \in GL_n(A), \beta \in GL_m(A)$.

Theorem 5.5.1 (Quillen)**.** *If* $\mathbf{S} \bigsqcup_{n \geqslant 0} GL_n(A)$, *then the K-space* $K^{\oplus}(\mathbf{S}) = B(\mathbf{S}^{-1}\mathbf{S})$ *is the group completion of* $B\mathbf{S} \simeq \bigsqcup_{n \geqslant 0} BGL_n(A)$, *and in fact we have*

$$B(\mathbf{S}^{-1}\mathbf{S}) \simeq \mathbb{Z} \times BGL(A)^+$$

Proof. The proof is based on Theorem 5.4.9 and Quillen's Recognition Criterion 3.3.3, which says that the map $BGL(A)^{\to}BGL(A)^{+}$ is universal among all maps $f\colon BGL(A) \to H$ to H-spaces H.

Lemma 5.5.2. *If for a map $f\colon BGL(A) \to H$ the induced map $f_*\colon \mathrm{H}_\bullet(BGL(A), \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}_\bullet(H, \mathbb{Z})$ is an isomorphism, then f is an acyclic map, and the induced map*

$$\overline{f}\colon BGL(A)^{+} \xrightarrow{\sim} H$$

is a homotopy equivalence.

Proof of Lemma. The proof follows immediately from the following standard result from homotopy theory.

Proposition 5.5.3 ([Whi78][Ch.IV)**.** *] If X, Y are H-spaces having the homotopy type of CW complexes, then any map $f\colon X \to Y$ inducing an isomorphism on homology*

$$f_*\colon \mathrm{H}_\bullet(X, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}_\bullet(Y, \mathbb{Z})$$

is a homotopy equivalence

For the proof of the proposition, one first shows that $\pi_1(Y)$ acts trivially on the homotopy fiber Ff. Then by Relative Hurewicz Theorem one comncludes that $\pi_*(Ff) = 0$. Hence $\pi_i(f)$ are isomorphisms for all i. Since X, Y are CW complexes, Whiteheads's Theorem implies that f is a homotopy equivalence. □

To apply Theorem 5.4.9 we need to construct a map $f\colon BGL(A) \to Y_{\mathbf{S}}$, where $Y_{\mathbf{S}}$ is the connected component of the identity (basepoint) of $B(\mathbf{S}^{-1}\mathbf{S})$, which satisfies the assumption of Lemma 5.5.2 (for $H = Y_{\mathbf{S}}$).

For each $n \geqslant 1$, we have natural group homomorphisms

$$\eta_n\colon GL_n(A) = \mathrm{Aut}_{\mathbf{S}}(A^n) \to \mathrm{Aut}_{\mathbf{S}^{-1}\mathbf{S}}(A^n, A^n)$$

given by $g \mapsto (g, 1)$. These homomorphisms fit into the following commutative diagram

$$\begin{tikzcd}
GL_n(A) \arrow[r, "\eta_n"] \arrow[d, "A\oplus"'] & \mathrm{Aut}_{\mathbf{S}^{-1}\mathbf{S}}(A^n, A^n) \arrow[d, "(A,A)\oplus"] \\
GL_{n+1}(A) \arrow[r, "\eta_{n+1}"] & \mathrm{Aut}_{\mathbf{S}^{-1}\mathbf{S}}(A^{n+1}, A^{n+1})
\end{tikzcd}$$

where $A\oplus$ and $(A, A)\oplus$ are translation maps in $\mathbf{S}$ and $\mathbf{S}^{-1}\mathbf{S}$, corresponding to the objects $A \in \mathrm{Ob}(\mathbf{S})$ and $(A, A) \in \mathrm{Ob}(\mathbf{S}^{-1}\mathbf{S})$.

The commutativity of the above diagram can be interpreted by saying that there is a sequence of functors

$$\begin{tikzcd}
\vdots \arrow[d] & \\
GL_n(A) \arrow[d, "A\oplus"'] \arrow[dr, "\eta_n"] & \\
GL_{n+1}(A) \arrow[r, "\eta_{n+1}"] \arrow[d, "A\oplus"'] & \mathbf{S}^{-1}\mathbf{S} \\
GL_{n+2}(A) \arrow[ur, "\eta_{n+2}"] \arrow[d] & \\
\vdots &
\end{tikzcd}$$

and natural transformations $\eta_n \Rightarrow \eta_{n+1} \circ (A\oplus)$ which induces a *homotopy commutative* diagram of spaces

$$\begin{tikzcd}
\vdots \arrow[d] & \\
BGL_n(A) \arrow[dd, "A\oplus"'] \arrow[dr, "B\eta_n"] & \\
 & B(\mathbf{S}^{-1}\mathbf{S}) \\
BGL_{n+1}(A) \arrow[ur, "B\eta_{n+1}"] \arrow[d] & \\
\vdots &
\end{tikzcd}$$

Here we, as usual, think of the groups $GL_n(A)$ as one-object categories. In the inductive limit, we get a well-defined map

$$BGL(A) = \operatorname{colim}_n BGL_n(A) \xrightarrow{\eta} B(\mathbf{S}^{-1}\mathbf{S})$$

It is easy to see that η lands, in fact, in the connected component $Y_{\mathbf{S}}$ of the identity in $B(\mathbf{S}^{-1}\mathbf{S})$, i.e.

$$\eta\colon BGL(A) \to B(\mathbf{S}^{-1}\mathbf{S})$$

Since $B(\mathbf{S}^{-1}\mathbf{S})$ is an H-space, so is its identity connected component $Y_{\mathbf{S}}$. Hence, to apply Lemma 5.5.2, we need to show that η induces a homology isomorphism $\eta_*\colon \mathrm{H}_\bullet(BGL(A)) \xrightarrow{\sim} \mathrm{H}_\bullet(Y_{\mathbf{S}})$.

Let $a = [A] \in \pi_0(B\mathbf{S}) = \pi_0(\mathbf{S})$ denote the class of $A \in \mathrm{Ob}(\mathbf{S})$. By Theorem 5.4.9, we have

$$\mathrm{H}_\bullet\left[B(\mathbf{S}^{-1}\mathbf{S})\right] \simeq \mathrm{H}_\bullet(B\mathbf{S})\left[\pi_0(\mathbf{S})^{-1}\right]$$

But $\pi_0(\mathbf{S}) = \{a^n\}_{n\geqslant 0}$ is generated by a, so that $\mathrm{H}_0(\mathbf{S}) = \mathbb{Z}[\pi_0(\mathbf{S})] \simeq \mathbb{Z}[a,a^{-1}]$. The localization in the RHS of the above equation is given then by the inductive limit:

$$\mathrm{H}_\bullet(B\mathbf{S})\left[\pi_0(\mathbf{S})^{-1}\right] = \operatorname{colim}\left\{\mathrm{H}_\bullet(B\mathbf{S}) \xrightarrow{a} \mathrm{H}_\bullet(B\mathbf{S}) \xrightarrow{a} \dots\right\}$$

where the maps are induced by the translation functor $A\oplus\colon \mathbf{S}\to\mathbf{S}$. Hence, we have

$$\mathrm{H}_\bullet(B(\mathbf{S}^{-1}\mathbf{S})) \simeq \mathrm{H}_\bullet(Y_{\mathbf{S}})\otimes\mathbb{Z}[a,a^{-1}]$$

and $\mathrm{H}_\bullet(Y_{\mathbf{S}}) \simeq \operatorname{colim}_n \mathrm{H}_\bullet(BGL_n(A)) \simeq \mathrm{H}_\bullet(BGL(A))$. The latter isomorphism implies that $f\colon BGL(A)\to Y_{\mathbf{S}}$ is a homology equivalence. By Lemma 5.5.2, ir induces a homotopy equivalence

$$\overline{f}\colon BGL(A)^+ \xrightarrow{\sim} Y_{\mathbf{S}}$$

which implies the theorem. □

Remark 5.5.4. A similar argument works for other categories $\mathbf{S}$. For example, let $\mathbf{S} := \mathbf{Iso}(\mathbf{Fin}) \simeq \coprod_{n\geqslant 0} S_n$ (the groupoid of finite sets). The transition functor $\{1\}\colon \mathbf{S}\to\mathbf{S}$ yields the inclusion $S_n \hookrightarrow S_{n+1}$ of symmetric groups (similar to $GL_n\hookrightarrow GL_{n+1}$) and these assemble together to give

$$BS_\infty \to B(\mathbf{S}^{-1}\mathbf{S})$$

where $S_\infty = \operatorname{colim}_n S_n$ is the infinite symmetric group. The above arguments show that

$$B(\mathbf{S}^{-1}\mathbf{S}) = K^{\oplus}(\mathbf{Iso}(\mathbf{Fin})) \simeq \mathbf{Z}\times BS_\infty^+.$$

Theorem 5.5.5 (Barratt–Priddy–Quillen–Segal)**.** *There are isomorphisms of groups* $K_i(\mathbf{Iso}(\mathbf{Iso})) \simeq \pi_i^{\mathrm{st}}(\mathbb{S})$, *where* $\pi_i^{\mathrm{st}}(\mathbb{S}) := \lim_{n\to\infty}\pi_{i+n}(\mathbb{S}^n)$ *are the stable homotopy groups of spheres.*

5.6 Proving the "plus=Q" theorem

Theorem 5.5.1 provides one step of proving the "plus=Q" construction, showing "plus=$\mathbf{S}^{-1}\mathbf{S}$" part. What is left is to show that "$Q=\mathbf{S}^{-1}\mathbf{S}$". This will follow from the following more general result of Quillen.

Theorem 5.6.1 (Quillen)**.** *Let* $\mathbf{C}$ *be a* split *exact category, and let* $\mathbf{S} = \mathbf{Iso}(\mathbf{C})$ *be the groupoid of isomorphisms in* $\mathbf{C}$*. Then*

$$\Omega BQ\mathbf{C} \simeq B(\mathbf{S}^{-1}\mathbf{S}) \tag{2.13}$$

Hence, $K_i(\mathbf{C}) \simeq K_i^{\oplus}(\mathbf{S})$ *for all* $i\geqslant 0$.

This section will be devoted to proving this theorem. The idea of the proof goes as follows. Instead of constructing the equivalence 2.13 directly, we will construct a homotopy fibration sequence of spaces

$$B(\mathbf{S}^{-1}\mathbf{S}) \to E \to BQ\mathbf{C}$$

with the space E being *contractible.* The space E will be obtained in the form $E = B(\mathbf{S}^{-1}\mathbf{X})$ where $\mathbf{S}^{-1}\mathbf{X} := \mathbf{S}\ltimes(\mathbf{S}\times\mathbf{X})$ arises from an action of $\mathbf{S},\oplus$ on a certain category $\mathbf{X}$. This category $\mathbf{X} = \mathbf{Ex}(\mathbf{C})$ is the category of exact diagrams (sequences) in $\mathbf{C}$.

Remark 5.6.2. The construction of $\mathbf{Ex}(\mathbf{C})$ generalizes the construction of the category of exact sequences $\mathbf{Ex}(\mathbf{A})$ of an *abelian* category $\mathbf{A}$, see [BMP14, Sect.4.14] (but the use of $\mathbf{Ex}(\mathbf{C})$ is different). It will be convenient for the construction of $\mathbf{Ex}(\mathbf{C})$ to give a more concrete characterization of exact categories in terms of abelian categories and (honest) exact sequences (like in [BMP14]).

Theorem 5.6.3 (see [BMP14])**.** *If $(\mathbf{C}, \mathcal{E})$ is an exact category, there is an embedding, i.e. a fully faithful additive functor, $\mathbf{C} \hookrightarrow \mathbf{A}$ of $\mathbf{C}$ into an* abelian *category $\mathbf{A}$ such that*

1. *$[a \hookrightarrow b \twoheadrightarrow c] \in \mathcal{E}$ if and only if $0 \to a \to b \to c \to 0$ is a SES in $\mathbf{A}$;*
2. *$\mathbf{C}$ is closed under extensions in $\mathbf{A}$. In other words, if $0 \to a \to b \to c \to 0$ is a SES in $\mathbf{A}$, and $a, c \in \mathrm{Ob}(\mathbf{C})$, then necessarily $b \in \mathrm{Ob}(\mathbf{C})$ and $[a \hookrightarrow b \twoheadrightarrow c] \in \mathcal{E}$.*

For an exact category $\mathbf{C}$ we define $\mathbf{Ex}(\mathbf{C})$ to be the category having

$$\mathrm{Ob}(\mathbf{Ex}(\mathbf{C})) = \mathcal{E} = \text{ the set of admissible exact diagrams } [a \hookrightarrow b \twoheadrightarrow c] \text{ in } \mathbf{C}$$

If $E = [a \hookrightarrow b \twoheadrightarrow c]$ and $E' = [a' \hookrightarrow b' \twoheadrightarrow c']$ are objects in $\mathbf{Ex}(\mathbf{C})$, we define a morphism $f \colon E \to E'$ in $\mathrm{Mor}\,[\mathbf{Ex}(\mathbf{C})]$ to be an equivalence class of diagrams of the form

$$\begin{tikzcd}
E' : \arrow[dd, "f"'] & a' \arrow[r, hook] & b' \arrow[r, two heads] & c' \\
& a \arrow[u, hook, "\alpha"] \arrow[r, hook] \arrow[d, equal] & b' \arrow[u, equal] \arrow[r, two heads] \arrow[d, hook, "\beta"] & c'' \arrow[u] \arrow[d, hook] \\
E : & a \arrow[r, hook] & b \arrow[r, two heads] & c
\end{tikzcd}$$

where the rows are exact. The equivalence relation is given by an isomorphism of such diagrams in $\mathbf{C}$, which is identity at all vertices except (possibly) fo the vertex c''. The composition is defined by pull-backs (resp. push-outs) of admissible epics (resp. monics).

Note that the rightmost column represents represents a morphism $\varphi(f) = [c' \twoheadleftarrow c'' \hookrightarrow c] \in \mathrm{Hom}_{Q\mathbf{C}}(c', c)$ in $Q\mathbf{C}$. Assigning $f \mapsto \varphi(f)$ gives a functor

$$\begin{aligned} \varphi \colon \mathbf{Ex}(\mathbf{C}) &\to Q\mathbf{C} \\ [a \hookrightarrow b \twoheadrightarrow c] &\mapsto c \end{aligned}$$

For an object $c \in \mathrm{Ob}(Q\mathbf{C}) = \mathrm{Ob}(\mathbf{C})$, we denote

$$\mathcal{E}_c := \varphi^{-1}(c),$$

the *fiber category* of φ. Recall, see 3.8.1, that by definition

$$\begin{aligned} \mathrm{Ob}(\mathcal{E}_c) &:= \{E \in \mathrm{Ob}(\mathbf{Ex}(\mathbf{C})) \mid \varphi(E) = c\} \\ \mathrm{Mor}(\mathcal{E}_c) &:= \{f \in \mathrm{Mor}(\mathbf{Ex}(\mathbf{C})) \mid \varphi(f) = \mathrm{id}_c\} \end{aligned}$$

A morphish in $\mathcal{E}_c$ is represented by a diagram

$$\begin{tikzcd}
a' \arrow[r, hook] & b' \arrow[r, two heads] & c \\
a \arrow[u, hook, "\alpha"] \arrow[r, hook] \arrow[d, equal] & b' \arrow[u, equal] \arrow[r, two heads] \arrow[d, hook, "\beta"] & c \arrow[u, equal] \arrow[d, equal] \\
a \arrow[r, hook] & b \arrow[r, two heads] & c
\end{tikzcd}$$

which we will denote, for notational brevity, as

$$\begin{tikzcd}
a' \arrow[r, hook] & b' \arrow[r, two heads] \arrow[d, hook, "\beta"] & c \arrow[d, equal] \\
a \arrow[u, hook, "\alpha"] \arrow[r, hook] & b \arrow[r, two heads] & c
\end{tikzcd} \tag{2.14}$$

Note that the morphisms α, β above are necessarily isomorphisms in $\mathbf{C}$. Thus, if $E = [a \hookrightarrow b \twoheadrightarrow c]$ and $E' = [a' \hookrightarrow b' \twoheadrightarrow c]$ are objects in $\mathcal{E}_c$, then

$$\mathrm{Hom}_{\mathcal{E}_c}(E', E) = \{(\alpha, \beta) \in \mathbf{Iso}(\mathbf{C}) \times \mathbf{Iso}(\mathbf{C}) \mid \text{ diagram (2.14) commutes}\}$$

In particular, $\mathcal{E}_c$ is a groupoid for every $c \in \mathrm{Ob}(Q\mathbf{C}) = \mathrm{Ob}(\mathbf{C})$.

Lemma 5.6.4. *Let $\mathbf{C}$ be an exact category, and $\mathbf{S} = \mathbf{Iso}(\mathbf{C})$ be the associated groupoid of isomorphisms in $\mathbf{C}$. Then the following holds.*

1. *For any $c \in \mathrm{Ob}(Q\mathbf{C})$, $\mathcal{E}_c$ is a is a symmetric monoidal category, and there is a faithful monoidal functor*
$$\eta_c \colon \mathbf{S} \to \mathcal{E}_c$$
sending $a \mapsto [a \hookrightarrow c \oplus c \twoheadrightarrow c]$.

2. *For the zero object $c = 0$, the functor $\eta_0 \colon \mathbf{S} \to \mathcal{E}_0$ is a homotopy equivalence. Thus*
$$\mathbf{S} := \mathbf{Iso}(\mathbf{C}) \simeq \mathcal{E}_0 := \varphi^{-1}(0).$$

Proof. Given $E_i = [a_i \hookrightarrow b_i \twoheadrightarrow c]$, $i = 1, 2$ in $\mathrm{Ob}(\mathcal{E}_c)$, we define the "join" operation:

$$E_1 * E_2 := [a_1 \oplus a_2 \hookrightarrow b_1 \times_c b_2 \twoheadrightarrow c]$$

which is a symmetric product, with identity being $e := [0 \hookrightarrow c \twoheadrightarrow c]$. We leave it as a (trivial) exercise to check that η_c is fully faithful and monoidal with respect to the "join" product. This proves the first statement.

For $c = 0$, the functor $\eta_0 \colon \mathbf{S} \to \mathcal{E}_0$ sends $a \mapsto [a \overset{=}{\hookrightarrow} a \twoheadrightarrow 0]$ and a morphism $a \xrightarrow{\alpha} a'$ to the diagram

$$\begin{tikzcd}
a' \arrow[r, hook, "="] & a' \arrow[r, two heads] \arrow[d, hook, "\alpha^{-1}"] & 0 \arrow[d, equal] \\
a \arrow[u, hook, "\alpha"] \arrow[r, hook, "="] & a \arrow[r, two heads] & 0
\end{tikzcd}$$

Obviously, η_0 is fully faithful, and every $E \in \mathrm{Ob}(\mathcal{E}_0)$ is isomorphic to one of the form $[a \hookrightarrow a \twoheadrightarrow 0]$, i.e. η_0 is also essentially surjective, and hence is an equivalence of categories. This finishes the proof of the lemma. $\square$

Lemma 5.6.4 allows us to define an action of the groupoid $\mathbf{S}$ on the category $\mathbf{Ex}(\mathbf{C})$. To this end, we note that the direct sum of exact sequences in $\mathbf{C}$ induce an operation $\oplus$ on $\mathbf{Ex}(\mathbf{C})$. By the second statement of Lemma 5.6.4, we can embed $\mathbf{S}$ into $\mathbf{Ex}(\mathbf{C})$ as the zero fiber category of $\varphi\colon \mathbf{Ex}(\mathbf{C}) \to Q\mathbf{C}$. Via this embedding, we define the action

$$\mathbf{S} \times \mathbf{Ex}(\mathbf{C}) \to \mathbf{Ex}(\mathbf{C})$$

by $a \oplus [a' \hookrightarrow b' \twoheadrightarrow c'] := [a \oplus a' \hookrightarrow a \oplus b' \twoheadrightarrow c']$. Note that $\varphi(a \oplus E) = \varphi(E)$ for all $E \in \mathrm{Ob}(\mathbf{Ex}(\mathbf{C}))$ which means that the action of $\mathbf{S}$ on $\mathbf{Ex}(\mathbf{C})$ preserves the fibers of φ. In particular, for any fixed $c \in \mathrm{Ob}(Q\mathbf{C})$ we have an action

$$\mathbf{S} \times \mathcal{E}_c \to \mathcal{E}_c$$

and the associated categories $\mathbf{S} \ltimes \mathcal{E}_c$ and $\mathbf{S}^{-1}\mathcal{E}_c := \mathbf{S} \ltimes (\mathbf{S} \times \mathbf{Ex}(\mathbf{C}))$.

Lemma 5.6.5. *If $\mathbf{C}$ is split exact, then the category $\mathbf{S} \ltimes \mathcal{E}_c$ is contractible for all $c \in \mathrm{Ob}(Q\mathbf{C}) = \mathrm{Ob}(\mathbf{C})$.*

Proof. Observe that if $\mathbf{C}$ is split exact, then every object of $\mathcal{E}_c$ is isomorphic to one coming from $\mathbf{S}$ (as an image of the functor η_c of Lemma 5.6.4). Hence, $\mathbf{S} \ltimes \mathcal{E}_c$ is connected, i.e. $\pi_0(\mathbf{S} \ltimes \mathcal{E}_c) = \{1\}$. Now, by Lemma 5.6.4, $\mathcal{E}_c$ has a symmetric monoidal structure. The monoidal product $*$ on $\mathcal{E}_c$ induces a monoidal product on $\mathbf{S} \ltimes \mathcal{E}_c$, which makes $B(\mathbf{S} \ltimes \mathcal{E}_c)$ an H-space. Since $\pi_0(\mathbf{S} \ltimes \mathcal{E}_c) = \{1\}$ is a trivial group, and $B(\mathbf{S} \ltimes \mathcal{E}_c)$ is a CW complex, by Theorem 5.3.5 the H-space $B(\mathbf{S} \ltimes \mathcal{E}_c)$ is group-like, i.e. it has a homotopy inverse (see Definition 5.3.7).

Assigning to each $E = [a \hookrightarrow b \twoheadrightarrow c] \in \mathrm{Ob}(\mathbf{S} \ltimes \mathcal{E}_c)$ the diagonal morphism in $\mathbf{S} \ltimes \mathcal{E}_c$,

$$\begin{tikzcd}
E: \arrow[d, "\delta_E"'] & a \arrow[r, hook] \arrow[d] & b \arrow[r, two heads] \arrow[d] & c \arrow[d, equal] \\
E * E: & a \oplus a \arrow[r, hook] & b \times_c b \arrow[r, two heads] & c
\end{tikzcd}$$

which is represented by the class of the pair $(a, \widetilde{\delta}_E) \in \mathrm{Hom}_{\mathbf{S}\ltimes\mathcal{E}_c}(E, E * E)$, where $\widetilde{\delta}_E$ is given by the diagram

$$\begin{tikzcd}
a \oplus a \arrow[r, hook] \arrow[d, equal] & b \oplus a \arrow[r, two heads] \arrow[d, hook, "\simeq"] & c \arrow[d, equal] \\
a \oplus a \arrow[r, hook] & b \times_c c \arrow[r, two heads] & c
\end{tikzcd}$$

We get a natural transformation $\delta\colon \mathrm{id}_{\mathbf{S}\ltimes\mathcal{E}_c} \Rightarrow *$

$$\delta = \{\delta_E \colon E \to E * E\}_{E \in \mathrm{Ob}(\mathcal{E}_c)}$$

On $B(\mathbf{S} \ltimes \mathcal{E}_c)$ the natural transformation δ induces a homotopy between the identity on $B(\mathbf{S} \ltimes \mathcal{E}_c)$ and the multiplication by 2, i.e. $x \simeq 2 \cdot x$ in $B(\mathbf{S} \ltimes \mathcal{E}_c)$. "Adding" the homotopy inverse $(-x)$ implies

$$(-x) + x \simeq -x + 2x \Rightarrow 0 \simeq x$$

Thus $B(\mathbf{S} \ltimes \mathcal{E}_c)$ is contractible. □

Corollary 5.6.6. *If $\mathbf{C}$ is split exact, the functor $\eta_c \colon \mathbf{S} \to \mathcal{E}_c$ of Lemma 5.6.4 induces a homotopy equivalence $\mathbf{S}^{-1}\mathbf{S} \overset{\sim}{\to} \mathbf{S}^{-1}\mathcal{E}_c$.*

Proof. This follows from Lemma 5.6.5 and the fact that

$$\mathbf{S}^{-1}\mathbf{S} \to \mathbf{S}^{-1}\mathcal{E}_c \xrightarrow{\pi} \mathbf{S} \ltimes \mathcal{E}_c$$

where π is the projection induced by $\mathbf{S} \times \mathcal{E}_c \twoheadrightarrow \mathcal{E}_c$ is a homotopy fibration. We leave the latter as an exercise. □

Exercise 2. Let $\mathbf{X}$ be a category in which every morphism is monic (e.g. isomorphism). Let $\mathbf{S}$ act on $\mathbf{X}$ so that, for each $x \in \mathbf{X}$, the translations $\mathrm{Aut}_{\mathbf{S}}(s) \overset{\oplus x}{\hookrightarrow} \mathrm{Aut}_{\mathbf{X}}(s \oplus x)$ are injective. Then

$$\mathbf{S}^{-1}\mathbf{S} \overset{\oplus x}{\longleftrightarrow} \mathbf{S}^{-1}\mathbf{X} \overset{\pi}{\longleftrightarrow} \mathbf{S} \ltimes \mathbf{X}$$

is a homotopy fibration for every $x \in \mathrm{Ob}(\mathbf{X})$. In particular, if $B(\mathbf{S} \ltimes \mathbf{X}) \simeq *$, then $\mathbf{S}^{-1}\mathbf{S} \overset{\oplus x}{\longleftrightarrow} \mathbf{S}^{-1}\mathbf{X}$ is a homotopy equivalence.

Next, we describe how the fibere category $\mathcal{E}_c$ varies as c varies in $\mathrm{Ob}(\mathbf{C})$.

Lemma 5.6.7. *For any $f \colon c' \to c$ in $Q\mathbf{C}$, there is a canonical functor $f^* \colon \mathcal{E}_c \to \mathcal{E}_{c'}$, and morphism of functors $\eta \colon f^* \Rightarrow \mathrm{in}$,*

$$\eta = \{\eta_E \colon f^*(E) \to E\}_{E \in \mathrm{Ob}(\mathcal{E}_c)}$$

where $\mathrm{in} \colon \mathcal{E}_c \hookrightarrow \mathbf{Ex}(\mathbf{C})$ *is the fiber inclusion functor.*

Proof. Let $f \in \mathrm{Hom}_{Q\mathbf{C}}(c', c)$ be represented by a diagram

$$\begin{tikzcd} & c'' \arrow[dl, two heads, "p"'] \arrow[dr, hook, "i"] & \\ c' & & c \end{tikzcd}$$

in $\mathbf{C}$ which we will denote by (i,p). To define f^* on $E = [a \hookrightarrow b \twoheadrightarrow c] \in \mathrm{Ob}(\mathcal{E}_c)$ we choose a pull-back $b' := b \times_c c''$ to get $E'' = [a \hookrightarrow b \times_c c'' \twoheadrightarrow c'']$ which is exact in $\mathbf{C}$ by the definition of an exact category. Then, compose $b' := b \times_c c'' \twoheadrightarrow c'' \twoheadrightarrow c'$ and take the kernel $a' := \mathrm{Ker}\,[b \times_c c'' \twoheadrightarrow c'' \twoheadrightarrow c']$. This yields an exact sequence in $\mathcal{E}_{c'}$:

$$f^*(E) := \left[a' \hookrightarrow b' \twoheadrightarrow c'\right]$$

Since $\mathcal{E}_c$ is a groupoid, it is easy to check that f^* is independent (up to isomorphism) of the choices of representatives made, and in fact, f^* defines a functor. Moreover, the construction of $f^*(E)$ yields a diagram

$$\begin{tikzcd}
a' \arrow[r, hook] & b' \arrow[r, two heads] & c' \\
a \arrow[u, hook, "\alpha"] \arrow[r, hook] \arrow[d, equal] & b' \arrow[u, equal] \arrow[r, two heads] \arrow[d, hook, "\beta"] & c'' \arrow[u] \arrow[d, hook] \\
a \arrow[r, hook] & b \arrow[r, two heads] & c
\end{tikzcd}$$

Note that β is an admissible monic, because $a \hookrightarrow b' = b \times_c c'' \twoheadrightarrow b$ is. This gives the required natural transformation $\eta_E\colon f^*(E) \to E$. $\square$

Corollary 5.6.8. *The functor $\varphi\colon \mathbf{Ex}(\mathbf{C}) \to Q\mathbf{C}$ is a fibered functor with base change f^*. Thus, the assignment $c \mapsto \mathcal{E}_c$ defines a contravariant functor $Q\mathbf{C} \to \mathbf{Cat}$.*

Now, recall that $\mathbf{S}$ acts on the category $\mathbf{Ex}(\mathbf{C})$ via the inclusion $\mathbf{S} \simeq \mathcal{E}_0 \hookrightarrow \mathbf{Ex}(\mathbf{C})$ and $\varphi(a \oplus E) = \varphi(E)$, see 5.6.4. Hence, we have the induced functor

$$\Phi := \mathbf{S}^{-1}\varphi\colon \mathbf{S}^{-1}\left(\mathbf{Ex}(\mathbf{C})\right) to Q\mathbf{C}$$

whose fiber over $0 \in \mathrm{Ob}(Q\mathbf{C})$ is $\mathbf{S}^{-1}\mathbf{S}$.

Proposition 5.6.9. *If $\mathbf{C}$ is split exact, and $\mathbf{S} := \mathbf{Iso}(\mathbf{C})$, then*

$$B(\mathbf{S}^{-1}\mathbf{S}) \to B\mathbf{S}^{-1}\mathbf{Ex}(\mathbf{C}) \xrightarrow{B\Phi} B(Q\mathbf{C})$$

is a homotopy fibration.

Proof. This will follow from from Quillen's Theorem B , see Theorem 3.9.1, if we show two things:

1. Φ is a fibered functor;
2. the base change functors $f^*\colon \mathcal{E}_{c'} \to \mathcal{E}_c$.

Now, the first statement follows from Corollary 5.6.8 and the following fact which we leave as an exercise.

Exercise 3. Let $\mathbf{S}$ act on $\mathbf{X}$, and let $F\colon \mathbf{Y} \to \mathbf{X}$ be a functor coequalizing the diagram

$$\begin{tikzcd}
\mathbf{S} \times \mathbf{X} \arrow[r, shift left, "\oplus"] \arrow[r, shift right, "\pi_{\mathbf{X}}"'] & \mathbf{X}
\end{tikzcd}$$

i.e. F satisfies $F \circ \oplus = F \circ \pi_{\mathbf{X}}$, where $\pi_{\mathbf{X}}\colon \mathbf{S} \times \mathbf{X} \to \mathbf{X}$ is the projection onto $\mathbf{X}$. Assume that the base change functors of F commute with the action of $\mathbf{S}$ on the fibers of F. Then, F is fibered implies that $\mathbf{S}^{-1}F\colon \mathbf{S}^{-1}\mathbf{X} \to \mathbf{Y}$ is also fibered.

To check that f^* are homotopy equivalences it suffices to look at maps f of the form $0 \hookrightarrow c$ and $c \twoheadrightarrow 0$. Note that if f is represented by $[0 \hookrightarrow c]$ then $f^*\colon \mathcal{E}_c \to \mathcal{E}_0 \simeq \mathbf{S}$ takes $[a \hookrightarrow b \twoheadrightarrow c] \mapsto a$. The composition of the equivalence $\mathbf{S}^{-1}\mathbf{S} \xrightarrow{\sim} \mathbf{S}^{-1}\mathcal{E}_c$ of Corollary 5.6.6 with f^* is the identity functor. Hence, f^* must be a homotopy equivalence.

On the other hand, if f is represented by $[c \twoheadrightarrow 0]$, then f^* maps $[a \hookrightarrow b \twoheadrightarrow c]$ to b. In this case, the composition of the homotopy equivalence $\mathbf{S}^{-1}\mathbf{S} \xrightarrow{\sim} \mathbf{S}^{-1}\mathcal{E}_c$ of Corollary 5.6.6 with f^* maps $a \mapsto a \oplus c$. Since there is a natural transformation $\mathrm{id} \Rightarrow \oplus c$, given by natural maps $a \to a \oplus c$, this composition is a homotopy equivalence, and hence f^* is a homotopy equivalence.

Applying Quillen's Theorem B finishes the proof. $\square$

Finally we can prove Theorem 5.6.1.

Proof of Theorem 5.6.1. The result will follow from Proposition 5.6.9 if we show that $\mathbf{S}^{-1}\mathbf{Ex}(\mathbf{C})$ is a contractible category. We indicate basic steps.

Step 1. We prove that $\mathbf{Ex}(\mathbf{C})$ is a contractible category. Let $(Q\mathbf{C})^{\mathrm{mon}} \subset Q\mathbf{C}$ be the subcategory of $Q\mathbf{C}$ whose objects are $\mathrm{Ob}((Q\mathbf{C})^{\mathrm{mon}}) = \mathrm{Ob}(Q\mathbf{C}) = \mathrm{Ob}(\mathbf{C})$, and morphisms are represented by admissible monics $[a \overset{=}{\twoheadleftarrow} a \hookrightarrow b]$ in $Q\mathbf{C}$. The category $\mathbf{Ex}(\mathbf{C})$ is equivalent to the Segal subdivision $\mathbf{Sub}\,[(Q\mathbf{C})^{\mathrm{mon}}]$ of the category $(Q\mathbf{C})^{\mathrm{mon}}$, which, in turn, is homotopy equivalent to $(Q\mathbf{C})^{\mathrm{mon}}$ itself (via the target functor). Recall that *Segal subdivision* $\mathbf{Sub}(\mathbf{A})$ of a category $\mathbf{A}$ is the category with $\mathrm{Ob}(\mathbf{Sub}(\mathbf{A})) = \mathrm{Mor}(\mathbf{A})$ and

$$\mathrm{Hom}_{\mathbf{Sub}(\mathbf{A})}(a \xrightarrow{f} b, a' \xrightarrow{f'} b) = \left\{(\alpha, \beta) \mid \alpha\colon a' \to a, \beta\colon b \to b',\ \text{s.t. } f' = \beta f \alpha\right\}$$

Thus,

$$\mathbf{Ex}(\mathbf{C}) \simeq \mathbf{Sub}\,[(Q\mathbf{C})^{\mathrm{mon}}] \simeq (Q\mathbf{C})^{\mathrm{mon}}$$

But $(Q\mathbf{C})^{\mathrm{mon}}$ is contractible, and so $\mathbf{Ex}(\mathbf{C})$ also is.

Step 2. Any action of $\mathbf{S}$ on a *contractible* category $\mathbf{X}$ is invertible, i.e. has a homotopy inverse. In this case, if all translations in $\mathbf{S}$ are faithful, then the functors $\mathbf{X} \to \mathbf{S}^{-1}\mathbf{X}$, $x \mapsto (s, x)$ are homotopy equivalences for any $s \in \mathrm{Ob}(\mathbf{S})$. Thus, by Step 1 and the assumption of the Theorem we have a homotopy equivalence

$$\mathbf{Ex}(\mathbf{C}) \xrightarrow{\sim} \mathbf{S}^{-1}\mathbf{Ex}(\mathbf{C})$$

Step 3. From Step 2 we conclude that $\mathbf{S}^{-1}\mathbf{Ex}(\mathbf{C})$ is contractible, which finishes the proof. $\square$

6 Algebraic K-theory of finite fields

Our main reference for the material of this section is [Hil81].

6.1 Basics of topological K-theory

Let X be a (para)compact topological space. Let $\mathbb{V}B_{\mathbb{C}}(X)$ be the set of isomorphism classes of *complex* vector bundles over X. Clearly, $\mathbb{V}B_{\mathbb{C}}(X)$ is a commutative unital semi-ring with *addition* induced by the Whitney sum $\oplus$ and *multiplication* induced by the tensor product of vector bundles:

$$\begin{aligned}\cdot, +\colon \mathbb{V}B_{\mathbb{C}}(X) \times \mathbb{V}B_{\mathbb{C}}(X) \to\colon \mathbb{V}B_{\mathbb{C}}(X)\\ [E_1] + [E_2] := [E_1 \oplus E_2]\\ [E_1] \cdot [E_2] := [E_1 \otimes E_2]\end{aligned}$$

Definition 6.1.1. *The* topological K-group $K^0(X)$ *is defined to be the group completion of* $(\mathbb{V}B_{\mathbb{C}}(X), +)$*: it is a commutative ring with* 1.

Lemma 6.1.2. *Whenever the two spaces $X \sim Y$ are homotopy equivalent, there is a ring isomorphism $K^0(X) \simeq K^0(Y)$.*

Thus, K^0 defines a contravariant functor

$$K^0 \colon \mathbf{Ho}(\mathbf{Top})^{\mathrm{op}} \longrightarrow \mathbf{ComRings},$$

where $K^0\left[X \xrightarrow{f} Y\right] =: f^* \colon K^0(Y) \to K^0(X)$ is induced by the pull-back of vector bundles $[E] \mapsto [f^*E]$.

For a fixed vector bundle E on X, the *dimension function* $X \to \mathbb{N}$ defined by $x \mapsto \dim_{\mathbb{C}}(E_x)$ is locally constant, and defines a semi-sing homomorphism

$$\begin{aligned}\dim\colon \mathbb{V}B_{\mathbb{C}}(X) \to [X, \mathbb{N}]\\ [E] \mapsto [\dim(E)\colon x \mapsto \dim_{\mathbb{C}}(E_x)]\end{aligned}$$

Here we use the usual notation $[-,-] := \mathrm{Hom}_{\mathbf{Ho}(\mathbf{Top})}(-,-)$. This extends to a unique ring homomorphism

$$\underline{\dim}\colon K^0(X) \to [X, \mathbb{Z}] = \mathrm{H}^0(X, \mathbb{Z}).$$

We let $\widetilde{K}^0(X) := \mathrm{Ker}\,[\underline{\dim}] \subset K^0(X)$. The map $\underline{\dim}$ actually has a canonical splitting, and so we have a natural decomposition

$$K^0(X) \simeq \widetilde{K}^0(X) \oplus \mathrm{H}^0(X, \mathbb{Z}).$$

Next, the functor $\widetilde{K}^0$ is representable by the classifying space of $GL(\mathbb{C})$ viewed as a *topological* group. More precisely,

Theorem 6.1.3. *For every compact space X,*

$$\begin{aligned}K^0(X) \simeq [X, \mathbb{Z} \times BGL(\mathbb{C})^{top}]\\ \widetilde{K}^0(X) \simeq [X, BGL(\mathbb{C})^{top}]\end{aligned}$$

In particular, there is an isomorphism $\pi_n\left[BGL(\mathbb{C})^{top}\right] \simeq \widetilde{K}^0(\mathbb{S}^n)$.

Remark 6.1.4. 1. For every $n \geqslant 1$, the canonical inclusion $U_n \hookrightarrow GL_n(\mathbb{C})$ is a deformation retract of $GL_n(\mathbb{C})^{top}$. Hence, we have a homotopy equivalence $BU_n \simeq BGL_n(\mathbb{C})^{top}$ for each n, and therefore, taking the inductive limit $n \to \infty$

$$BU \simeq BGL(\mathbb{C})^{top}.$$

Thus, we can replace $BGL(\mathbb{C})$ in the above theorem by BU.

2. It follows from the last claim of the theorem that

$$K^0(\mathbb{S}^d) = \mathbb{Z} \oplus \widetilde{K}^0(\mathbb{S}^d) = \begin{cases} \mathbb{Z}, & d \equiv 1 \mod 2 \\ \mathbb{Z} \oplus \mathbb{Z}, & d \equiv 0 \mod 2 \end{cases}$$

In particular, for $n \geq 1$,

$$\pi_n(BGL(\mathbb{C})^{top}) = \begin{cases} 0, & n \equiv 1 \mod 2 \\ \mathbb{Z}, & n \equiv 0 \mod 2 \end{cases}$$

The last isomorphism is called the *Bott periodicity.*

6.2 λ-structures and Adams operations

Definition 6.2.1. *If K is a* commutative *ring, a* λ-structure *on K is a family of maps of sets $\lambda^k \colon K \to K$, $k \geqslant 0$, satisfying the following conditions*

1. $\lambda^0(x) = 1$, $\forall x \in K$;
2. $\lambda^k(x+y) = \sum_{i=0}^{k} \lambda^i(x)\lambda^{k-i}(y)$, *for all $x, y \in K$;*

Remark 6.2.2. If $W(K) := 1 + tK[\![t]\!]$, then the two conditions above translate to the map

$$\begin{aligned} \lambda_t \colon (K, +) &\to (W(K), \cdot) \\ x &\mapsto \sum_{k \geqslant 0} \lambda^k(x) t^k \end{aligned}$$

is a group homomorphism, i.e. $\lambda_t(x+y) = \lambda_t(x) \cdot \lambda_t(y)$.

Example 6.2.3. There is a natural λ-structure on the rings $\mathbb{Z}$ and $\mathbb{Q}$ given by

$$\lambda^k \colon \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto \binom{n}{k} := \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k!}$$

This motivates the following

Definition 6.2.4. *Let K be a commutative ring, and let $K_{\mathbb{Q}} := K \otimes_{\mathbb{Z}} \mathbb{Q}$ be the corresponding $\mathbb{Q}$-algebra. Define $\lambda^k \colon K_{\mathbb{Q}} \to K_{\mathbb{Q}}$ by*

$$\lambda^k(x) := \binom{x}{k} := \frac{x(x-1)\dots(x-k+1)}{k!}$$

We say that K is a binomial ring *if*

$$\forall x \in K, \forall k \geqslant 1, \binom{x}{k} \in K \subset K_{\mathbb{Q}}$$

So for example, $\mathbb{Z}$ is a binomial subring of the binomial ring $\mathbb{Q}$.

Example 6.2.5. If A is a commutative ring, then $K_0(A)$ has a natural λ-structure given by

$$\begin{aligned} \lambda^k \colon K_0(A) &\to K_0(A) \\ [P] &\mapsto [\Lambda^k_A P] \end{aligned}$$

Moreover, there is a λ-ring homomorphism given by rank

$$\begin{aligned} \underline{\mathrm{rk}} \colon K_0(A) &\to \mathrm{H}_0(A) := [\mathrm{Spec}(A), \mathbb{Z}] \\ [\Lambda^k_A P] &\mapsto \left[\mathfrak{p} \mapsto \binom{\mathrm{rk}\, P_{\mathfrak{p}}}{k}\right] \end{aligned}$$

Example 6.2.6. If X is a topological space, then $K^0(X)$ has a natural λ-structure given by

$$\begin{aligned} \lambda^k \colon K^0(X) &\to K^0(X) \\ [E] &\mapsto \left[\Lambda^k E\right] \end{aligned}$$

Again, we have a homomorphism of λ-rings

$$\begin{aligned} \underline{\dim} \colon K^0(X) &\to \mathrm{H}^0(X; \mathbb{Z}) \\ [\Lambda^k E] &\mapsto \left[x \mapsto \binom{\dim_{\mathbb{C}} E_x}{k}\right] \end{aligned}$$

Definition 6.2.7. *A* positive structure *on a λ-ring K is given by the following data:*

- *a λ-ring homomorphism $\varepsilon \colon K \to \mathrm{H}^0$ (called the* augmentation map*) admitting a splitting, and so $K \simeq \widetilde{K} \oplus \mathrm{H}^0$ (here H^0 is an abelian group);*
- *a subset $P \subset K$ called the set of* positive elements

satisfying the following properties:

(1) $\mathbb{N} = \{0, 1, 2, \dots\} \subseteq P \subset K$*;*

(2) P is closed under $+$, $\cdot$ and λ^k for all $k \geq 1$. In other words, P is a λ-sub-semiring of K;

(3) every element of $\widetilde{K} := \operatorname{Ker}\varepsilon$ can be written as $p - q$ for some $p, q \in P$;

(4) for any element $p \in P$,

$$\begin{aligned} \varepsilon(p) &= n \in \mathbb{N} \\ \lambda^i(p) &= 0, \ \forall i > n \\ \lambda^n(p) &\in K^\times \end{aligned}$$

Remark 6.2.8. Condition (2) says that the group completion $P^{-1}P$ of $(P, +)$ is a λ-subring of K. And therefore, by the condition (3), $P^{-1}P \simeq \mathbb{Z} \oplus \widetilde{K}$, where $\widetilde{K} = \operatorname{Ker}\varepsilon$.

Definition 6.2.9. *An element $l \in P$ is called a* line element *if $\varepsilon(l) = 1$, $\lambda^1(l) = l$ and $l \in K^\times$. The set of line elements form a subgroup $L \subset K^\times$.*

Example 6.2.10. The λ-ring $K_0(A)$ has

$$\begin{aligned} \mathrm{H}^0 &:= \mathrm{H}_0(A) = [\operatorname{Spec} A, \mathbb{Z}] \\ P &:= \{[P] \mid \operatorname{rk}(P) = \text{const}\} \\ \varepsilon &:= \operatorname{rk}\colon K_0(A) \to \mathrm{H}_0(A) \text{ is the rank function.} \end{aligned}$$

The line elements form a subgroup $L = \operatorname{Pic}(A) \subset K_0(A)^\times$ of projective modules of rank 1, which correspond to line bundles on $\operatorname{Spec} A$.

Example 6.2.11. Suppose X is a compact space with $\pi_1(X)$ a *finite* group. The λ-ring $K^0(X)$ has a canonical positive structure, with

$$\begin{aligned} \mathrm{H}^0 &:= \mathrm{H}^0(X) = [X, \mathbb{Z}] \\ P &:= \{[E] \mid \dim E = \text{const}\} \\ \varepsilon &:= \underline{\dim}\colon K^0(X) \to \mathrm{H}^0(X). \end{aligned}$$

The set L of line elements in $K^0(X)$ are classes of (topological complex) line bundles over X. We have $L \simeq \mathrm{H}^1(X; \mathcal{O}_X^\times)$.

6.3 Witt vectors and special λ-rings

For a commutative ring R, the abelian group $W(R) := 1 + tR[\![t]\!]$ has a (unique) structure of a unital commutative ring characterized by the following properties:

1. $W\colon \textbf{ComRings} \to \textbf{ComRings}$ sending $R \mapsto W(R)$ is a functor;
2. $1_W := 1 + t$ is the multiplicative identity in $(W(R), *)$;
3. for any $f(t) \in W(R)$ and $r \in R$

$$(1 + rt) * f(t) = f(rt)$$

This ring is called *the ring of (big) Witt vectors in R.*

Remark 6.3.1. Note that every element $f(t) \in W(R)$ can be formally factored as $\prod_{i=1}^{\infty}(1 + r_i t^i)$.

There is a (unique) λ-ring structure on $W(R)$ which is natural in R and satisfies

$$\lambda^k(1 - rt) = 0, \ \forall k \geqslant 2$$

Thus, we can also regard W as a functor $W\colon \mathbf{ComRings} \to \lambda\mathbf{Rings}$. The following proposition can be found in [Haz09, Sect.16.1].

Proposition 6.3.2. *The functor W is right adjoint to the forgetful functor $\lambda\mathbf{Rings} \to \mathbf{ComRings}$.*

Remark 6.3.3. The multiplication $*$ and the maps λ^k on $W(R)$ are completely determined by certain "'universal" (i.e. independent of R) polynomials $\{P_n(x_1, \dots, x_n; y_1, \dots, y_n)\}_{n\geqslant 0}$ and $\{P_{n,k}(z_1, \dots, z_{nk})\}_{k\geqslant 1, n\geqslant 0}$ such that

$$\left(\sum_{i\geqslant 0} a_i t^i\right) * \left(\sum_{j\geqslant 0} b_j t^j\right) = \sum_{n\geqslant 0} c_n t^n$$

where $c_n = P_n(a_1, \dots, a_n; b_1, \dots, b_n)$, $\forall a_i, b_j \in R$, and

$$\lambda^k\left(\sum_{i\geqslant 0} a_i t^i\right) = \sum_{n\geqslant 0} b_n^{(k)} t^n$$

where $b_n^{(k)} = P_{n,k}(a_1, \dots, a_{nk})$.

Remark 6.3.4. If $\mathbb{Q} \subseteq R$, then there is an isomorphism $\prod_{n=1}^{\infty} R \xrightarrow{\sim} W(R)$ of rings, given by $(r_1, r_2, \dots) \mapsto \prod_{n=1}^{\infty} e^{1 - r_n t^n/n}$.

Definition 6.3.5. *A λ-ring K is called* special *if the map $\lambda_t\colon K \to W(K)$,*

$$\lambda_t(x) = \sum_{k\geqslant 0} \lambda^k(x) t^k$$

is a λ-ring homomorphism (w.r.t. the above λ-structure on $W(K)$).

Thus, for a special λ-ring K we have

- $\lambda^k(1) = 0$ for $k \neq 0, 1$;

- $\lambda^n\left(\lambda^k(x)\right) = P_{n,k}(\lambda^1(x), \ldots, \lambda^{nk}(x))$.

Exercise 1. The assignment $\lambda^n(s_1) = s_n$ defines a special λ-ring structure on the ring of polynomials $U = \mathbb{Z}[s_1, \ldots, s_n, \ldots]$.

Theorem 6.3.6. *A λ-ring K with a positive structure P is special if and only if the following Splitting Property holds: for any $p \in P$, there exists $K' \supset K$ (λ-ring embedding) such that p is a sum of line elements of K'.*

6.4 Adams operations

Let K be a λ-ring given with an augmentation $\varepsilon\colon K \to \mathrm{H}^0 \hookrightarrow K$.

Definition 6.4.1. *An* Adams operations $\Psi^k\colon K \to K$, $k \geqslant 0$, *are defined by the following inductive formulas:*

$$\begin{aligned}
\Psi^0(x) &= \varepsilon(x) \\
\Psi^1(x) &= x \\
\Psi^2(x) &= x^2 - \varepsilon(x)\lambda^2(x) \\
&\vdots \\
\Psi^k(x) &= \lambda^1(x)\Psi^{k-1}(x) - \lambda^2(x)\Psi^{k-2}(x) + \cdots + (-1)^{k+1}\lambda^k(x)\varepsilon(x)
\end{aligned}$$

Assume that K has a positive structure $(K, \mathrm{H}^0, P, \varepsilon)$. Then K satisfies the following properties.

(1) for any line element $l \in L$, $\Psi^k(l) = l^k$;

(2) if $I \subset K$ is a λ-ideal with $I^2 = 0$, then

$$\Psi^k(x) = (-1)^{k-1}k\lambda^k(x),\ \forall x \in I$$

(3) for any binomial ring H (with $\lambda^k(x) = \binom{x}{k}$), we have $\Psi^k(x) = x$ for all $k \geq 0$. This formula follows from the obvious formula

$$x \cdot \sum_{i=0}^{k-1} (-1)^i \binom{x}{i} = (-1)^{k+1} x \textit{ choosek}$$

Now, the key property is given by the following

Proposition 6.4.2. *If K is a* special *λ-ring with a positive structure, then each $\Psi^k\colon K \to K$ is a ring homomorphism, and we have*

$$\Psi^j \circ \Psi^k = \Psi^{jk},\ \forall j, k \geqslant 0$$

Proof. The additivity $\Psi^k(x+y) = \Psi^k(x) + \Psi^k(y)$ follows from the definition of Adams operations Ψ^k. The multiplicativity follows from the fact that on linear elements, $\Psi^k(l) = l^k$. **NEED TO TYPE UP THE PROOF.** $\square$

6.5 Higher topological K-theory and Bott periodicity

Recall from Theorem 6.1.3 and remark after it that for a compact space X, we have

$$\begin{aligned} K^0(X) &\simeq [X, \mathbb{Z} \times BU] \\ &\simeq [X, \mathbb{Z} \times BGL(\mathbb{C})^{top}] \end{aligned}$$

The higher *topological K-groups* $K^{-n}(X)$, $n > 0$ are defined via the suspension

$$K^{-n} := K^0(\Sigma^n X) \simeq [\Sigma^n X, \mathbb{Z} \times BU].$$

Since the suspension functor Σ is left adjoint to the loop group functor, we have

$$K^{-n}(X) \simeq [X, \Omega^n(\mathbb{Z} \times BU)],\ \forall n \geq 0$$

Thus, the functor K^{-n} is representable by $\Omega^n(\mathbb{Z} \times BU)$. In particular, if $X = *$ is just a one-point space,

$$K^{-n}(*) \simeq [\Sigma^n(*), \mathbb{Z} \times BU] = [\mathbb{S}^n, \mathbb{Z} \times BU] = \pi_n(\mathbb{Z} \times BU)$$

Hence $\widetilde{K}^{-n}(*) = \begin{cases} \mathbb{Z}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$. Now, a refinement of Bott periodicity says

Theorem 6.5.1 (Refined Bott periodicity)**.** *There is a natural homotopy equivalence*

$$\Omega U \simeq \mathbb{Z} \times BU. \tag{2.15}$$

In particular, $\Omega^2 U \simeq U$.

Since $\Omega(BU) \simeq U$, the homotopy equivalence (2.15) implies

$$\Omega^2(\mathbb{Z} \times BU) \simeq \Omega\Omega BU \simeq \Omega U \simeq \mathbb{Z} \times BU$$

Thus, $\Omega^2(\mathbb{Z} \times BU) \simeq \mathbb{Z} \times BU$, which in turn implies the *2-periodicity* of the (complex) topological K-theory:

$$K^{-n-2}(X) \simeq K^{-n}(X),\ \forall n \in \mathbb{Z}$$

Remark 6.5.2. For *real* (topological) vector bundles, the analogue of Bott periodicity equivalence is

$$\Omega^7 O \simeq \mathbb{Z} \times BO$$

where $O = O_\infty(\mathbb{R})$ is the infinite orthogonal group. This equivalence implies $\Omega^8(\mathbb{Z} \times BO) \simeq \mathbb{Z} \times BO$ and $\Omega^8 O \simeq O$.

6.6 Quillen Theorem for Finite fields

Recall that the Adams operations

$$\Psi^k(X)\colon \widetilde{K}^0(X) \to \widetilde{K}^0(X),\ \forall k \geq 0 \tag{2.16}$$

define a family of ring homomorphisms which are natural in X. Since $\widetilde{K}^0(X) \simeq [X, BU]$, we have the *universal* Adams operations

$$\Psi^k\colon BU \to BU$$

inducing (2.16) for every X.

Note that $\widetilde{K}^0(X)$ is a commutative ring, so BU is a (homotopy) commutative H-space (see Section 5.3) and each Ψ^k is an H-space map.

Now, let $\mathbb{F}_q$ be a finite field with q elements. Consider $GL_n(\mathbb{F}_q)$ and let $\mathbb{1}_n$ and ρ_n denote the *trivial* and *natural* n-dimensional representations of $GL_n(\mathbb{F}_q)$ on $\mathbb{F}_q^n$. One can lift (Brower lifting) these representations to *complex* n-dimensional representations. We denote these lifted representations by $\widetilde{\mathbb{1}}_n$ and $\widetilde{\rho}_n$.

Denote by $B(\widetilde{\mathbb{1}}_n)$ and $B(\widetilde{\rho}_n)$ the corresponding maps of CW-complexes

$$BGL_n(\mathbb{F}_q) \to BGL(\mathbb{C})$$

Since $BGL(\mathbb{C})$ is a (homotopy) commutative H-space, we can define the map

$$b_n := B(\widetilde{\rho}_n) - B(\widetilde{\mathbb{1}}_n)\colon BGL_n(\mathbb{F}_q) \to BGL(\mathbb{C}) \simeq BU$$

One can check that the following telescoping diagram commutes (up to homotopy)

$$\begin{tikzcd}
\vdots \arrow[d] & \\
BGL_n(\mathbb{F}_q) \arrow[d, hook, "\text{in}"'] \arrow[dr, "b_n"] & \\
BGL_{n+1}(\mathbb{F}_q) \arrow[r, "b_{n+1}"'] \arrow[d] & BGL(\mathbb{C}) \\
\vdots &
\end{tikzcd}$$

where the map in is induced by the standard embedding $GL_n(\mathbb{F}_q) \hookrightarrow GL_{n+1}(\mathbb{F}_q)$. Hence, we get a map (well-defined up to homotopy):

$$b\colon BGL(\mathbb{F}_q) \to BGL(\mathbb{C}) \simeq BU$$

Since BU is an H-space, this map induces a map

$$\bar{b}\colon BGL(\mathbb{F}_q)^+ \to BU \tag{2.17}$$

such that

$$\begin{tikzcd}
BGL(\mathbb{F}_q) \arrow[rr, "b"] \arrow[dr, "i"'] & & BU \\
& BGL(\mathbb{F}_q)^+ \arrow[ur, "\bar{b}"'] &
\end{tikzcd}$$

where $i\colon BGL(\mathbb{F}_q) \to BGL(\mathbb{F}_q)^+$ is the plus-construction on $BGL(\mathbb{F}_q)$, see Section 3.2.

Theorem 6.6.1 (Quillen)**.** *The map (2.17) identifies $BGL(\mathbb{F}_q)^+$ with the homotopy fiber of the map $\Psi^q - \mathrm{id}\colon BU \to BU$, i.e. we have a homotopy fibration:*

$$BGL(\mathbb{F}_q)^+ \xrightarrow{\bar{b}} BU \xrightarrow{\Psi^q-\mathrm{id}} BU$$

Now, since $\pi_{2i}(BU) = [\mathbb{S}^{2i}, BU] \simeq \widetilde{K}^0(\mathbb{S}^{2i}) \simeq \mathbb{Z}$, and $\Psi^q\colon \widetilde{K}^0(\mathbb{S}^{2i}) \to \widetilde{K}^0(\mathbb{S}^{2i}) \simeq \mathbb{Z}$ is given by the multiplication by q^i, we conclude from the long exact homotopy sequence that

$$\pi_n\left[BGL(\mathbb{F}_q)^+\right] \simeq \begin{cases} 0, & n \text{ even} \\ \mathbb{Z}/(q^i-1), & n = 2i-1 \end{cases}$$

Corollary 6.6.2. *For every finite field $\mathbb{F}_q$ and $n \geq 1$ we have*

$$K_n(\mathbb{F}_q) \simeq \begin{cases} \mathbb{Z}/(q^i-1), & \textit{if } n = 2i-1 \\ 0, & \textit{if } n \textit{ is even} \end{cases}$$

Remark 6.6.3. This shows that, unlike topological K-theory, the algebraic K-theory is **not** periodic.

If $\mathbb{F}_q \subset \mathbb{F}_{q'}$, then the induced maps $K_n(\mathbb{F}_q) \hookrightarrow K_n(\mathbb{F}_{q'})$ are injective, with

$$K_n(\mathbb{F}_q) \simeq K_n(\mathbb{F}_{q'})^G \subseteq K_n(\mathbb{F}_{q'})$$

where $G = \mathrm{Gal}(\mathbb{F}_{q'}/\mathbb{F}_q)$ is the Galois group.

Remark 6.6.4. All products in the ring $K_\bullet(\mathbb{F}_q)$ are trivial.

6.7 λ-operations in Higher K-theory

Recall from Section 6.2 that for a commutative ring A, the Grothendieck group $K_0(A)$ is equipped with λ-operations

$$\begin{aligned} \lambda^k\colon K_0(A) &\to K_0(A) \\ [P] &\mapsto [\Lambda_A^k P] \end{aligned}$$

making $K_0(A)$ a special λ-ring, see Definition 6.3.5. Our goal is to extend this λ-structure to higher K-group: $\lambda^k\colon K_n(A) \to K_n(A)$, $\forall n \geq 0$. We will use the "plus"-construction to define these operations.

General construction. Let G be an arbitrary group, P a finitely generated A-module, a faithful representation $\rho\colon G \to \operatorname{Aut}_A(P)$ of G into P, and we also choose an isomorphism

$$P \oplus Q \simeq A^N$$

We will identify G with its image under ρ. For any element $g \in G$, we can extend $g\colon P \to P$ to an automorphism $\widetilde{g}\colon A^N \to A^N$ by putting $\widetilde{g} = g \oplus \mathrm{id}_Q$. This gives a map of spaces

$$q(\rho)\colon BG \to B\operatorname{Aut}_A(P) \to BGL_N(A) \hookrightarrow BGL(A) \to BGL(A)^+ \tag{2.18}$$

A different choice of embedding $P \hookrightarrow A^N$ gives a map homotopic to the above, because the two maps only differ by conjugation, and $BGL(A)^+$ is an H-space. Thus, $q(\rho)$ is well-defined up to homotopy.

Remark 6.7.1. If H is any H-space, the action of $\pi_1(H)$ on $[X, H]$ is trivial for any space X, see [Whi78, III.4.18].

Example 6.7.2. If P is a finitely generated projective module of rank n over a commutative ring A, then $\Lambda^k_A(P)$ is a finitely generated projective A-module of rank $\binom{n}{k}$. Being a functor, Λ^k_A defines a map

$$\Lambda^k\colon \operatorname{Aut}_A(P) \to \operatorname{Aut}_A(\Lambda^k_A P).$$

In other words, it defines a representation of $G := \operatorname{Aut}_A(P)$ on $\Lambda^k_A(P)$ for every k. Therefore, it induces

$$\Lambda^k_P := q(\lambda^k)\colon B\operatorname{Aut}_A(P) \to BGL(A)^+$$

Note that $\Lambda^0_P = \{*\}$ is a constant map, because $\Lambda^0\colon \operatorname{Aut}_A(P) \to \operatorname{Aut}_A(A) = A^\times$ is a trivial map.

Any (connected) H-space H (in particular, $H = BGL(A)^+$) has a multiplicative inverse i, defined up to homotopy. Then, given a map $f\colon X \to H$, we can define

$$(-f)\colon X \xrightarrow{f} H \xrightarrow{i} H$$

and take a formal $\mathbb{Z}$-linear combination of maps $X \to H$.

Definition 6.7.3. *If P a projective module of $\operatorname{rk}(P) = n$ over a commutative ring A, we define $\lambda^k_P\colon B\operatorname{Aut}_A(P) \to BGL(A)^+$ by*

$$\lambda^k_P := \sum_{i=0}^{k-1} (-1)^i \binom{n+i-1}{i} \Lambda^{k-i}_P$$

To show that the maps λ^k_P are compatible with inclusions $P \hookrightarrow P \oplus Q \simeq A^N$ inducing the λ-maps $\lambda^k\colon BGL(A)^+ \to BGL(A)^+$, we will use the representation ring $R_A(G)$.

Recall (cf. [Hil81, Sect.1, p.242]) that $R_A(G) = K_0(A[G], A)$, where $K_0(A[G], A)$ is the group completion of the monoid $\mathsf{Rep}(A[G], A)$ of $A[G]$-modules P which are finitely generated projective modules over A. Here $A[G] = A \otimes_{\mathbb{Z}} \mathbb{Z}[G]$ is the group ring of G over A. The Λ^k-operation makes $R_A(G)$ into a *special λ-ring* with a positive structure.

We will think of elements of $K_0(A[G], A)$ as stable equivalence classes of pairs $[(P, \rho)]$, where P is a finitely generated projective A-module and $\rho\colon G \to \operatorname{Aut}_A P$ is an action of G on P.

Proposition 6.7.4. *If*

$$0 \to (P', \rho') \to (P, \rho) \to (P'', \rho'') \to 0$$

is an exact sequence of $A[G]$-modules, then

$$q(\rho) = q(\rho') + q(\rho'') \text{ in } [BG, BGL(A)^+]$$

for the map q defined by (2.18)*. Hence, there is a natural map*

$$\begin{aligned} q\colon R_A(G) &\to [BG, BGL(A)^+] \\ [(P, \rho)] &\mapsto [q(\rho)\colon BG \to BGL(A)^+] \end{aligned}$$

Proof. For the proof see [Hil81, Proposition 1.1]. □

Now, we proceed to define operations λ^k on $K_n(A)$ for any $n \geq 0$. Let $i_n\colon GL_n(A) \hookrightarrow GL_{n+1}(A)$ be the standard embedding, and let

$$i_n^*\colon R_A\left[GL_{n+1}(A)\right] \to R_A\left[GL_n(A)\right]$$

be the induced maps of representation rings. Thus,

$$i_n^*\colon \left[GL_{n+1}(A) \xrightarrow{\rho} \operatorname{Aut}_A P\right] \mapsto \left[GL_n(A) \overset{i_n}{\hookrightarrow} GL_{n+1}(A) \xrightarrow{\rho} \operatorname{Aut}_A P\right]$$

In other words, we have $i_n^*[\rho] := \rho \circ i_n$. Note that $i_n^*(\mathrm{id}_{n+1}) = \mathrm{id}_n \oplus 1$. Here id_n denotes the module A^n with the obvious $GL_n(A)$-module structure. Then, the *virtual characters*

$$\rho_n := \mathrm{id}_n - n \cdot 1 \in R_A(GL_n(A))$$

satisfy

$$\begin{aligned} i_n^* \rho_{n+1} &= i_n^*\left(\mathrm{id}_{n+1} - (n+1) \cdot 1\right) \\ &= \mathrm{id}_n \oplus 1 - (n+1) \cdot 1 \\ &= \mathrm{id}_n - n \cdot 1 \\ &= \rho_n \text{ for all } n \geqslant 0 \end{aligned}$$

Next, observe that the maps $i_n^*\colon R_A\left[GL_{n+1}(A)\right] \to R_A\left[GL_n(A)\right]$ are in fact homomorphisms of (special) λ-rings. Thus, by Proposition 6.7.4, we have a family of homotopy classes of maps:

$$\lambda_n^k := q(\lambda^k(\rho_n)) \in \left[BGL_n(A), BGL(A)^+\right]$$

which are compatible up to homotopy in the sense that the following diagram commutes (up to homotopy):

$$\begin{tikzcd} R_A\left[GL_{n+1}(A)\right] \arrow[r, "q"] \arrow[d, "i_n^*"'] & \left[BGL_{n+1}(A), BGL(A)^+\right] \arrow[d, "i_n^*"] \\ R_A\left[GL_n(A)\right] \arrow[r, "q"] & \left[BGL_n(A), BGL(A)^+\right] \end{tikzcd} \qquad \begin{tikzcd} \lambda^k(\rho_{n+1}) \arrow[r, mapsto] \arrow[d, mapsto] & \lambda^k_{n+1} \arrow[d, mapsto] \\ \lambda^k(\rho_n) \arrow[r, mapsto] & \lambda^k_n \end{tikzcd}$$

Since $BGL_n(A) \overset{i_n}{\hookrightarrow} BGL_{n+1}(A)$ are cofibrations of CW complexes (cofibrant objects), one can inductively construct maps $\lambda^k \colon BGL_n(A) \to BGL(A)^+$ which are *strictly compatible*, so that in the limit $n \to \infty$, they determine a continuous map

$$\lambda^k_\infty \colon BGL(A) \to BGL(A)^+.$$

These maps induce the required λ-operations

$$\lambda^k \colon BGL(A)^+ \to BGL(A)^+.$$

Remark 6.7.5. The map $q\colon R_A(G) \to [BG, BGL(A)^+]$ of Proposition 6.7.4 allows one to *define* operations in $[BG, BGL(A)^+]$ by applying q to operations in $R_A(G)$. It plays the same "universal" role as the Weil algebra in the theory of connections on principal bundles. The stabilization procedure described above allows to define a map

$$\varprojlim R_A\left[GL_n(A)\right] \to \varprojlim \left[BGL_n(A), BGL(A)^+\right] \simeq \left[BGL(A)^+, BGL(A)^+\right].$$

The Adams operations can be *defined* in this way.

Definition 6.7.6. *If X is any space with a basepoint, and $f\colon X \to BGL(A)^+$ is any map, we define*

$$\lambda^k(f) := \lambda^k \circ f \colon X \xrightarrow{f} BGL(A)^+ \xrightarrow{\lambda^k} BGL(A)^+.$$

When $X = \mathbb{S}^n$, $[X, BGL(A)^+] \simeq \pi_n\left(BGL(A)^+\right) = K_n(A)$. Hence, we get $\lambda^k \colon K_n(A) \to K_n(A)$, $\forall n \geq 1$.

Recall that the (homotopy commutative) H-space structure "+" on $BGL(A)^+$ is induced by the maps $\Box\colon GL_n(A) \times GL_m(A) \to GL_{n+m}(A) \hookrightarrow GL(A)$ by applying the classifying space functor B:

$$\begin{tikzcd} BGL_n(A) \times BGL_m(A) \arrow[r] \arrow[dr] & BGL(A) \arrow[r] & BGL(A)^+ \\ & BGL(A)^+ \wedge BGL(A)^+ \arrow[ur, dashed] & \end{tikzcd}$$

Thus, the set $[X, BGL(A)^+]$ is an abelian group (in the usual sense) for every space X. Now, $BGL(A)^+$ carries a natural product

$$\gamma\colon BGL(A)^+ \wedge BGL(A)^+ \to BGL(A \otimes A)^+ \to BGL(A)^+$$

called the Loday product, see Section 3.5 for details. This allows to make $[X, BGL(A)^+]$ into a commutative associative ring via the convolution product: if $f, g\colon X \to BGL(A)^+$, then $f \bullet g$ is given by

$$\begin{tikzcd} X \arrow[r, "f\bullet g"] \arrow[d, "\mathrm{diag}"'] & BGL(A)^+ \\ X \wedge X \arrow[r, "f\wedge g"] & BGL(A)^+ \wedge BGL(A)^+ \arrow[u, "\gamma"'] \end{tikzcd}$$

Remark 6.7.7. If $X = \mathbb{S}^n$ or $X\Sigma Y$, then for any $n \geq 1$ this product on $[X, BGL(A)^+]$ is *zero*, because in these cases the diagonal map $X \to X \wedge X$ is homotopic to the constant map. For $X = \mathbb{S}^1$ it is obvious. For $X = \Sigma Y$ an example of the required based homotopy $h\colon \Sigma Y \wedge I_+ \to \Sigma Y \wedge \Sigma Y$ is given by $((s, x), t) \mapsto ((s(1-t), x), (s(1-t), x))$.

Recall that in any λ-ring K, $\lambda^0(x) = 1$ for all x, which requires the existence of 1_k. In our case, $\lambda^0(x) = 0$ on $[X, BGL(A)^+]$. To fix this, we extend λ^k-operations to the K-theory space $K(A) = K_0(A) \times BGL(A)^+$ (see Definition 3.3.4) by the rule

$$\lambda^k(a, x) := \Big(\lambda^k(a), \lambda^k(x) + a \cdot \lambda^{k-1}(x) + \cdots + \lambda^i(a)\lambda^{k-i}(x) + \cdots + \lambda^{k-1}(a)x\Big)$$

so that $\lambda^0(a, x) = (\lambda^0(a), \lambda^0(x)) = (1, 0)$ as usual.

Theorem 6.7.8. *For any based space X, the operations λ^k make $K_0(A) \times [X, BGL(A)^+]$ a special λ-ring.*

Proof. It suffices to consider the universal case: $X = BGL(A)^+$. Recall that in this case, $\pi_1(X) = K_1(A) \simeq GL(A)/[GL(A), GL(A)]$. This gives a canonical diagram

$$\begin{tikzcd} R_A\,[K_1(A)] \arrow[r] & R_A\,[GL(A)] & {[BGL(A), BGL(A)^+]} \arrow[d, "\simeq"] \\ & {[X, X]} \arrow[r, equal] & {[BGL(A)^+, BGL(A)^+]} \end{tikzcd}$$

where the last map comes from Quillen Recognition Criterion 3.3.3 (saying that the canonical map $i\colon BGL(A) \to BGL(A)^+$ is universal among maps from $BGL(A)$ to H-spcaes in the homotopy category of spaces).

Since $GL_n(A) \hookrightarrow GL(A) \twoheadrightarrow K_1(A)$, the above map factores as

$$R_A(K_1(A)) \to \varprojlim R_A(GL_n(A)) \xrightarrow{q} [X, X]$$

Therefore, we can check the required identities in $R_A(GL_n(A))$, making sure they are compatible under the stabilization. For example, we know that

$$\lambda^k(x + y) = \sum \lambda^i(x)\lambda^{k-i}(y)$$

arises from $\lambda^k \circ \oplus = \sum \lambda^i \odot \lambda^{k-i}$ in $R_A\,[GL_m(A) \times GL_n(A)]$, and similarly the identities for $\lambda^k(xy)$ and $\lambda^n\left(\lambda^k(x)\right)$ hold in $R_A(G)$ (for the appropriate group G) and hence in $[X, X]$ via q. $\square$

For any based space X, we have an aspherical space $FX \simeq B[\pi_1(X)]$ together with a natural map

$$f\colon X \to FX$$

such that $f_*\colon \pi_1(X) \simeq \pi_1(FX)$. By composition, we have

$$R_A(\pi_1(X)) \xrightarrow{q} \left[B\pi_1(X), BGL(A)^+\right] \xrightarrow{f^*} \left[X, BGL(A)^+\right]$$

This defines a natural transformation of functors

$$q\colon R_A\left[\pi_1(-)\right] \to \left[-, BGL(A)^+\right]$$

from $\mathbf{Ho}(\mathbf{Top}_*)$ to **Groups**. The following theorem is proved in [Hil81, Cor. 2.3].

Theorem 6.7.9 (Quillen)**.** *The natural transformation q is universal for morphisms*

$$\eta_H\colon R_A\left[\pi_1(-)\right] \to [-, H]$$

to representable functors of H-spaces. Namely, for any connected H-space H, a natural transformation η_H as above induces a (unique up to homotopy) map $g\colon BGL(A)^+ \to H$ such that $\eta_H = g^* \circ q$, where g^* is the induced natural transformation $[-, BGL(A)^+] \xrightarrow{g^*} [-, H]$.

Example 6.7.10. The above construction allows one to construct λ-operations on $[X, BU]$, where $BU \simeq BGL(\mathbb{C})^{top}$. It follows that the natural map $[X, BGL(\mathbb{C})^+] \to [X, BU]$, relating algebraic and topological K-theory, commutes with λ^k and Ψ^k-operations.

Application. Let $\mathbb{F}_q$ be a finite field, and $\mathbb{F}_q^\times \to \mathbb{C}^\times$ a group homomorphism. This induces a map of representation rings $R_{\mathbb{F}_q}(G) \xrightarrow{\mathrm{Br}} R_{\mathbb{C}}(G)$ called the *Brower lift*. By composition, we have

$$R_{\mathbb{F}_q}\left[\pi_1(X)\right] \xrightarrow{\mathrm{Br}} R_{\mathbb{C}}\left[\pi_1(X)\right] \xrightarrow{q^{\mathbb{C}}} \left[X, BGL(\mathbb{C})^+\right]$$

Hence, by Theorem 6.7.9, there is a natural map

$$BGL(\mathbb{F}_q)^+ \to BGL(\mathbb{C})^+ \to BU$$

This map is exactly the map $\bar{b}$ appearing in Quillen Theorem 6.6.1 identifying $BGL(\mathbb{F}_q)^+$ with the homotopy fiber of $\Psi^q - 1\mathrm{id}\colon BU \to BU$. One can show that the Brower lifting br is actually a map of λ-rings so that the maps

$$\left[X, BGL(\mathbb{F}_q)^+\right] \to \left[X, BGL(\mathbb{C})^+\right] \to [X, BU]$$

are also maps of λ-rings for all based spaces X. This was used by Quillen (for $X = \mathbb{S}^n$) to compute $K_\bullet(A)$.

7 Final remarks on Algebraic K-theory

7.1 Historic remarks on algebraic and topological K-theory

Recall from Section 6.1 that for a (compact) topological space X one defines $K^0(X)$ as the group completion of the abelian monoid $\pi_0(\mathbb{V}B(X))$ of isomorphism classes of complex vector bundle on X with the monoid operation given by the direct sum $\oplus$ of vector bundles. For $i > 0$, one defines $K^{-i}(X) := K^0(\Sigma^i X)$, where Σ denotes the suspension of X.

Theorem 7.1.1 (Bott Periodicity)**.** *For all $i \geqslant 0$ there are isomorphisms $K^{-i}(X) \simeq K^{-i-2}(X)$.*

This allows one to define $K^i(X)$ for *all integers $i \in \mathbb{Z}$* simply by periodicity. The functor $K^{-i}\colon \mathbf{Spaces}^{\mathrm{op}} \to \mathbf{Ab}$ given by $X \mapsto K^{-i}(X)$ satisfies the following properties.

1. It is homotopy invariant in X.
2. If $X = X_1 \cup X_2$ for two open subsets $X_1, X_2 \subseteq X$, and $X_{12} := X_1 \cap X_2$, then there exists the following long exact sequence of groups
$$\cdots \to K^{-i-1}(X) \to K^{-i}(X_{12}) \to K^{-i}(X_1) \oplus K^{-i}(X_2) \to K^{-i}(X) \to \dots$$
called *Mayer–Vietoris sequence.*
3. There is an isomorphism $K^i(X) \otimes \mathbb{Q} \simeq \bigoplus_{k \equiv i \mod 2} \mathrm{H}^k(X, \mathbb{Q})$.

So $K^\bullet$ is a generalized cohomology theory.

One could try (as people did) to simply replace $\mathbf{Spaces}^{\mathrm{op}}$ by $\mathbf{Rings}$ and construct functors $K_i\colon \mathbf{Rings} \to \mathbf{Ab}$ by requiring the following analog of Mayer–Vietoris sequence. One replaces the diagram

$$\begin{tikzcd} X_{12} \arrow[r, hook] \arrow[d, hook] & X_2 \arrow[d, hook] \\ X_1 \arrow[r, hook] & X \end{tikzcd}$$

in **Spaces** by the following diagram in **Rings**

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[d, two heads] & A_1 \arrow[d, two heads] \\ A_2 \arrow[r, two heads] & A_{12} \end{tikzcd}$$

and then requires existence of the following long exact sequence of abelian groups

$$\cdots \to K_{i+1}(A_{12}) \to K_i(A) \to K_i(A_1) \oplus K_i(A_2) \to K_i(A_{12}) \to \dots$$

Unfortunately, as was proven by Swan in 1968, there is **no** non-trivial such cohomology theory on **Rings**!

What Quillen suggested was, instead of looking for a cohomology theory on **Rings**, one should *fix* a ring A, and then associate to it a functor $k_A\colon \mathbf{Spaces} \to \mathbf{Ab}$ which is homotopy invariant and satisfies Mayer-Vietoris property (so that $k_A(\mathbb{S}^n) \simeq K_n(A)$).

This idea was realized by Max Karoubi in 1987. The rest of this section will be devoted to his construction.

Fix a unital associative ring A, and let X be a topological space (CW complex).

Definition 7.1.2. *A (flat) A-bundle on X is a covering $P \to X$ s.t. every fiber P_x over each point $x \in X$ has a structure of finitely generated projective A-module.*

Let $\mathbf{VB}_A(X)$ be the category of such flat A-bundles with morphisms being *isomorphisms* of A-bundles (morphisms of A-bundles are defined in the obvious way). So $\mathbf{VB}_A(X)$ is a groupoid. Thus we get a functor

$$\begin{aligned} VB_A\colon \mathbf{Spaces}^{\mathrm{op}} &\to \mathbf{AbMon} \\ X &\mapsto \pi_0\left[\mathbf{VB}_A(X)\right] \end{aligned}$$

This is a contravariant homotopy functor (in X) satisfying Mayer-Vietoris property. This functor turns out to be representable,

$$VB_A(X) \simeq \mathrm{Hom}_{\mathbf{Ho(Top)}}(X, B\mathbf{S}) = [X, B\mathbf{S}],$$

where $[X, B\mathbf{S}]$ denotes the set of homotopy classes of maps from X to $B\mathbf{S}$, and $\mathbf{S} = \mathbf{Iso}(\mathbb{P}(A))$ is the groupoid associated to the category $\mathbb{P}(A)$ of finitely generated projective modules over A.

Definition 7.1.3. *A* virtual A-bundle *on X is a pair $(Y \xrightarrow{f} X, E \twoheadrightarrow Y)$ where $f\colon Y \to X$ is an acyclic map (see Definition 3.1.4) and $E \twoheadrightarrow Y$ is a flat A-bundle on Y.*

We will denote virtual bundles by $(E \twoheadrightarrow Y \to X)$. Two virtual bundles $(E \twoheadrightarrow Y \to X)$ and $(E' \twoheadrightarrow Y' \to X)$ are called *equivalent* if there exists commutative up to homotopy diagram of spaces of the form

$$\begin{tikzcd} & Z \arrow[dd, "g"] & \\ Y \arrow[ur, "h"] \arrow[dr, "f"'] & & Y' \arrow[ul, "h'"'] \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

where g is acyclic, and there exists a flat A-bundle $F \twoheadrightarrow Z$ such that $h^*F \simeq E$ and $(h')^*F \simeq E'$.

Given two virtual bundles $(E \twoheadrightarrow Y \to X)$ and $(E' \twoheadrightarrow Y' \to X)$ define $E \oplus E'$ by first defining $Z := \mathrm{holim}\{Y \to X \leftarrow Y'\}$, and then

$$E' \oplus E := \pi_Y^* E \oplus \pi_{Y'}^* E'$$

where $\pi_Y \colon Z \to Y$ and $\pi_{Y'} \to Y'$ are the natural maps.

Define $k_A(X) := \pi_0^{-1}\pi_0$, the group completion of the abelian monoid $\pi_0 := \pi_0\left[\mathbf{VB}^{\mathrm{vir}}(X)\right]$, where $\mathbf{VB}^{\mathrm{vir}}(X)$ similar as above denotes the groupoid of virtual A-bundles on X and their isomorphisms.

Theorem 7.1.4 (Karoubi'87,**REFERENCE?**). *The functor k_A is homotopy representable by $B(\mathbf{S}^{-1}\mathbf{S})$, i.e.*

$$k_A(X) \simeq [X, B(\mathbf{S}^{-1}\mathbf{S})] \simeq [X, K_0(A) \times BGL(A)^+]$$

where $\mathbf{S} = \mathbf{Iso}(\mathbb{P}(A))$.

Notice that $k_A(\mathbb{S}^n) \simeq K_0(A) \oplus K_n(A)$, $\forall n \geqslant 0$.

7.2 Remarks on delooping

Recall that given two based spaces (X, x_0), (Y, y_0) we define their bouquet $X \vee Y$ by

$$X \vee Y = X \sqcup Y \Big/ x_0 \sim y_0$$

Note that $X \vee Y$ is just the push-out, and hence there is a natural map $\iota \colon X \vee Y \hookrightarrow X \times Y$

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & Y \arrow[d] \arrow[ddr, bend left, "\iota_Y"] & \\
X \arrow[r] \arrow[drr, bend right, "\iota_X"'] & X \vee Y \arrow[dr, dashed, "\iota"] & \\
& & X \times Y
\end{tikzcd}$$

This allows us to define the *smash* product $X \wedge Y$ of X and Y by putting

$$X \wedge Y := (X \times Y)\Big/(X \vee Y)$$

Denoting by $\mathbf{Top}_*$ the category of pointed topological spaces, we have the following natural bijection

$$\mathrm{Hom}_{\mathbf{Top}_*}(X \wedge Y, Z) \simeq \mathrm{Hom}_{\mathbf{Top}_*}(X, \mathrm{Hom}_{\mathbf{Top}_*}(Y, Z))$$

for any pointed spaces X, Y, Z. Here $\mathrm{Hom}_{\mathbf{Top}_*}(Y, Z)$ is a topological space with compact-open topology.

Finally, we define loop space and suspension of a topological space X by $\Sigma X := X \wedge \mathbb{S}^1$ and $\Omega X = \mathrm{Hom}_{\mathbf{Top}_*}(\mathbb{S}^1, X)$, respectively. The constructions of loop space and suspension are obviously functorial, giving two functors $\Omega, \Sigma \colon \mathbf{Top}_* \to \mathbf{Top}_*$. Moreover, they are actually adjoint to one another, giving the adjunction

$$\mathrm{Hom}_{\mathbf{Top}_*}(\Sigma X, Y) \simeq \mathrm{Hom}_{\mathbf{Top}_*}(X, \Omega Y)$$

The bijection of Hom-sets above is not only a bijection, but is an actual homeomorphism of topological spaces. Taking π_0 of both sides gives the bijection

$$[\Sigma X, Y] \simeq [X, \Omega Y]$$

Definition 7.2.1. *A* sppectrum $\mathbb{X}$ *is a sequence* $\mathbb{X} = \{X_0, X_1, \dots\}$ *of pointed topological spaces given together with bonding maps* $\sigma_i \colon \Sigma X_i \to X_{i+1}$, $\forall i \geqslant 0$.

Example 7.2.2. For any topological space $X \in \mathbf{Top}_*$ one obtains the *suspesion spectrum* $\Sigma^\infty X := \{X, \Sigma X, \Sigma^2 X, \dots\}$ with all bonding maps σ_i being identity maps. The suspension spectrum construction gives a functor $\Sigma^\infty \colon \mathbf{Spaces} \to \mathbf{Spectra}$ (we do not specify what *is* really the category **Spectra**).

Note that using the adjunction $\mathrm{Hom}_{\mathbf{Top}_*}(\Sigma X, Y) \simeq \mathrm{Hom}_{\mathbf{Top}_*}(X, \Omega Y)$, bonding maps σ_i give rise to some maps $\sigma_i^* \colon X_i \to \Omega X_{i+1}$.

Definition 7.2.3. *An* Ω*-spectrum* $\mathbb{X}$ *is a spectrum s.t. the adjoint bonding maps*

$$\sigma_i^* \colon X_i \xrightarrow{\sim} \Omega X_{i+1}$$

are homotopy equivalences.

A space X is called an *infinite loop space* if it is the 0-th component of an Ω-spectrum. The process of constructing other components X_i, $i \geqslant 1$ of this spectrum is called *delooping* of $X = X_0$.

Example 7.2.4. Recall that a K-space for an associative unital ring A is

$$K(A) = K_0(A) \times BGL(A)^+ =: X_0$$

The Q-construction applied to the category $\mathbb{P}(A)$ produces another space, $BQ\mathbb{P}(A) =: X_1$. Then "plus=Q" theorem tells that

$$X_0 \simeq \Omega X_1$$

Thus it is natural to ask whether the K-space X_0 is actually the 0-th term of an Ω-spectrum, whose first term is given by the space $X_1 = BQ\mathbb{P}(A)$.

Theorem 7.2.5 (Reference?). *There is a sequences of spaces starting with* $X_0 = K(A)$ *and* $X_1 = BQ\mathbb{P}(A)$ *forming an* Ω*-spectrum* $\mathbb{K}(A)$,

$$\mathbb{K} \colon \mathbf{Rings} \to \mathbf{Spectra}$$

Remark 7.2.6. There is a generalization of Q-construction which produces all the X_i's.

Chapter 3

Introduction to model categories

Our main reference for the material of this chapter is a very nice survey paper by Dwyer and Spaliński [DS95]. Besides that, we will use Quillen's original paper [Qui67], the book [MP12] of May and Ponto, and Appendix A of the paper [BFR14] by Berest, Felder and Ramadoss. For a deeper discussion of the subject we refer to books [Hov99] by Hovey and [Hir03] by Hirschhorn.

1 Model categories

1.1 Axioms

A *(closed) model category* $\mathbf{C}$ is a category with three distinguished classes of morphisms, denoted $\mathrm{Fib}, \mathrm{Cof}, \mathrm{WE}$ and called the *fibrations, cofibrations and weak equivalences* respectively. We will usually denote the fibrations by $\twoheadrightarrow$, the cofibrations by $\hookrightarrow$ and the weak equivalences by $\xrightarrow{\sim}$. The fibrations that are at the same time weak equivalences are called *acyclic fibrations* and will be denoted by $\overset{\sim}{\twoheadrightarrow}$. Similarly, the cofibrations which are weak equivalences will be called *acyclic cofibrations*, and will be denoted by $\overset{\sim}{\hookrightarrow}$. We will sometimes denote model categories as a tuple $(\mathbf{C}, \mathrm{WE}, \mathrm{Fib}, \mathrm{Cof})$ to emphasize the additional structure, but most of the time we will just use the notation $\mathbf{C}$, assuming that the specific classes $\mathrm{WE}, \mathrm{Fib}, \mathrm{Cof}$ are understood.

The classes $\mathrm{Fib}, \mathrm{Cof}, \mathrm{WE}$ are closed under compositions, contain all the isomorphisms, and moreover the following five axioms are satisfied.

MC1 The category $\mathbf{C}$ has all finite limits and colimits. In particular, $\mathbf{C}$ has both initial and terminal objects, denoted by e and $*$ respectively.

MC2 If $f\colon X \to Y$ and $g\colon Y \to Z$ are two composable morphisms in $\mathbf{C}$, then if two out of the three morphisms f, g, gf are in WE, then so is the third morphism. This axiom is called "*2-out-of-3 axiom.*"

MC3 All three classes $\mathrm{WE}, \mathrm{Fib}, \mathrm{Cof}$ are closed under taking retracts. Recall that $f\colon X \to Y$ is called a *retract* of $g\colon X' \to Y'$ if there is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r] & Y' \arrow[r] & Y \end{tikzcd}$$

in which the rows compose to the identity morphisms id_X and id_Y, respectively.

MC4 Consider the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "f"'] & X6 - g \arrow[d, two heads] \\ B \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

with $f \in \mathrm{Cof}$ and $g \in \mathrm{Fib}$. If one of the maps f or g is in addition a *weak equivalence*, then there exists a lifting $h\colon B \to X$ making the diagram commutative.

MC5 Every morphism $f\colon A \to X$ in $\mathbf{C}$ can be factored as a composition $A \overset{\sim}{\hookrightarrow} B \twoheadrightarrow X$ of an acyclic cofibraion followed by a fibration, and as a composition $A \hookrightarrow Y \overset{\sim}{\twoheadrightarrow} X$ of a cofibration followed by an acyclic fibration.

Whenever a lifting in Axiom MC4 exists, we say that f has a *left lifting property* (LLP) with respect to g, and g has a *right lifting property* (RLP) with respect to f. We denote the class of all morphisms that have LLP w.r.t. the acyclic fibrations by $\mathrm{Fib} \cap \mathrm{WE}$ by $LLP(\mathrm{Fib} \cap \mathrm{WE})$. Similarly, the class of morphisms having RLP w.r.t. the acyclic cofibrations will be denoted by $RLP(\mathrm{Cof} \cap \mathrm{WE})$. Then Axiom MC4 says that $\mathrm{Cof} \subseteq LLP(\mathrm{Fib} \cap \mathrm{WE})$ and $\mathrm{Fib} \subseteq RLP(\mathrm{Cof} \cap \mathrm{WE})$.

Remark 1.1.1. A model category is called *bicomplete* when it has all small limits and colimits. A model category is called *factorable* if the factorizations provided by Axiom MC5 are functorial. Most model categories are bicomplete and factorable, and these conditions are often incorporated into the axioms (see [Hir03] and [Hov99]).

Let $e \in \mathrm{Ob}(\mathbf{C})$ and $* \in \mathrm{Ob}(\mathbf{C})$ denote the initial and terminal object of $\mathbf{C}$. Axiom MC1 assures they always exist. Then for any $A \in \mathrm{Ob}(\mathbf{C})$ there are unique morphisms $e \to A$ and $A \to *$.

Definition 1.1.2. *We say that an object $A \in \mathrm{Ob}(\mathbf{C})$ is* cofibrant *if the unique morphism $e \to A$ is a cofibration. Similarly, we say that A is* fibrant *if the morphism $A \to *$ is a fibration.*

We say that a model category $\mathbf{C}$ is cofibrant *(resp.* fibrant*) if every object $A \in \mathrm{Ob}(\mathbf{C})$ is cofibrant (resp. fibrant).*

1.2 Examples of model categories

Topological spaces

The category **Top** of topological spaces has several natural model structures.

1. **Quillen model structure.** Let WE be the class of *weak homotopy equivalences*, i.e. maps $f\colon X \to Y$ of topological spaces inducing isomorphisms $f_*\colon \pi_i(X, x_0) \xrightarrow{\sim} \pi_i(Y, f(x_0))$, $i \geqslant 0$ on all homotopy groups.

   We take Fib to be the class of all Serre fibrations (see Definition 3.0.1). In other words, Fib is the class of maps of topological spaces which have LLP with respect to inclusions of CW complexes. We take Cof to be the class Cof $= LLP(\mathrm{Fib} \cap \mathrm{WE})$.

   Theorem 1.2.1. *Cofibrations in the Quillen model structure on* **Top** *are the so-called* generalized relative CW inclusions*, i.e. maps $X_0 \to \operatorname{colim}_n X_n$, where*

   $$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \dots X_n \hookrightarrow \dots$$

   such that (X_n, X_{n+1}) is a relative CW complex, $\forall n \geqslant 0$. In other words, X_{n+1} is obtained from X_n by attaching cells.

   Corollary 1.2.2. *(a) The category* **Top** *with the Quillen model structure is a* fibrant *model category.*

   (b) Cofibrant objects are CW complexes.

   (c) **Ho**(**Top**) *is the usual homotopy category.*

2. **Strøm model structure.** Let WE be the class of the usual homotopy equivalences. So the class WE is strictly bigger than the class of weak equivalences in Quillen model structure on **Top**.

   Let Fib be the class of Hurewich fibrations, see Remark 3.0.3. Again we put Cof to be Cof $= LLP(\mathrm{Fib} \cap \mathrm{WE})$.

   Theorem 1.2.3 (Strøm,[Str72])**.** *The class* Cof *of cofibrations consists precisely of Hurewich cofibration. In other words, cofibrations are closed embeddings $f\colon A \hookrightarrow B$ such that for any $Y \in \mathrm{Ob}(\mathbf{Top})$ there exists a lifting*

   $$\begin{tikzcd} B \times \{0\} \sqcup A \times I \arrow[r] \arrow[d, "\mathrm{id}\times i_0"', "f\times\mathrm{id}"] & Y \arrow[d] \\ B \times I \arrow[r] \arrow[ur, dashed, "\exists h"'] & * \end{tikzcd}$$

   With Strøm model structure, the category **Top** is both fibrant and cofibrant.

3. **Mixed model structure.** There is actually third model structure on **Top**, which is in some sense a mixture of the previous two. We will not discuss it here, and we refer reader to the book [MP12] for the details.

Complexes

Let A be a unital associative ring, and take $\mathbf{C} = \mathbf{Com}^+(A)$ be the category of non-negatively graded chain (i.e. with differential of degree -1) complexes of A-modules.

Let WE be the class of all quasi-isomorphisms, i.e. morphisms $f\bullet\colon X_\bullet \to Y_\bullet$ of complexes inducing isomorphism in homology.

Let Fib be the class if maps $f_\bullet\colon X_\bullet \to Y_\bullet$ such that $f_n\colon X_n \to B_n$ is a surjective map of A-modules, in every strictly positive degree $n > 0$. Once again, we put $\mathrm{Cof} = LLP(\mathrm{Fib} \cap \mathrm{WE})$.

Theorem 1.2.4. *The class* Cof *consists of all maps $f\bullet\colon X_\bullet \to Y_\bullet$ of complexes such that $f_n\colon X_n \to Y_n$ is* injective *and* $\mathrm{Coker}(f_n)$ *is a projective A-module, for all $n \geqslant 0$.*

Corollary 1.2.5. *The category $\mathbf{Com}^+(A)$ with the model structure described above is a fibrant model category. Cofibrant objects are precisely the complexes $X_\bullet$ with all components X_n being projective A-modules.*

This model structure is called *projective* model structure on $\mathbf{Com}^+(A)$. There is a dual model structure, called *injective*, but we will not discuss it here.

1.3 Natural constructions

Duality

Notice that the axioms MC1–MC5 are symmetric with respect to fibrations and cofibrations. As an immediate corollary we obtain that if $(\mathbf{C}, \mathrm{WE}, \mathrm{Fib}, \mathrm{Cof})$ is a model category, then the opposite category $(\mathbf{C}^{\mathrm{op}}, \mathrm{WE}, \mathrm{Cof}, \mathrm{Fib})$ is again a model category. This phenomenon is sometimes called the *Eckmann–Hilton duality.*

Comma categories

If $d \in \mathrm{Ob}(\mathbf{C})$ is an object of a model category $\mathbf{C}$, then both comma categories $d\backslash\mathbf{C}$ and $\mathbf{C}/d$ have a natural model structure. Namely, a morphism

$$\begin{tikzcd} & d \arrow[dl] \arrow[dr] & \\ A \arrow[rr] & & B \end{tikzcd}$$

in $d\backslash\mathbf{C}$ is in $\mathrm{WE}, \mathrm{Fib}, \mathrm{Cof}$ if and only if $A \to B$ is in $\mathrm{WE}, \mathrm{Fib}, \mathrm{Cof}$ in $\mathbf{C}$, respectively. Similarly for the category $\mathbf{C}/d$.

Morphism category

If $\mathbf{C}$ is a model category, then the morphism category $\mathbf{Mor}(\mathbf{C})$ of morphisms in $\mathbf{C}$ has a natural model structure. Consider a morphism $(\alpha, \beta) \in \mathrm{Mor}(\mathbf{Mor}(\mathbf{C}))$ given by the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r, "\beta"] & B' \end{tikzcd}$$

Then we define $(\alpha, \beta) \in \mathrm{WE}(\mathbf{Mor}(\mathbf{C}))$ (resp. $\mathrm{Fib}(\mathbf{Mor}(\mathbf{C}))$) if and only if $\alpha, \beta \in \mathrm{WE}(\mathbf{C})$ (resp. $\mathrm{Fib}(\mathbf{C})$). Cofibrations in $\mathbf{Mor}(\mathbf{C})$ have a bit trickier description. Namely, $(\alpha, \beta) \in \mathrm{Cof}(\mathbf{Mor}(\mathbf{C}))$ if and only if $\alpha \in \mathrm{Cof}(\mathbf{C})$ and the map g defined by the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook, "\alpha"'] & B \arrow[d] \arrow[ddr, bend left, "\beta"] & \\
A' \arrow[r] \arrow[drr, bend right, "f'"'] & A \sqcup_{A'} B \arrow[dr, dashed, "g"] & \\
 & & B'
\end{tikzcd}$$

is also a cofibration in $\mathbf{C}$.

Remark 1.3.1. Note that $\mathbf{Mor}(\mathbf{C})$ is actually a functor category,

$$\mathbf{Mor}(\mathbf{C}) = \mathbf{Fun}(\bullet \to \bullet, \mathbf{C})$$

.

Push-out data

Let $\mathbf{D}$ be the category with $\mathrm{Ob}(\mathbf{D}) = \{a, b, c\}$ and the only non-identity morphisms in $\mathbf{D}$ being the two morphisms $b \to a$ and $b \to c$. We denote schematically this category by $\mathbf{D} = \{a \leftarrow b \to c\}$. Let $\mathbf{C}$ be a category, and consider the functor category $\mathbf{C}^{\mathbf{D}} = \mathbf{Fun}(\mathbf{D}, \mathbf{C})$. This category is called the *category of push-out data*.

If $\mathbf{C}$ has a model structure, then the category $\mathbf{C}^{\mathbf{D}}$ also inherits a natural model structure. A morphism $[f \colon X \to Y] \in \mathrm{Mor}(\mathbf{C}^{\mathbf{D}})$ is represented by a diagram

$$\begin{tikzcd}
X(a) \arrow[d, "f_a"'] & X(b) \arrow[l] \arrow[r] \arrow[d, "f_b"'] & X(c) \arrow[d, "f_c"'] \\
Y(a) & Y(b) \arrow[l] \arrow[r] & Y(c)
\end{tikzcd}$$

Then $f \in \mathrm{Mor}(\mathbf{C}^{\mathbf{D}})$ is a *weak equivalence* (resp. a fibration) in $\mathbf{C}^{\mathbf{D}}$ if and only if all three f_a, f_b, f_c are weak equivalences (resp. fibrations) in $\mathbf{C}$. Finally, $f \in \mathrm{Cof}(\mathbf{C}^{\mathbf{D}})$ if and only if the induced maps

$$f_{ab} \colon X(a) \sqcup_{X(c)} Y(b) \to Y(a)$$
$$f_{bc} \colon X(c) \sqcup_{X(b)} Y(b) \to Y(c)$$

are cofibrations in $\mathbf{C}$ and also f_b is a cofibration in $\mathbf{C}$.

There is a similar model structure on the category of *pull-back data* $\mathbf{C}^{\mathbf{D}}$ with $\mathbf{D} = \{a \to b \leftarrow c\}$.

Remark 1.3.2. In general, for arbitrary model category $\mathbf{C}$ not every functor category $\mathbf{C}^{\mathbf{D}}$ (even for $\mathbf{D}$ finite!) has a model structure. But for "good" categories $\mathbf{C}$ this is the case.

Moreover, if $\mathbf{D}$ is *very finite*, i.e. in any chain of composable morphisms

$$X_0 \to X_1 \to \cdots \to X_n$$

if n is bigger then some fixed number N, then for *any* model category $\mathbf{C}$ the functor category $\mathbf{C}^{\mathbf{D}}$ has an induced model structure.

2 Formal consequences of axioms

2.1 Lifting properties

Proposition 2.1.1. *In any model category* $(\mathbf{C}, \mathrm{WE}, \mathrm{Fib}, \mathrm{Cof})$*,*

(1) $\mathrm{Cof} = LLP(\mathrm{Fib} \cap \mathrm{WE})$*;*

(1') $\mathrm{Cof} \cap \mathrm{WE} = LLP(\mathrm{Fib})$*;*

(2) $\mathrm{Fib} = RLP(\mathrm{Cof} \cap \mathrm{WE})$*;*

(2') $\mathrm{Fib} \cap \mathrm{WE} = RLP(\mathrm{Cof})$*.*

Proof. We will only prove (1). The other equalities are proved similarly. The inclusion $\mathrm{Cof} \subseteq LLP(\mathrm{Fib} \cap \mathrm{WE})$ is part of Axiom MC4. So we need to show that

$$LLP(\mathrm{Fib} \cap \mathrm{WE}) \subseteq \mathrm{Cof}$$

Assume $[f\colon X \to Y] \in LLP(\mathrm{Fib} \cap \mathrm{WE})$. By Axiom MC5, we can factor f as $f\colon X \hookrightarrow A \overset{\sim}{\twoheadrightarrow} Y$. Call the corresponding maps $i\colon X \hookrightarrow A$ and $p\colon A \overset{\sim}{\twoheadrightarrow} Y$. By the assumption, there exists a map $g\colon Y \to A$ lifting the identity map id_Y:

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "f"'] & A \arrow[d, two heads, "p"', "\sim"] \\ Y \arrow[r, equal] \arrow[ur, dashed, "\exists g"] & Y \end{tikzcd}$$

Hence, we have the following commutative diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r, equal] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & A \arrow[r, "p"] & Y \end{tikzcd}$$

where $pg = \mathrm{id}_Y$. Hence, f is a retract of i, and so by Axiom MC3 must be a cofibration since so is i. $\square$

Remark 2.1.2. Proposition 2.1.1 shows that the axiomatics of model categories is not minimal. Indeed, the two classes $\{\mathrm{Cof}, \mathrm{WE}\}$ determine Fib, and $\{\mathrm{Fib}, \mathrm{WE}\}$ determine Cof. In practice, one usually describes either $\{\mathrm{Cof}, \mathrm{WE}\}$ or $\{\mathrm{Fib}, \mathrm{WE}\}$, whichever is easier to describe.

Proposition 2.1.3. *In any model category* $\mathbf{C}$*,*

(1) Cof *and* $\mathrm{Cof} \cap \mathrm{WE}$ *are preserved under arbitrary push-outs;*

(2) Fib *and* $\mathrm{Fib} \cap \mathrm{WE}$ *are preserved under arbitrary pull-backs.*

Proof. Note that (2) follows from (1) by duality, i.e. by applying (1) to the category $\mathbf{C}^{\mathrm{op}}$. To prove (1), for any $f\colon X \to Y$ we need to show that in the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \arrow[d, hook, "j"] \\ A \arrow[r, "q"] & A \sqcup_X Y \end{tikzcd} \tag{3.1}$$

the map $i \in \mathrm{Cof}$ implies that j is also in Cof. Note that $A \sqcup_X Y$ exists in $\mathbf{C}$ by the axiom MC1. We show that $j \in LLP(\mathrm{Fib} \cap \mathrm{WE})$ and then use part (1) of Proposition 2.1.1 to conclude $j \in \mathrm{Cof}$. Consider the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "a"] \arrow[d, "j"'] & E \arrow[d, "\sim"', "p"] \\ A \sqcup_X Y \arrow[r, "b"] & B \end{tikzcd}$$

where $p \in \mathrm{Fib} \cap \mathrm{WE}$. Enlarge this diagram by adjoining diagram (3.1) on the left:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \arrow[r, "a"] \arrow[d, "j"] & E \arrow[d, "\sim"', "p"] \\ A \arrow[r, "q"'] \arrow[urr] & A \sqcup_X Y \arrow[r, "b"'] \arrow[ur, dashed, "\widetilde{h}"] & B \end{tikzcd}$$

Since we assumed i to be a cofibration, axiom MC4 implies that there exists a map $h\colon A \to E$ lifting $q \circ b$ and making the diagram commutative, i.e. $p \circ h = b \circ g$ and $h \circ i = a \circ f$.

By the universal property of push-outs, the maps $h\colon A \to E$ and $a\colon Y \to E$ induce a map $\widetilde{h}\colon A \sqcup_X Y \to E$

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "h"'] & A \sqcup_X Y \arrow[d, dashed, "\widetilde{h}"] & Y \arrow[l, "j"'] \arrow[dl, "a"] \\ & E & \end{tikzcd}$$

Hence $j \in LLP(\mathrm{Fib} \cap \mathrm{WE})$ as required. $\square$

Remark 2.1.4. In practice, it is often desirable to have *all* WE's preserved under push-outs and pull-backs. This *does not* happen automatically, i.e. it does not formally follow from the axioms.

A model category $\mathbf{C}$ is called *left proper* if for any $i \in \mathrm{Cof}$ and $f \in \mathrm{WE}$ the corresponding push-forward $\widetilde{f}$ given by

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "f"', "\sim"] & Y \arrow[d, "\widetilde{f}", "\sim"'] \\ A \arrow[r] & A \sqcup_X Y \end{tikzcd}$$

is also a weak equivalence, $\widetilde{f} \in \mathrm{WE}$. Similarly, a model category $\mathbf{C}$ is called *right proper* if for any fibration $p \in \mathrm{Fib}$ and any weak equivalence $f \in \mathrm{WE}$ the corresponding pull-back map $\widetilde{f}$ defined by the diagram

$$\begin{tikzcd} X \times_A Y \arrow[r] \arrow[d, "\widetilde{f}"', "\sim"] & X \arrow[d, "f"', "\sim"] \\ Y \arrow[r, two heads, "p"] & A \end{tikzcd}$$

is also a weak equivalence, $\widetilde{f} \in \mathrm{WE}$. A *proper* model category is a category which is both left and right proper. Most interesting model categories are proper but this usually needs a proof.

2.2 Homotopy equivalence relations

Recall that for any $A \in \mathrm{Ob}(\mathbf{C})$, the *folding map* $\nabla\colon A \sqcup A \to A$ is defined as

$$\begin{tikzcd} e \arrow[r] \arrow[d] & A \arrow[d, "j_2"] \arrow[ddr, bend left, "\mathrm{id}_A"] & \\ A \arrow[r, "j_1"] \arrow[drr, bend right, "\mathrm{id}_A"'] & A \sqcup A \arrow[dr, "\nabla"] & \\ & & A \end{tikzcd}$$

Factorizing ∇, we define a *cylinder object* $Cyl(A)$ over A by

$$A \sqcup A \underset{i}{\to} Cyl(A) \underset{p}{\overset{\sim}{\to}} A$$

Notice that we do not require necessarily that i is a cofibration or that p is a fibration, only that p is a weak equivalence. Moreover, we do not require $Cyl(A)$ to be *functorial* in A, even though the notation might suggest the opposite.

Definition 2.2.1. *A cylinder object $Cyl(A)$ is called* good *if $i \in \mathrm{Cof}$. It's called* very good *if both $i \in \mathrm{Cof}$ and $p \in \mathrm{Fib} \cap \mathrm{WE}$.*

Axiom MC5 implies that a very good cylinder exist for any $A \in \mathrm{Ob}(\mathbf{C})$ but it doesn't have to be neither canonical not functorial in A.

Notation 2.2.2. We write $i_1 := i \circ j_1$ and $i_2 := i \circ j_2$ for the corresponding maps $A \to Cyl(A)$.

Lemma 2.2.3. *If A is cofibrant and $Cyl(A)$ is good, then $i_1, i_2 \in \mathrm{Cof} \cap \mathrm{WE}$, i.e. we have*

$$\begin{tikzcd} & A \sqcup A \arrow[d, hook, "i"] & \\ A \arrow[ur, "j_1"] \arrow[r, hook, "\sim", "i_1"'] & Cyl(A) & A \arrow[ul, "j_2"'] \arrow[l, hook', "\sim"', "i_2"] \end{tikzcd}$$

Proof. We have, by definition,

$$\mathrm{id}_A \colon\; A \xrightarrow[i_k]{} Cyl(A) \xrightarrow[p]{\sim} A, \qquad (k = 1, 2)$$

By the axiom MC4, since $p \in \mathrm{WE}$ and $\mathrm{id}_A \in \mathrm{WE}$ we conclude that both $i_1, i_2 \in \mathrm{WE}$.

Note that j_1 and j_2 are defined formally by the push-out square

$$\begin{tikzcd} e \arrow[r] \arrow[d] & A \arrow[d, "j_1"] \\ A \arrow[r, "j_2"'] & A \sqcup A \end{tikzcd}$$

Hence, by Proposition 2.1.3, A is cofibrant implies that both j_1, j_2 are cofibrations. on the other hand, $Cyl(A)$ is good, which by definition means that $i \in \mathrm{Cof}$. Combining this together, we conclude

$$i_1 := i \circ j_1 \in \mathrm{Cof}$$
$$i_2 := i \circ j_2 \in \mathrm{Cof}$$

□

Remark 2.2.4. In **Top**, every $A \in \mathrm{Ob}(\mathbf{Top})$ has a *canonical* cylinder $Cyl(A) = A \times [0, 1]$. However, unless A is a CW complex, this cylinder is *not good.*

Definition 2.2.5. *If $f, g \in \mathrm{Hom}_{\mathbf{C}}(A, X)$, a* left homotopy *from f to g is defined to be a map $H \colon Cyl(A) \to X$ such that the following diagram commutes:*

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[dr, "f"'] & Cyl(A) \arrow[d, "H"'] & A \arrow[l, "i_2"'] \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

A left homotopy H is called good *(resp.,* very good*) if $Cyl(A)$ is good (resp. very good). We write $f \overset{l}{\sim} g$ if there exists a left homotopy $f \overset{H}{\rightsquigarrow} g$.*

It is easy to show that

(1) If $f \overset{l}{\sim} g\colon A \to X$, then there is a *good* left homotopy $f \overset{H}{\rightsquigarrow} g$.

(2) If X is fibrant, and $f \overset{l}{\sim} g$, then there is a *very good* left homotopy $f \overset{H}{\rightsquigarrow} g$.

Notation 2.2.6. For any $A, X \in \mathrm{Ob}(\mathbf{C})$ we write $\pi^l(A, X) := \mathrm{Hom}_{\mathbf{C}}(A, X) \big/ \sim$, where $\sim$ is the equivalence relation *generated by* $\overset{l}{\sim}$.

Theorem 2.2.7 (Properties of left homotopy)**.** *(1) Assume that A is cofibrant. Then*

(1a) For any $X \in \mathrm{Ob}(\mathbf{C})$, "$\overset{l}{\sim}$" is an equivalence relation on $\mathrm{Hom}_{\mathbf{C}}(A, X)$, and of course

$$\mathrm{Hom}_{\mathbf{C}}(A, X) \big/ \overset{l}{\sim} \; = \pi^l(A, X)$$

(1b) Any $p\colon Y \overset{\sim}{\twoheadrightarrow} X$ induces a bijection

$$p_*\colon \pi^l(A, Y) \xrightarrow{\simeq} \pi^l(A, X)$$

(2) Assume that X is fibrant. Then

(2a) if $f \overset{l}{\sim} g\colon A \to X$ and $h\colon A' \to A$, then

$$fh \overset{l}{\sim} gh\colon A' \to X$$

(2b) if $f \overset{l}{\sim} g\colon B \to X$ and $h \overset{l}{\sim} k\colon A \to B$, then

$$fh \overset{l}{\sim} gk\colon A \to X$$

Moreover, the composition in $\mathbf{C}$ induces

$$\begin{aligned} \pi^l(A, B) \times \pi^l(B, X) &\to \pi^l(A, X) \\ ([h], [f]) &\mapsto [fh] \end{aligned}$$

Recall that the diagonal map $\Delta\colon X \to X \times X$ is defined by

$$\begin{tikzcd}
X \arrow[dr, "\Delta"] \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\
& X \times X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \arrow[d] \\
& X \arrow[r] & *
\end{tikzcd}$$

Definition 2.2.8. *A* path object PX *on* X *is defined by factorization of* Δ*:*

$$\Delta\colon X \underset{j}{\xrightarrow{\sim}} PX \underset{q}{\rightarrow} X \times X$$

We write

$$q_1 := p_1 \circ q\colon PX \underset{q}{\rightarrow} X \times X \underset{p_1}{\longrightarrow} X$$
$$q_2 := p_2 \circ q\colon PX \underset{q}{\rightarrow} X \times X \underset{p_2}{\longrightarrow} X$$

A path object PX is said to be *good* if $q \in \mathrm{Fib}$. It is said to be *very good* if $q \in \mathrm{Fib}$ and $j \in \mathrm{Cof} \cap \mathrm{WE}$.

Definition 2.2.9. *A* right homotopy $f \overset{K}{\rightsquigarrow} g\colon A \to X$ *is defined to be a map* $K\colon A \to PX$ *such that*

$$\begin{tikzcd} X & PX \arrow[l, "q_1"'] \arrow[r, "q_2"] & X \\ & A \arrow[ul, "f"] \arrow[u, "K"] \arrow[ur, "g"'] & \end{tikzcd}$$

We write $f \overset{r}{\sim} g$ if such K does exist, and denote

$$\pi^r(A, X) := \mathrm{Hom}_{\mathbf{C}}(A, X) \big/ \sim$$

where "$\sim$" is the equivalence relation generated by $\overset{r}{\sim}$.

Theorem 2.2.10 (Properties of right homotopy)**.** *(1) Assume that* X *is fibrant. Then*

(1a) For any $A \in \mathrm{Ob}(\mathbf{C})$*, "*$\overset{r}{\sim}$*" is an equivalence relation on* $\mathrm{Hom}_{\mathbf{C}}(A, X)$*, and*

$$\mathrm{Hom}_{\mathbf{C}}(A, X) \big/ \overset{r}{\sim} \; = \pi^r(A, X)$$

(1b) Any $i\colon A \overset{\sim}{\hookrightarrow} B$ *induces a bijection*

$$i^*\colon \pi^r(B, X) \xrightarrow{\simeq} \pi^r(A, X)$$

(2) Assume that A *is cofibrant. Then*

(2a) if $f \overset{r}{\sim} g\colon A \to X$ *and* $h\colon X \to X'$*, then*

$$hf \overset{r}{\sim} hg\colon A \to X'$$

(2b) if $f \overset{r}{\sim} g\colon A \to X$ *and* $h \overset{r}{\sim} k\colon X \to Y$*, then*

$$hf \overset{r}{\sim} kg\colon A \to Y$$

Moreover, the composition in $\mathbf{C}$ *induces*

$$\begin{aligned} \pi^r(A, X) \times \pi^r(X, Y) &\to \pi^r(A, X) \\ ([f], [h]) &\mapsto [hf] \end{aligned}$$

Proposition 2.2.11. *Let* $f, g \in \mathrm{Hom}_{\mathbf{C}}(A, X)$*. Then*

(1) If A *is cofibrant, then*

$$f \overset{l}{\sim} g \Rightarrow f \overset{r}{\sim} g$$

(2) If X *is fibrant, then*

$$f \overset{r}{\sim} g \Rightarrow f \overset{l}{\sim} g$$

The following key definition is a consequence of Theorem 2.2.7, Theorem 2.2.10 and Proposition 2.2.11.

Definition 2.2.12. *Suppose* $A \in \mathrm{Ob}(\mathbf{C})$ *is a cofibrant, and* $X \in \mathrm{Ob}(\mathbf{C})$ *is a fibrant object of* $\mathbf{C}$*. Then there is an equivalence relation* $\sim$ *on* $\mathrm{Hom}_{\mathbf{C}}(A, X)$ *such that*

$$f \overset{l}{\sim} g \Leftrightarrow f \sim g \Leftrightarrow f \overset{r}{\sim} g$$

We say that f *and* g *are* homotopic *if* $f \sim g$*, and write*

$$\pi(A, X) := \mathrm{Hom}_{\mathbf{C}}(A, X) \Big/ \sim$$

Definition 2.2.13. *If* A *and* X *are* both *fibrant* and *cofibrant, then* $f\colon A \to X$ *is called a* homotopy equivalence *if there exists* $g\colon X \to A$ *such that* $gf \sim \mathrm{id}_A$ *and* $fg \sim \mathrm{id}_X$*.*

Next, Theorems 2.2.7 and 2.2.10 imply

Proposition 2.2.14. *Assume* A *is a cofibrant,* X *is both fibrant and cofibrant, and* Y *is a fibrant object of* $\mathbf{C}$*. Then*

$$f \sim g\colon A \to X,\ h \sim k\colon X \to Y \ \Rightarrow hf \sim kg\colon A \to Y$$

and we have the induced map

$$\pi(A, X) \times \pi(X, Y) \to \pi(X, Y).$$

2.3 Whitehead Theorem

The last result we would like to state and prove is an abstract version of the famous theorem of Whitehead, asserting that a *weak* homotopy equivalence between CW complexes is actually a homotopy equivalence. It is remarkable that this is an abstract model categorical result, "having nothing to do with topology."

Theorem 2.3.1. *Suppose A, X are both fibrant and cofibrant in* $\mathbf{C}$. *Then*

$$(A \xrightarrow{f} X) \in \mathrm{WE} \Leftrightarrow f \text{ is a homotopy equivalence}$$

Proof. Suppose first that $f\colon A \to X$ belongs to WE. By MC5 we can factor it as

$$f\colon A \underset{\sim}{\xrightarrow{q}} C \overset{p}{\twoheadrightarrow} X \tag{3.2}$$

with q being a acyclic cofibration and p being a fibration. By MC2, since both $f, q \in \mathrm{WE}$ then also $p \in \mathrm{WE}$. Since $q \in \mathrm{Cof} \cap \mathrm{WE}$ and A is fibrant, we have

$$\begin{tikzcd}
A \arrow[r, equal, "\mathrm{id}_A"] \arrow[d, hook, "q"', "\sim"] & A \arrow[d, two heads] \\
C \arrow[r] \arrow[ur, dashed, "r"'] & *
\end{tikzcd}$$

By MC4, there exists a lifting $r\colon C \to A$ such that $rq = \mathrm{id}_A$. By Theorem 2.2.10, since C is cofibrant, q induces a bijection

$$q^*\colon \pi^r(C, C) \xrightarrow{\simeq} \pi^r(A, C)$$

given by $[g] \mapsto [gq]$. Since

$$q^*([qr]) = [qrq] = [q\mathrm{id}_A] = [q] = [\mathrm{id}_C q]$$

we conclude that $qr \sim \mathrm{id}_C$. Thus, r is a two-sided homotopy inverse of q.

A dual argument shows that $p\colon C \overset{\sim}{\twoheadrightarrow} X$ has a homotopy inverse $s\colon X \to C$. Thus, by Theorems 2.2.7 and 2.2.10, $rs\colon C \overset{\sim}{\to} A$ is a homotopy inverse of $f = pq$.

To prove the converse statement, assume f has a homotopy inverse. Fator $f = pq$ as in 3.2. Note that C is both fibrant and cofibrant. It suffices to prove that $p \in \mathrm{WE}$. Then by MC2 we would have that f is also a weak equivalence.

Let $g\colon X \to A$ be a homotopy inverse of f. Let

$$\begin{tikzcd}
X \arrow[r, hook, "i_1"] \arrow[dr, "fg"'] & Cyl(X) \arrow[d, "H"] & X \arrow[l, hook', "i_2"'] \arrow[dl, "\mathrm{id}_X"] \\
& X &
\end{tikzcd}$$

be the corresponding homotopy. By MC4 and Lemma 2.2.3, there exists a map $\widetilde{H}\colon Cyl(X) \to C$ lifting H along p:

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook, "qg'"'] & C \arrow[d, "p"] \\ Cyl(X) \arrow[r, "H"] \arrow[ur, dashed, "\widetilde{H}"] & X \end{tikzcd}$$

Put $s := \widetilde{H} \circ i_2\colon X \to C$. Then $ps = p\widetilde{H}i_2 = Hi_2 = \mathrm{id}_X$. Since $q \in \mathrm{WE}$, by the proof of the implication $\Rightarrow$, q has a homotopy inverse, say $r\colon C \to A$. Then

$$pq = f \Rightarrow pqr = fr \Rightarrow p \sim fr$$

Since $s \sim qg$ (via $\widetilde{H}$), we have

$$sp \sim qgp \sim qgfr \sim qr \sim id_C$$

This implies that $sp \in \mathrm{WE}$. By the following (trivial) commutative diagram

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d, "p"'] & C \arrow[r, equal] \arrow[d, "sp"] & C \arrow[d, "p"] \\ X \arrow[r, "s"] & C \arrow[r, "p"] & X \end{tikzcd}$$

p is a retract of sp. Therefore, MC3 implies that $p \in \mathrm{WE}$ which finishes the proof. □

3 Homotopy category

According to Quillen, the term "model category" is short for "a category of models for a homotopy theory." The homotopy theory associated to a model category $\mathbf{C}$ is the homotopy category $\mathbf{Ho}(\mathbf{C})$, which often comes with extra structure (e.g. triangulation, loop and suspension functors, etc.) depending on the model category $\mathbf{C}$ only up to homotopy. The key idea is that the *same* homotopy category may have *different* models, which can be used for different problems and in many different situations. The prototypical important example is the classical homotopy theory (with base point). It has three well-known models:

(1) the model category of connected pointed spaces;

(2) the model category of reduced simplicial sets;

(3) the model category of simplicial groups.

These model categories are quite different. For example, (1) and (3) are fibrant model categories, while (2) is cofibrant. But still they are in some sense equivalent, since they have equivalent homotopy categories. Our goal is to make this notion of equivalence and the conditions which imply it precise. This was one of the original goal of Quillen and still is one of the most useful applications of the model category theory.

3.1 Definition of a homotopy category

Let us denote by $\mathbf{C}_c, \mathbf{C}_f, \mathbf{C}_{cf}$ the full subcategories of $\mathbf{C}$ consisting respectively of all cofibran, all fibrant and cofibrant-fibrant objects of $\mathbf{C}$. Next, define the categories $\pi\mathbf{C}_c, \pi\mathbf{C}_f, \pi\mathbf{C}_{cf}$ with the same objects as $\mathbf{C}_c, \mathbf{C}_f$ and $\mathbf{C}_{cf}$ and the morphisms being the right homotopy, left homotopy and homotopy classes of morphisms, respectively. In other words, the morphisms are given by

$$\begin{aligned}
\mathrm{Hom}_{\pi\mathbf{C}_c}(A, X) &:= \pi^r(A, X)\\
\mathrm{Hom}_{\pi\mathbf{C}_f}(A, X) &:= \pi^l(A, X)\\
\mathrm{Hom}_{\pi\mathbf{C}_{cf}}(A, X) &:= \pi(A, X)
\end{aligned}$$

Recall that by MC5, for every object $A \in \mathrm{Ob}(\mathbf{C})$ the canonical morphisms $e \to A$ and $A \to *$ can be factored as:

$$\begin{aligned}
e \hookrightarrow QA &\overset{\sim}{\twoheadrightarrow} A\\
A \overset{\sim}{\hookrightarrow} RA &\twoheadrightarrow *
\end{aligned}$$

where QA is a cofibrant and RA is a fibrant object in $\mathbf{C}$. The acyclic fibration $p_A \colon QA \overset{\sim}{\twoheadrightarrow} A$ and the acycic cofibration $i_A \colon A \overset{\sim}{\hookrightarrow} RA$ are called *cofibrant* and *fibrant* models (or replacements) of A, respectively. Fix QA and RA for each A. We will assume that $QA = A$ if A is cofibrant, and that $RA = A$ in the case A is fibrant.

Note that QA and RA are *not unique*. However, we have the following

Lemma 3.1.1. 1. *The assignment $A \mapsto QA$ extends to a functor $Q \colon \mathbf{C} \to \pi\mathbf{C}_c$. The restriction of Q to $\mathbf{C}_f \subseteq \mathbf{C}$ induces a functor $Q' \colon \pi\mathbf{C}_f \to \pi\mathbf{C}_{cf}$.*

2. *The assignment $A \mapsto RA$ extends to a functor $R \colon \mathbf{C} \to \pi\mathbf{C}_f$. The restriction of R to $\mathbf{C}_c \subseteq \mathbf{C}$ induces a functor $R' \colon \pi\mathbf{C}_c \to \pi\mathbf{C}_{cf}$.*

Proof. Given $f \colon A \to X$, choose $p_A \colon QA \overset{\sim}{\twoheadrightarrow} A$ and $p_X \colon QX \overset{\sim}{\twoheadrightarrow} X$ cofibrant models and consider the diagram

$$\begin{tikzcd}
e \arrow[rr, hook] \arrow[d, hook] & & QX \arrow[d, two heads, "\sim"', "p_X"] \\
QA \arrow[r, dashed, two heads, "\sim", "p_A"'] \arrow[urr, dashed, "\widetilde{f}"] & A \arrow[r, "f"'] & X
\end{tikzcd}$$

Since QA is cofibrant and $p_X \in \mathrm{WE} \cap \mathrm{Fib}$, by MC4 there exists $\widetilde{f} \colon QA \to QX$ lifintg $f \colon A \to X$. We define $Q(f) := [\widetilde{f}]_r \in \mathrm{Mor}(\pi\mathbf{C}_c)$.

By Theorem 2.2.7, if A is cofibrant and $p \colon Y \overset{\sim}{\twoheadrightarrow} X$ is an acyclic fibration, then p induces a bijection

$$p_* \colon \pi^l(A, Y) \overset{\sim}{\to} \pi^r(A, X)$$

In our situation, this implies that $p_X\colon QX \overset{\sim}{\twoheadrightarrow} X$ induces a bijection

$$(p_X)_*\colon \pi^l(QA, QX) \xrightarrow{\sim} \pi^l(QA, X)$$

given by $[\widetilde{f}] \mapsto [p_X \circ \widetilde{f}] = [f \circ p_A]$. Hence $\widetilde{f}$ is uniquely determined by $[f]$ up to *left* homotopy. We at the same time, since QA is cofibrant, Proposition 2.2.11 implies that

$$\widetilde{f} \overset{l}{\sim} \widetilde{f}'\colon QA \to QX \ \Rightarrow \widetilde{f} \overset{r}{\sim} \widetilde{f}'\colon QA \to QX$$

Hence, $\widetilde{f}$ is uniquely determined by $[f]$ up to *right* homotopy as well.

Note also that MC2, $\widetilde{f} \in \mathrm{WE}$ if and only if $f \in \mathrm{WE}$. This shows that Q is a well-defined functor $Q\colon \mathbf{C} \to \pi\mathbf{C}_c$. We leave the rest as a (trivial) exercise to the reader.

The proof of the second statement is similar to the first one. □

Definition 3.1.2. *The* homotopy category $\mathbf{Ho}(\mathbf{C})$ *of a model category* $\mathbf{C}$ *is the category with* $\mathrm{Ob}(\mathbf{Ho}(\mathbf{C})) = \mathrm{Ob}(\mathbf{C})$ *and morphisms between any two objects* A, X *given by*

$$\mathrm{Hom}_{\mathbf{Ho}(\mathbf{C})}(A, X) := \mathrm{Hom}_{\pi\mathbf{C}_{cf}}(R'QA, R'QX) = \pi(RQA, RQX)$$

Proposition 2.2.14 ensures that this is well-defined.

3.2 Homotopy category as a localization

There is a natural functor $\gamma\colon \mathbf{C} \to \mathbf{Ho}(\mathbf{C})$ which is the identity on $\mathrm{Ob}(\mathbf{C})$ and sends

$$(f\colon A \to X) \mapsto R'Q(f)\colon R'Q(A) \to R'Q(X)$$

IF A and X are both fibrant and cofibrant, i.e. $A, X \in \mathrm{Ob}(\mathbf{C}_{cf})$, then

$$\gamma\colon\ \mathrm{Hom}_{\mathbf{C}}(A, X) \twoheadrightarrow \mathrm{Hom}_{\mathbf{Ho}(\mathbf{C})}(A, X)$$

is onto, and induces a bijection

$$\pi(A, X) \simeq \mathrm{Hom}_{\mathbf{Ho}(\mathbf{C})}(A, X)$$

because in this case $RQA = RA = A$ and $RQX = RX = X$.

Remark 3.2.1. At first glance, the definition of $\mathbf{Ho}(\mathbf{C})$ looks asymmetric: for example, we can replace $R'(QA)$ by $Q'(RA)$, etc. This, however, will give the same category as the following key theorem shows.

Theorem 3.2.2. *The functor* $\gamma\colon \mathbf{C} \to \mathbf{Ho}(\mathbf{C})$ *is the localization of* $\mathbf{C}$ *with respect to the class of weak equivalences* WE. *Thus,*

$$\mathbf{Ho}(\mathbf{C}) \simeq \mathbf{C}\left[\mathrm{WE}^{-1}\right]$$

In particular, $\gamma(f)$ *is an isomorphism in* $\mathbf{Ho}(\mathbf{C})$ *if and only if* $f \in \mathrm{WE}$.

Remark 3.2.3. Theorem 3.2.2 shows that $\mathbf{Ho}(\mathbf{C})$ depends only on the class WE of weak equivalences, not on fibrations and cofibrations. This suggests that in a model category the weak equivalences play a primary role: they carry the fundamental homotopy-theoretic information, while Cof and Fib are auxiliary classes needed to make various constructions.

Proof of Theorem 3.2.2. We need to verify the universal property of localization of categories:

(1) $\gamma(f)$ is an isomorphism for any $f \in \mathrm{WE}$;

(2) given $F\colon \mathbf{C} \to \mathbf{D}$ such that $F(f) \in \mathrm{Iso}(\mathbf{D})$ for any $f \in \mathrm{WE}$ then

$$\begin{tikzcd}
\mathbf{C} \arrow[rr, "F"] \arrow[dr, "\gamma"'] & & \mathbf{D} \\
& \mathbf{Ho}(\mathbf{C}) \arrow[ur, dashed, "\exists!\overline{F}"'] &
\end{tikzcd}$$

For (1), if $(f\colon A \to X) \in \mathrm{WE}$ then $R'Q(f)$ is represented by $\widetilde{f}\colon RQA \to RQX$ such that

$$\begin{tikzcd}
QA \arrow[r, "Qf"] \arrow[d, hook, "i_{QA}"', "\sim"] & QX \arrow[r, "\sim", "i_{QX}"'] & RQX \arrow[d, two heads] \\
RQA \arrow[urr, dashed] \arrow[rr, two heads] & & *
\end{tikzcd}
\qquad
\begin{tikzcd}
e \arrow[rr, hook] \arrow[d, hook] & & QX \arrow[d, two heads, "\sim"', "p_X"] \\
QA \arrow[urr, "Qf"] \arrow[r, two heads, "\sim", "p_A"'] & A \arrow[r, "f"] & X
\end{tikzcd}$$

From the diagrams above, we have

$$f \in \mathrm{WE} \Leftrightarrow Qf \in \mathrm{WE} \Leftrightarrow RQf \in \mathrm{WE}$$

By Whitehead theorem 2.3.1, $[\widetilde{f}]$ represents an isomorphism in $\pi\mathbf{C}_{cf}$ which is exactly $\gamma(f)$. On the other hand, if $\gamma(f)$ is an isomorphism in $\mathbf{Ho}(\mathbf{C})$, then $\widetilde{f}$ has inverse up to homotopy and hence again is a weak equivalence by Whitehead theorem.

To prove (2), given $F\colon \mathbf{C} \to \mathbf{D}$ such that $F(f) \in \mathrm{Iso}(\mathbf{D})$ for any $f \in \mathrm{WE}$ we can construct $\overline{F}\colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{D}$ as follows. Since $\mathrm{Ob}(\mathbf{Ho}(\mathbf{C})) = \mathrm{Ob}(\mathbf{C})$, we define

$$\overline{F}(X) = F(X),\ \forall X \in \mathrm{Ob}(\mathbf{Ho}(\mathbf{C}))$$

Given $f \in \mathrm{Hom}_{\mathbf{Ho}(\mathbf{C})}(A, X)$, pick $\widetilde{f}\colon RQ(A) \to RQ(X)$ representing f. It is easy to check that $F(\widetilde{f})$ depends only on the class $[\widetilde{f}]$, i.e. on f. Then define $\overline{F}(f)$ by

$$\begin{tikzcd}
A \arrow[d, "f"'] & QA \arrow[l, two heads, "p_A"] \arrow[r, hook, "\sim", "i_{QA}"'] \arrow[d, dashed] & RQA \arrow[d, "\widetilde{f}"] \\
X & QX \arrow[l] \arrow[r, hook, "\sim", "i_{QX}"'] & RQX
\end{tikzcd}$$

where we put

$$\overline{F}(f) := F(p_X) \circ F(i_{QX})^{-1} \circ F(\widetilde{f}) \circ F(i_{QA}) \circ F(p_A)^{-1}$$

It is easy to check that $\overline{F}$ respects identity maps and compositions, and that $\overline{F} \circ \gamma = F$. $\square$

In [DK80], Dwyer and Kan give a simple description of morphisms in $\mathbf{Ho}(\mathbf{C})$ in terms of equivalence classes of certain diagrams in $\mathbf{C}$. More precisely, they prove the following

Proposition 3.2.4 ([DK80])**.** *Let $\mathbf{C}$ be a model category, $X, Y \in \mathrm{Ob}(\mathbf{C})$. Then $\mathrm{Hom}_{\mathbf{Ho}(\mathbf{C})}(\gamma X, \gamma Y)$ is isomorphic to the set of equivalence classes of diagrams in $\mathbf{C}$ of the form*

$$\begin{tikzcd} X & A \arrow[l, "\sim"'] \arrow[r] & B \arrow[r, "\sim"] & Y \end{tikzcd}$$

where the equivalence relation is generated by the diagrams of the form

$$\begin{tikzcd} & A_1 \arrow[dl, "\sim"'] \arrow[r] \arrow[d] & B_1 \arrow[dr, "\sim"] \arrow[d] & \\ X & A_2 \arrow[l, "\sim"] \arrow[r] & B_2 \arrow[r, "\sim"'] & Y \end{tikzcd}$$

3.3 Derived functors

If $\mathbf{C}$, $\mathbf{D}$ are model categories, a functor $F\colon \mathbf{C} \to \mathbf{D}$ is called *homotopy invariant* (or *homotopical*) if $F(\mathrm{WE}_{\mathbf{C}}) \subseteq \mathrm{WE}_{\mathbf{D}}$. In this case, it extends (or descends) to a functor $\mathbf{Ho}(F)$ between the homotopy categories

$$\begin{tikzcd} \mathbf{C} \arrow[r, "F"] \arrow[d, "\gamma_{\mathbf{C}}"'] & \mathbf{D} \arrow[d, "\gamma_{\mathbf{D}}"] \\ \mathbf{Ho}(\mathbf{C}) \arrow[r, "\mathbf{Ho}(F)"] & \mathbf{Ho}(\mathbf{D}) \end{tikzcd}$$

This rarely happens in practice: interesting functors are usually *not* homotopical. The idea is that we should define derived functors

$$\mathbb{L}F, \mathbb{R}F\colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{D})$$

which give the best possible homotopical approximation to "$\mathbf{Ho}(F)$." The key question is: when do $\mathbb{L}F$ and $\mathbb{R}F$ exist and under which conditions do they induce an equivalence of categories $\mathbf{Ho}(\mathbf{C}) \simeq \mathbf{Ho}(\mathbf{D})$.

Definition 3.3.1. *Let $\mathbf{C}$ be a model category, $F\colon \mathbf{C} \to \mathbf{D}$ a functor from $\mathbf{C}$ to any category $\mathbf{D}$. A* left derived functor *$\mathbb{L}F$ of the functor F is a* right *Kan extension of F along $\gamma_{\mathbf{C}}$, i.e. $\mathbb{L}F := \mathrm{Ran}_{\gamma_{\mathbf{C}}}(F)$, see Section 2.4*

$$\begin{tikzcd} \mathbf{C} \arrow[rr, "F"] \arrow[dr, "\gamma_{\mathbf{C}}"'] & \arrow[d, phantom, "\Downarrow"] & \mathbf{D} \\ & \mathbf{Ho}(\mathbf{C}) \arrow[ur, dashed, "\mathbb{L}F"'] & \end{tikzcd}$$

Thus $\mathbb{L}F$ comes with a morphism of functors

$$t\colon \mathbb{L}F \circ \gamma_{\mathbf{C}} \Rightarrow F$$

which is universal among all pairs $(G\colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{D}, s\colon G \circ \gamma_{\mathbf{C}} \Rightarrow F)$. In other words, for any such pair there exists unique natural transformation $\overline{s}\colon G \Rightarrow \mathbb{L}F$ making the following diagram commute:

$$\begin{tikzcd} G \circ \gamma_{\mathbf{C}} \arrow[rr, "s"] \arrow[dr, "\overline{s}\circ\gamma_{\mathbf{C}}"'] & & F \\ & \mathbb{L}F \circ \gamma_{\mathbf{C}} \arrow[ur, dashed, "t"'] & \end{tikzcd}$$

Similarly one defines right derived functor *of a functor F to be a* left *Kan extension $\mathbb{R}F = \mathrm{Lan}_{\gamma_{\mathbf{C}}}(F)$.*

Definition 3.3.2. *If $F\colon \mathbf{C} \to \mathbf{D}$ is a functor between two model categories, then its* (total) left derived fucntor $\mathbb{L}F$ *is the left derived functor of the composition $\gamma_{\mathbf{D}} \circ F\colon \mathbf{C} \to \mathbf{Ho}(\mathbf{D})$, i.e. $\mathbb{L}F = \mathrm{Ran}_{\gamma_{\mathbf{C}}}(\gamma_{\mathbf{D}} \circ F)$. Similarly,* (total) right derived functor *of F is the right derived functor of the composition $\gamma_{\mathbf{D}} \circ F$.*

Lemma 3.3.3. *Let $F\colon \mathbf{C} \leftrightarrows \mathbf{D}\colon G$ be a pair of adjoint functors between two model categories. The following conditions are equivalent:*

1. *$F(\mathrm{Cof}_{\mathbf{C}}) \subseteq \mathrm{Cof}_{\mathbf{D}}$ and $F(\mathrm{Cof}_{\mathbf{C}} \cap \mathrm{WE}_{\mathbf{C}}) \subseteq \mathrm{Cof}_{\mathbf{D}} \cap \mathrm{WE}_{\mathbf{D}}$;*
2. *$G(\mathrm{Fib}_{\mathbf{D}}) \subseteq \mathrm{Fib}_{\mathbf{C}}$ and $G(\mathrm{Fib}_{\mathbf{D}} \cap \mathrm{WE}_{\mathbf{D}}) \subseteq \mathrm{Fib}_{\mathbf{C}} \cap \mathrm{WE}_{\mathbf{C}}$;*
3. *$F(\mathrm{Cof}_{\mathbf{C}}) \subseteq \mathrm{Cof}_{\mathbf{D}}$ and $G(\mathrm{Fib}_{\mathbf{D}}) \subseteq \mathrm{Fib}_{\mathbf{C}}$;*
4. *$F(\mathrm{Cof}_{\mathbf{C}} \cap \mathrm{WE}_{\mathbf{C}}) \subseteq \mathrm{Cof}_{\mathbf{D}} \cap \mathrm{WE}_{\mathbf{D}}$ and $G(\mathrm{Fib}_{\mathbf{D}} \cap \mathrm{WE}_{\mathbf{D}}) \subseteq \mathrm{Fib}_{\mathbf{C}} \cap \mathrm{WE}_{\mathbf{C}}$;*
5. *F preserves cofibrations between cofibrant objects and $F(\mathrm{Cof}_{\mathbf{C}} \cap \mathrm{WE}_{\mathbf{C}}) \subseteq \mathrm{Cof}_{\mathbf{D}} \cap \mathrm{WE}_{\mathbf{D}}$;*
6. *G preserves fibrations between fibrant objects and $G(\mathrm{Fib}_{\mathbf{D}} \cap \mathrm{WE}_{\mathbf{D}}) \subseteq \mathrm{Fib}_{\mathbf{C}} \cap \mathrm{WE}_{\mathbf{C}}$.*

Definition 3.3.4. *Pairs of adjoint functors (F, G) satisfying one of the equivalent conditions of Lemma 3.3.3 are called* Quillen pairs*. In this case we call F a* left Quillen *and G a* right Quillen *functor respectively.*

3.4 Quillen's Adjunction Theorem

Theorem 3.4.1 (Quillen's adjunction Theorem)**.** *If (F, G) is a Quillen pair, then $\mathbb{L}F$ and $\mathbb{R}G$ exist and form an adjoint pair*

$$\mathbb{L}F\colon \mathbf{Ho}(\mathbf{C}) \leftrightarrows \mathbf{Ho}(\mathbf{D}) : \mathbb{R}G$$

The functors $\mathbb{L}F$ and $\mathbb{R}G$ are given by the formulas

$$\mathbb{L}F(A) = \gamma_{\mathbf{D}}F(QA)$$
$$\mathbb{R}G(X) = \gamma_{\mathbf{C}}G(RX)$$

where $p_A\colon QA \overset{\sim}{\twoheadrightarrow} A$ is any cofibrant replacement of A in $\mathbf{C}$ and $i_X\colon X \overset{\sim}{\hookrightarrow} RX$ is any fibrant replacement of X in $\mathbf{D}$.

Remark 3.4.2. The condition for a functor F (resp., G) to be a left Quillen (resp., right Quillen) functor is a sufficient, but not a *necessary* condition for $\mathbb{L}F$ (resp. $\mathbb{R}G$) to exist. In fact, there are many interesting functors, which are *not* Quillen but still have derived functors (see later).

Remark 3.4.3. The total derived functors of *Quillen* functors are "better" that of non-Quillen ones in the sense that they satisfy a *stronger* universal property. Namely, they are actually *absolute Kan extensions*, see [Mal07]. This means that for any functor $H\colon \mathbf{Ho}(\mathbf{D}) \to \mathbf{E}$ the composition $H \circ \mathbb{L}F$ is the right Kan extension of $H \circ \gamma_{\mathbf{D}} \circ F$ along $\gamma_{\mathbf{C}}$:

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "F"] \arrow[d, "\gamma_{\mathbf{C}}"'] & \mathbf{D} \arrow[r, "\gamma_{\mathbf{D}}"] & \mathbf{Ho}(\mathbf{D}) \arrow[r, "H"] & \mathbf{E} \\
\mathbf{Ho}(\mathbf{C}) \arrow[urr, "\mathbb{L}F"] \arrow[urrr, "\mathrm{Ran}_{\gamma_{\mathbf{C}}}(H\circ\gamma_{\mathbf{D}}\circ F)"'] & & &
\end{tikzcd}$$

One can also prove the following

Lemma 3.4.4. *If $F\colon \mathbf{C} \leftrightarrows \mathbf{D}\colon G$ are adjoint, $\mathbb{L}F$ and $\mathbb{R}G$ exist and are* absolute *derived functors, then*

$$\mathbb{L}F\colon \mathbf{Ho}(\mathbf{C}) \leftrightarrows \mathbf{Ho}(\mathbf{D})\colon \mathbb{R}G$$

Remark 3.4.5. Let $F, G\colon \mathbf{C} \to \mathbf{D}$ be two functors between model categories s.t. $\mathbb{L}F$ and $\mathbb{L}G\colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{D})$ exist. Then, any natural transformation $\varphi\colon F \Rightarrow G$ induces a (unique) natural transformation $\mathbb{L}\varphi\colon \mathbb{L}F \Rightarrow \mathbb{L}G$. This follows easily from the universal properties.

Indeed, recall that $\mathbb{L}G\colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{D})$ is defined by

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "G"] \arrow[d, "\gamma_{\mathbf{C}}"'] & \mathbf{D} \arrow[r, "\gamma_{\mathbf{D}}"] & \mathbf{Ho}(\mathbf{D}) \\
\mathbf{Ho}(\mathbf{C}) \arrow[ur, Rightarrow, "t_G"] \arrow[urr, "\mathbb{L}G"'] & &
\end{tikzcd}$$

with the universal property that for any pair (G', s') with $G'\colon \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{D})$ and $s'\colon G' \circ \gamma_{\mathbf{C}} \Rightarrow \gamma \circ G$ there is unique transformation $s\colon G' \Rightarrow \mathbb{L}G$ making the following diagram commute:

$$\begin{tikzcd}
G' \circ \gamma_{\mathbf{C}} \arrow[rr, "s"] \arrow[dr, "s\circ\gamma_{\mathbf{C}}"'] & & \gamma_{\mathbf{D}} \circ G \\
& \mathbb{L}G \circ \gamma_{\mathbf{C}} \arrow[ur, dashed, "t_G"'] &
\end{tikzcd}$$

Take $G' = \mathbb{L}F$ and $s' = \gamma_{\mathbf{C}}\varphi \circ t_F\colon \mathbb{L}F \circ \gamma_{\mathbf{C}} \overset{t_F}{\Rightarrow} \gamma_{\mathbf{D}} \circ F \overset{\gamma_{\mathbf{D}}\circ\varphi}{\Rightarrow} \gamma_{\mathbf{D}} \circ G$. Then, there exists unique $\widetilde{\varphi}\colon \mathbb{L}F \Rightarrow \mathbb{L}G$ s.t.

$$\begin{tikzcd}
\mathbb{L}F \circ \varphi \arrow[r, Rightarrow, "t_F"] \arrow[dr, Rightarrow, "\widetilde{\varphi}\gamma"'] & \gamma \circ F \arrow[r, Rightarrow, "\gamma\varphi"] & \gamma \circ G \\
& \mathbb{L}G \circ \gamma \arrow[ur, "t_G"'] &
\end{tikzcd}$$

3.5 Quillen's Equivalence Theorem

Theorem 3.5.1 (Quillen's Equivalence Theorem)**.** *Let $F\colon \mathbf{C} \leftrightarrows \mathbf{D}\colon G$ be a Quillen pair. Assume that*

($*$) *for every $A \in \mathrm{Ob}(\mathbf{C}_c)$ and $X \in \mathrm{Ob}(\mathbf{D}_f)$,*

$$(f\colon A \to G(X)) \in \mathrm{WE}_{\mathbf{C}} \Leftrightarrow (f^*\colon F(A) \to X) \in \mathrm{WE}_{\mathbf{D}}$$

Then $\mathbb{L}F\colon \mathbf{Ho}(\mathbf{C}) \simeq \mathbf{Ho}(\mathbf{D})\colon \mathbb{R}G$ are mutually inverse equivalences of categories.

Question (Marcelo Aguiar): How to state the condition ($*$) in terms of the adjunction morphisms for (F, G)?

Let $\Psi_{A,X}\colon \mathrm{Hom}_{\mathbf{D}}(F(A), X) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{C}}(A, G(X)),\ f \mapsto f^*$ be the adjunction isomorphism. Then, putting $X = F(A)$ and $A = G(X)$, we get a *unit morphism*

$$\eta_A := \Psi^{-1}_{A,F(A)}(\mathrm{id}_{F(A)})\colon A \to GF(A)$$

and a *counit morphism*

$$\varepsilon_X := \Psi^{-1}_{G(X),X}(\mathrm{id}_{G(X)})\colon FG(X) \to X$$

Note that for any maps $f\colon F(A) \to X$ and $g\colon A \to G(X)$,

$$\Psi_{A,X}(f) = [A \xrightarrow{\eta_A} GF(A) \xrightarrow{G(f)} G(X)]$$
$$\Psi^{-1}_{A,X}(g) = [F(A) \xrightarrow{F(g)} FG(X) \xrightarrow{\varepsilon_X} X]$$

Lemma 3.5.2. *Quillen's condition ($*$) is equivalent to the following one:*

($**$) *For every $A \in \mathrm{Ob}(\mathbf{C}_c)$ and $X \in \mathrm{Ob}(\mathbf{D}_f)$, the following compositions are weak equivalences:*

$$[A \xrightarrow{\eta_A} GF(A) \xrightarrow{G(i_{FA})} G\left(R(FA)\right)] \in \mathrm{WE}_{\mathbf{C}}$$
$$[F\left(Q(GX)\right) \xrightarrow{F(p_{GX})} FG(X) \xrightarrow{\varepsilon_X} X] \in \mathrm{WE}_{\mathbf{D}}$$

where $i_{FA}\colon FA \overset{\sim}{\hookrightarrow} R(FA)$ is a fibrant replacement of FA and $p_{GX}\colon Q(GX) \overset{\sim}{\twoheadrightarrow} GX$ is a cofibrant replacement of GX.

Proof. ($*$) $\Rightarrow$ ($**$). Note that the compositions in ($**$) are just adjoints of (co)fibrant models. In other words,

$$i^*_{FA} := \Psi_{A,R(FA)}(i_{FA}) = G(i_{FA}) \circ \eta_A$$

and

$$p^*_{GX} := \Psi^{-1}_{Q(GX),X}(p_{GX}) = \varepsilon_X \circ F(p_{GX})$$

Hence, if A is cofibrant, $R(FA)$ is fibrant and $i_{FA} \in \mathrm{WE}$ the condition $(*)$ implies that $i_{FA}^* \in \mathrm{WE}$. Similarly, if X is fibrant, $Q(GX)$ is cofirant and $p_{GX} \in \mathrm{WE}$ then $(*)$ implies that $p_{GX}^* \in \mathrm{WE}$.

$(**) \Rightarrow (*)$. Assume (F, G) satisfies $(**)$. Let $(f\colon FA \to X) \in \mathrm{WE}_{\mathbf{D}}$, where $A \in \mathrm{Ob}(\mathbf{C}_c)$ and $X \in \mathrm{Ob}(\mathbf{D}_f)$. Then, by definition,

$$\Psi_{A,X}(f) = [A \xrightarrow{\eta_A} GF(A) \xrightarrow{G(f)} G(X)]$$

and we have the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, equal] & GF(A) \arrow[r, "G(f)"] \arrow[d, "G(i_{FA})"] & GX \arrow[d, "G(i_X)"] \\ A \arrow[r] & G\,[R(FA)] \arrow[r] & G(RX) \end{tikzcd}$$

where $i_X\colon X \overset{\sim}{\hookrightarrow} RX$ and $i_{FA}\colon FA \overset{\sim}{\hookrightarrow} R(FA)$ are fibrant replacements, and Rf is a (functorial) extension of f to a morphism $Rf\colon R(FA) \to RX$.

Here for simplicity, we assume that there is a *functorial* factorization in MC5, so that QA and RX are natural in their arguments.

Now, note that $f \in \mathrm{WE}$ implies that $Rf \in \mathrm{WE}$ and, since G preserves $\mathrm{Fib} \cap \mathrm{WE}$, also $G(Rf) \in \mathrm{WE}$ and $G(i_X) \in$ (because X is fibrant and $i_X \in \mathrm{WE}$).

Since $(**)$ holds, the map $A \to G\,[R(FA)]$ is a weak equivalence. Hence the bottom horizontal composite map and the right-most vertical maps in the above diagram are both weak equivalences. It follows that the top horizontal map (which is f^*) is also a weak equivalence.

A similar asrgument shows that $f^* \in \mathrm{WE}$ implies $f \in \mathrm{WE}$. $\square$

3.6 About proofs of of Quillen's Adjunction and Equivalence Theorems

Informally, the reason why a non-homotopical functor (i.e. a functor that does *not* preserve weak equivalences) admit a derived functor is that it still preserves all weak equivalences between *some* "good" objects. This is made precise by the following

Proposition 3.6.1. *Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor from a model category to a category $\mathbf{D}$, satisfying the following property:*

(P) if f is a weak equivalence between cofibrant objects in $\mathbf{C}$, then $F(f)$ is an isomorphism in $\mathbf{D}$.

Then, F has the left derived functor $(\mathbb{L}F, t)$ (defined as a right Kan extension of F along $\gamma_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{Ho}(\mathbf{C})$), and for every $A \in \mathrm{Ob}(\mathbf{C}_c)$ the map

$$t_A\colon \mathbb{L}F(A) \overset{\sim}{\to} F(A)$$

is an isomorphism in $\mathbf{D}$.

In practice, to verify property (P) in the case when $\mathbf{D}$ is a homotopy category of another model category, the following lemma is often used.

Lemma 3.6.2 (K.Brown)**.** *Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor between* model *categories. Assume that F maps acyclic* cofibrations *between cofibrant objects in $\mathbf{C}$ to weak equivalences in $\mathbf{D}$. Then F maps* all weak equivalences *between cofibrant objects in $\mathbf{C}$ to weak equivalences in $\mathbf{D}$. Thus, $\gamma_{\mathbf{D}} \circ F\colon \mathbf{C} \to \mathbf{D} \to \mathbf{Ho}(\mathbf{D})$ satisfies property (P), and hence $\mathbb{L}F$ exists.*

Proof. Let $A, B \in \mathrm{Ob}(\mathbf{C})$ be two cofibrant objects in $\mathbf{C}$, and let $f\colon A \to B$ be a weak equivalence. Consider the map $f \sqcup \mathrm{id}_B$ defined by

$$
\begin{tikzcd}
e \arrow[r] \arrow[d] & B \arrow[d, "i_1"] \arrow[ddr, bend left, "\mathrm{id}_B"] & \\
A \arrow[r, "i_0"] \arrow[drr, bend right, "f"'] & A \sqcup B \arrow[dr, "f \sqcup \mathrm{id}_B"] & \\
& & B
\end{tikzcd}
$$

and apply MC5 to factor it as $f \sqcup \mathrm{id}_B = p \circ i$:

$$
\begin{tikzcd}
B \arrow[dr, hook, "i_1"] & & & \\
& A \sqcup B \arrow[r, hook, "i"] & C \arrow[r, two heads, "\sim", "p"'] & B \\
A \arrow[ur, hook, "i_0"'] & & &
\end{tikzcd}
$$

Since both A, B are cofibrant, i_0 and i_1 are cofibrations, since they are push-outs of the cofibrations $e \hookrightarrow B$ and $e \hookrightarrow A$, respectively. Hence $i \circ i_0$ and $i \circ i_1$ are also cofibrations. But $p \circ i \circ i_0 = f$ is a weak equivalence, and $p \circ i \circ i_1 = \mathrm{id}$ is a weak equivalence, together with p being also a weak equivalence by its definition. Hence, by the "two-out-of-three" axiom MC2, both $i \circ i_0$ and $i \circ i_1$ are in WE, and therefore in $\mathrm{Cof} \cap \mathrm{WE}$.

By the Lemma's assumption, $F(i \circ i_0) \in \mathrm{WE}$ and $F(i \circ i_1) \in \mathrm{WE}$. Also $F(p) \circ F(ii_1) = F(pii_1) = F(\mathrm{id}) \in \mathrm{WE}$. Applying MC2 again, we get that $F(p) \in \mathrm{WE}$, and therefore

$$F(f) = F(pii_0) = F(p) \circ F(ii_0) \in \mathrm{WE}$$

□

Remark 3.6.3. Brown's lemma is very useful in practice, because it allows one to effectively verify (P), and hence prove the existence of derived functors even when Quillen's Theorem doesn't apply (i.e. for non-Quillen functors). Typically, it works as follows.

Let $\mathbf{C}$ be a *fibrant* model category, and let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor which is expected (or suspected) to have a left derived functor. Let $f\colon A \overset{\sim}{\hookrightarrow} B$ be an acyclic cofibration, where

A, B both are cofibrant objects of $\mathbf{C}$. Then, by applying MC4 to the lifting problem

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, hook, "f"', "\sim"] & A \arrow[d] \\ B \arrow[r] \arrow[ur, dashed, "g"] & * \end{tikzcd}$$

we find that there is a lifting $g\colon B \to A$ such that $g \circ f = \mathrm{id}_A$. Notice that we used our assumption that $\mathbf{C}$ was a fibrant model category when we said that the map $A \twoheadrightarrow *$ is actually a fibration. Since $f, g \in \mathrm{WE}$ and A, B are both cofibrant (and automatically fibrant), by Whitehead Theorem 2.3.1 we also know that

$$f \circ g \sim \mathrm{id}_B$$

Then $F(g)F(f) = \mathrm{id}_{F(A)}$ implies that $\gamma F(f)$ has a left inverse in $\mathbf{Ho}(\mathbf{D})$. So, if it happens that $F(f \circ g) \sim F(\mathrm{id}_B) = \mathrm{id}_{F(B)}$ in $\mathbf{D}$, then $F(f)$ would be a weak equivalence, and Brown's lemma implies the existence of $\mathbb{L}F$.

But the condition $F(f \circ g) \sim \mathrm{id}_{F(B)}$ is easy to verify if F is defined by an explicit formula, and there is a concrete way to describe homotopy equivalence relations in $\mathbf{C}$ and $\mathbf{D}$ (like in chain complexes, topological spaces, DG algebras).

Example 3.6.4. Let $\mathbf{C} = \mathbf{DGA}_k$ be the category of DG algebras. Define $\Omega_{As} := T_k(\mathbb{V}) = k\langle x, dx\rangle$ to be the tensor algebra of the chain complex $\mathbb{V} = [0 \to k.x \xrightarrow{d} k.dx \to 0]$ with the variable x being of degree 0, dx being of degree -1 and the differential given by $d(x) := dx$. For any $a \in k$, we have DG evaluation maps

$$\begin{aligned} ev_a\colon \Omega_{As} &\to k \\ x &\mapsto a \\ dx &\mapsto 0 \end{aligned}$$

Then (see [BKR13, Prop. B2]) we have the following. If $f, g\colon A \to X$ with A cofibrant, then

$$f \sim g \Leftrightarrow \exists h\colon B * \Omega_{As} \text{ such that } h(0) := ev_0 \circ h = f, h(1) := ev_1 \circ h = g$$

3.7 Example: representation functor

Let k be a commutative ring. We assume that $\mathbb{Q} \subset k$. Let $\mathbf{Alg}_k$ be the category of associative unital algebras over k.

Recall that $M \in \mathrm{Ob}(\mathbf{Alg}_k)$ is called an *Azumaya algebra* if

- M is a faithfully projective k-module (i.e. M is a progenerator in $\mathbf{Mod}(k)$);
- the algebra map

$$\begin{aligned} A \otimes A^{\mathrm{op}} &\xrightarrow{\sim} \mathrm{End}_k(A) \\ a \otimes b &\mapsto [f(a \otimes b)\colon x \mapsto axb] \end{aligned}$$

  is an isomorphism.

Example 3.7.1. Let k be a field. Then $M = \mathbb{M}_n(k)$ the algebra of $n \times n$ matrices over k is an Azumaya algebra.

Example 3.7.2. Let X be a normal irreducible affine variety over a a field $\mathbb{F}$ of characteristic 0. Take $k = \mathbb{F}[X]$ to be the ring of regular functions on X, and take $\mathcal{F}$ to be an algebraic vector bundle on X. Then the module of global section $P = \mathcal{F}(X)$ is a finitely generated projective k-module, and $M = \mathrm{End}_k(P)$ is an Azumaya algebra.

Remark 3.7.3. In general, Azumaya algebras are *étale twisted forms* of matrix algebras $\mathbb{M}_n(k)$ over $X = Spec(k)$.

Every *Zariski* twisted form of $\mathbb{M}_n(k)$ is an Azumaya algebra as in Example 3.7.2. Such Azumaya algebras are called *trivial*. One can define the *Brower group* $\mathrm{Br}(k)$ to be the set of equivalence classes $[M]$ of Azumaya algebras over k with

$$[M] \sim [M'] \Leftrightarrow M \otimes_k M' \simeq \mathrm{End}_k(P)$$

for some projective k-module P, with the group multiplication given by

$$[M] \cdot [M'] := [M \otimes_k M']$$

Theorem 3.7.4 (O.Gabber)**.** $\mathrm{Br}(k) \simeq \mathrm{H}^2_{\acute{e}t}(X, \mathbb{G}_m)_{\mathrm{torsion}}$

There are many equivalent characterizations of Azumaya algebras. We need a categorical one, due to M.Knus and M.Ojanguren, see [KO74].

Let M be an algebra over k, and let $\mathbf{Mod}(k)$ and $\mathbf{Bimod}(M)$ be the categories of k-modules and M-bimodules, respectively. Define a pair of adjoint functors

$$F\colon \mathbf{Mod}(k) \leftrightarrows \mathbf{Bimod}(M)\colon G$$

where $F(N) := M \otimes_k N$ with M-bimodule structure given by $a(b \otimes n)c := abc \otimes n$. The functor G is given by $G(B) = B^M := \{b \in B \mid bm = mb, \forall m \in M\}$.

Theorem 3.7.5. *The algebra M is an Azumaya algebra if and only if the adjoint functors (F, G) are mutually inverse equivalences of categories.*

Corollary 3.7.6. *If M is an Azumaya algebra over k, then the functors F, G defined above induce mutually inverse equivalences of categories*

$$M \otimes -\colon \mathbf{Alg}_k \leftrightarrows \mathbf{Alg}_M\colon (-)^M$$

For a fixed algebra $A \in \mathbf{Alg}_k$ and an Azumaya algebra M over k, define

$$\mathsf{Rep}(A, M)\colon \mathbf{ComAlg}_k \to \mathbf{Sets}$$

by $\mathsf{Rep}(A, M)(B) = \mathrm{Hom}_{\mathbf{Alg}_k}(A, M \otimes_k B)$.

Theorem 3.7.7. *The functor* $\mathsf{Rep}(A, M)$ *is corepresented by the commutative algebra*

$$A_M := \Big[(A \sqcup_k M)^M\Big]_{\mathrm{ab}} \tag{3.3}$$

Proof. The functor $B \mapsto M \otimes_k B$ can be factored as

$$\mathbf{ComAlg}_k \overset{\mathrm{inc}}{\hookrightarrow} \mathbf{Alg}_k \xrightarrow{M\otimes -} \mathbf{Alg}_M \xrightarrow{\mathrm{forget}} \mathbf{Alg}_k$$

Each of the composite factors has left adjoint, namely

$$\mathbf{ComAlg}_k \xleftarrow{(-)_{\mathrm{ab}}} \mathbf{Alg}_k \xleftarrow{(-)^M} \mathbf{Alg}_M \xleftarrow{M\sqcup_k -} \mathbf{Alg}_k$$

The key fact is that $(-)^M$ is left adjoint to $M \otimes_k -$ since it is inverse to the equivalence functor $M \otimes_k -$, see Corollary 3.7.6. The composition of left adjoints is left adjoint to the composition of functors. Hence the result. $\square$

Definition 3.7.8. *For a given Azumaya k-algebra M, we call the functor*

$$\begin{aligned} (-)_M \colon \mathbf{Alg}_k &\to \mathbf{ComAlg}_k \\ A &\mapsto A_M \end{aligned}$$

the representation functor *in M.*

Formula 3.3 allow us to extend representation functor to a functor between model categories of DG algebras:

$$(-)_M \colon \mathbf{DGA}_k \leftrightarrows \mathbf{CDGA}_k \colon M \otimes_k - \tag{3.4}$$

From now on, assume that $\mathbb{Q} \subset k$.

Theorem 3.7.9 (cf.[BKR13])**.** *The pair of adjoint functors 3.4 is a Quillen pair. Hence $(-)_M$ has a total derived functor*

$$\mathbb{L}(-)_M \colon \mathbf{Ho}(\mathbf{DGA}_k) \to \mathbf{Ho}(\mathbf{CDGA}_k)$$

which is called derived representation functor *in M.*

Proof. Fibrations in both $\mathbf{DGA}_k$ and $\mathbf{CDGA}_k$ are (degree-wise) surjective morphisms and weak equivalences are quasi-isomorphisms. Since M is an Azumaya algebra, it is projective as a k-module, and so $M \otimes_k -$ is an *exact functor* on k-modules. It follows that $M \otimes_k -$ preserves WE and Fib$\,\cap\,$WE. Hence 3.4 is a Quillen pair. $\square$

Appendix A

Topological and geometric background

1 Connections on principal bundles

Let G be a Lie group and M be a fixed C^∞-manifold.

Definition 1.0.1. *A* principal G-bundle *over M is a smooth map $\pi\colon P \to M$ s.t. P is equipped with a* right *G-action $P \times G \to P$, $(p, g) \mapsto p \cdot g$ satisfying*

(1) $\forall x \in M$, $\pi^{-1}(x)$ is a G-orbit;

(2) π is a locally trivial fibre bundle in the sense that $\forall x \in M$, $\exists x \in U \subset M$ open in M and $\exists \varphi\colon \pi^{-1}(U) \xrightarrow{\sim} U \times G$ with properties

(3) φ is a diffeomorphism;

(4) the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi", "\sim"'] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "\mathrm{can}"] \\ & U & \end{tikzcd}$$

(5) $\forall p \in \pi^{-1}(U)$ we have $\varphi(p \cdot g) = \varphi(p) \cdot g$, i.e. φ is G-equivariant.

Remark 1.0.2. It follows from the definition that

- π is onto, and $\forall x \in M$, $\pi^{-1}(x) \simeq G$;
- G acts freely on P, and π induces a diffeomorphism $\pi\colon P/G \xrightarrow{\sim} M$.

Let $\pi\colon P \to M$ be a principal G-bundle over M. For $p \in P$ denote by T_pP the tangent space at p. The differential of π at p defines a projection $d\pi|_p\colon T_pP \to T_{\pi(p)}M$. By definition, we call $\mathrm{Ker}(d\pi|_p) \subseteq T_pP$ the *space of vertical vectors*, and we denote it by $V_p(P)$.

Definition 1.0.3. *A* connection *on P is a choice of* horizontal *subspace $H_p(P) \subseteq T_pP$ at each point $p \in P$ (depending smoothly on p) so that there is a direct sum decomposition*

$$T_pP \simeq V_p(P) \oplus H_p(P) \tag{A.1}$$

which is G-invariant (under G-action on P). In other words, $p \mapsto H_p(P)$ defines a distribution on P (= sub-bundle of the tangent bundle) which is G-invariant and transverse to the fibers of the principal bundle $\pi\colon P \to M$.

Now, for every point $p \in P$, the vector space $V_p(P)$ is canonically isomorphic to the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ which is the tangent space T_eG to the fibre $\pi^{-1}(\pi(p)) \simeq G$. Therefore, to give a decomposition (A.1) is equivalent to giving a projection

$$T_pP \twoheadrightarrow V_p(P) \simeq \mathfrak{g}, \quad v \mapsto \bar{v}$$

satisfying certain compatibility conditions (see below). This defines a $\mathfrak{g}$-valued 1-form $\theta \in \Omega^1(P, \mathfrak{g}) = \mathfrak{g} \otimes \Omega^1(P)$ on P satisfying

(1) G-invariance: $R_g^*\theta = \mathrm{Ad}_g^{-1} \circ \theta$, $\forall g \in G$, where $R_g\colon G \to G$ denotes the right multiplication by $g \in G$.

(2) $\theta(v) = \bar{v}$, $\forall v \in V_p(P)$, i.e. maps vertical vectors to their image under the isomorphism $V_p(P) \simeq \mathfrak{g}$.

Remark 1.0.4. The last condition is equivalent to the following. For every $p \in P$, the inclusion $\iota_p\colon G \simeq P_p \hookrightarrow P$ satisfies $\iota_p^*\theta = \theta_{MC} \in \Omega^1(G, \mathfrak{g})$, where θ_{MC} denotes the so-called Maurer–Cartan form assigning to each tangent vector its left invariant extension.

If $\theta \in \Omega^1(P, \mathfrak{g})$ is a connection, then the horizontal space is given by

$$H_p(P) = \{v \in T_p(P) \mid \theta(v) = 0\}.$$

Definition 1.0.5. *A* curvature *of a connection $\theta \in \Omega^1(P, \mathfrak{g})$ is a $\mathfrak{g}$-valued* 2*-form $\Omega \in \Omega^2(P, \mathfrak{g}) = \mathfrak{g} \otimes \Omega^2(P)$ satisfying the structure equation*

$$d\theta = -\frac{1}{2}[\theta, \theta] + \Omega \tag{A.2}$$

Remark 1.0.6. To define the commutator $[-,-]$ in (A.2) it is convenient to think of $\Omega^k(P, \mathfrak{g})$ as a subspace of the current Lie algebra $\mathfrak{g}(\Omega^\bullet(P))$ via inclusion

$$\Omega^k(P, \mathfrak{g}) \simeq \Omega^k(P) \otimes \mathfrak{g} \hookrightarrow \Omega^\bullet(P) \otimes \mathfrak{g} \simeq \mathfrak{g}(\Omega^\bullet(P)).$$

Here for the graded algebra $A = \Omega^\bullet(P)$ with multiplication $\wedge$ we denote by $\mathfrak{g}(A)$ its current Lie algebra $\mathfrak{g} \otimes A$ with the Lie bracket defined by

$$[\xi \otimes a, \eta \otimes b] := [\xi, \eta] \otimes (a \wedge b)$$

So $[\theta, \theta]$ in Equation A.2 is the bracket in the current Lie algebra $\mathfrak{g}(\Omega^\bullet(P))$.

Definition 1.0.7. *A connection* $\theta \in \Omega^1(P, \mathfrak{g})$ *is called* flat *if its curvature* $\Omega \equiv 0$.

Example 1.0.8. Let $\pi\colon M \times G \twoheadrightarrow G$ is the obvious projection. It is a principal G-bundle, and the pull-back $\theta = \pi^* \theta_{MC}$ of the Maurer–Cartan form is an example of flat connection. Its flatness follows from flatness of θ_{MC}.

Theorem 1.0.9 (Frobenius)**.** *Connection θ is flat if and only if the corresponding distribution $H(P)$ of horizontal vectors is* completely integrable. *This means that there exists a submanifold $N \subseteq P$ called* integral manifold *such that $\forall p \in P$, $H_p(P) \simeq T_pN$.*

Equivalently, θ is flat if $\forall x \in M$, $\exists U_x \subset M$ open with a trivialization $P|_{U_x} \xrightarrow{\sim} U_x \times G$ such that $\theta|_{U_x}$ is the pull-back of Maurer–Cartan from θ_{MC} on G via the obvious projection $U_x \times G \twoheadrightarrow G$. The following characterization of flat connections is very useful.

Corollary 1.0.10. *Let $\pi\colon P \to M$ be a principal G-bundle. The following are equivalent:*

- *P admits a flat connection;*
- *there exists $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ open covering of M for which the transition functions for P $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ are* constant *for all $\alpha, \beta \in I$.*

Definition 1.0.11. *A rrincipal bundle is called* flat *if it admits a flat connection.*

Suppose $f\colon H \to G$ is a homomorphism of Lie groups, $\xi\colon F \to M$ is a principal H-bundle, and $\pi\colon P \to M$ is a principal G-bundle. We say that ξ is a *reduction* of π (relative to f) if there is a C^∞-map $\varphi\colon F \to P$ s.t.

- $\varphi(F_x) \subseteq P_x$, $\forall x \in M$;
- $\varphi(p \cdot h) = \varphi(p) \cdot f(h)$, $\forall p \in F$, $\forall h \in H$.

Exercise 1. Show that $\pi\colon P \to M$ has a reduction to $\xi\colon F \to M$ *iff* there exists a covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of M and a set of transition functions for P of the form $\{f \circ h_{\alpha\beta}\}$ with $h_{\alpha\beta}\colon U_\alpha \cap U_\beta \to H$ and $f\colon H \to G$ a homomorphism.

Thus, $\pi\colon P \to M$ is flat *iff* it has reduction to G^δ, where G^δ is the group G equipped with *discrete topology*, and the homomorphism $f\colon G^\delta \to G$ is the identity map.

2 Coverings of spaces

Assume first that all the spaces we are dealing with are path connected. Later this assumption will be dropped.

Definition 2.0.1. *A map $p\colon E \to B$ is called a* covering *of B if every point $c \in B$ has an open neighborhood $U_c \subseteq B$ (called a fundamental neighborhood of c) such that every connected component of p^{-1} is open and is mapped homeomorphically to U_c.*

Terminology 2.0.2. We call E the total space, and B the base space of a the covering $p\colon E \to B$. Preimage $p^{-1}(c)$ of any point $c \in B$ is called the fiber of p over c and will be denoted by E_c.

Theorem 2.0.3. *Let $p\colon E \to B$ be a covering, $c \in B$ and $e, e' \in E$ such that $p(e) = p(e') = c$. Then*

(1) *Any path $f\colon [0,1] \to B$ with $f(0) = c$ lifts uniquely to a path $\widetilde{f}\colon [0,1] \to E$ such that $f(0) = e$ and $p \circ \widetilde{f} = f$.*

(1) *If two paths $f, g\colon [0,1] \to B$ are homotopy equivalent, their lifts $\widetilde{f} \sim \widetilde{g}$ are also homotopy equivalent with $\widetilde{f}(0) = \widetilde{g}(0)$ and hence also $\widetilde{f}(1) = \widetilde{g}(1)$.*

(1) *The induced homomorphism $p_*\colon \pi_1(E, e) \to \pi_1(B, c)$ is injective.*

(1) *The subgroups $p_*(\pi_1(E, e))$ and $p_*(\pi_1(E, e'))$ are conjugate inside $\pi_1(B, c)$.*

(1) *As e' runs over the entire fiber E_c, the subgroup $p_*(\pi_1(E, e'))$ runs over all subgroups conjugate to $p_*(\pi_1(E, e))$ in $\pi_1(B, c)$.*

Definition 2.0.4. *A covering $p\colon E \to B$ is called* universal *if $\pi_1(E) = \{1\}$, i.e. if E is simply-connected.*

For a space B, define its *fundamental groupoid* $\mathbf{\Pi}(B)$ as follows. Its object set $\mathrm{Ob}(\mathbf{\Pi}(B))$ is the space B itself, and $\mathrm{Hom}_{\mathbf{\Pi}(B)}(x, y) = \{\text{homotopy classes of paths } f\}$. For a path f we denote by $[f]$ its homotopy class.

Definition 2.0.5. *Define the* fiber translation functor *$E\colon \mathbf{\Pi}(B) \to \mathbf{Sets}$ as follows. On objects, for $c \in \mathrm{Ob}(\mathbf{\Pi}(B)) = B$ we put $E(c) := E - c = p^{-1}(c)$. If $[f]\colon c \to c'$ is a homotopy class of paths from c to c', pick a representative f of this class. It is a path $f\colon [0,1] \to B$ with $f(0) = c$ and $f(1) = c'$. Now define $E[f]$ to be the map $E_c \to E_{c'}$ sending $e \in E_c$ to $e' \in E_{e'}$, where e' is the target of the lift $\widetilde{f}$ of f s.t. $\widetilde{f}(0) = e$.*

For each point $c \in B = \mathrm{Ob}(\mathbf{\Pi}(B))$,

$$\mathrm{Hom}_{\Pi(B)}(c, c) = \mathrm{Aut}_{\Pi(B)}(c) = \pi_1(B, c).$$

The restriction of the fiber translation functor E to $\mathrm{Hom}_{\mathbf{\Pi}(B)}(c, c)$ gives an action of $\pi_1(B, c)$ on the fiber $E(c) = E_c$. This action is transitive, and for every $e \in E_c$ the isotropy group $\mathrm{Stab}_{\pi_1(B,c)}(e)$ coincides with $p_*(\pi_1(E, e))$. Thus, there is an isomorphism of $\pi_1(B, c)$-sets

$$E_c \simeq \pi_1(B, c)/p_*(\pi_1(E, e))$$

In particular, if $p\colon E \to B$ is the universal covering, the group $\pi_1(B, c)$ acts freely on E_c and so $E_c \simeq \pi_1(B, c)$ as $\pi_1(B, c)$-sets.

We now define the category $\mathbf{Cov}(B)$ of coverings of B as follows. Its objects are coverings $p\colon E \to B$. Morphisms from a covering $p\colon E \to B$ to a covering $p'\colon E' \to B$ are maps $\alpha\colon E \to E'$ such that $p = p' \circ \alpha$:

$$\begin{tikzcd} E \arrow[rr, "\alpha"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

Assigning to each $[p\colon E \to b] \in \mathrm{Ob}(\mathbf{Cov}(B))$ its fiber functor $E(p)\colon \mathbf{\Pi}(B) \to \mathbf{Sets}$ gives a functor

$$E(-)\colon \mathbf{Cov}(B) \to \mathbf{Fun}(\mathbf{\Pi}(B), \mathbf{Sets})$$

Definition 2.0.6. *A functor $T\colon \mathbf{\Pi}(B) \to \mathbf{Sets}$ is called* transitive *if for every $c \in B$ the homomorphism $T(c)\colon \mathrm{Aut}_{\mathbf{\Pi}(B)}(c) \to \mathrm{Aut}_{\mathbf{Sets}}(T(c))$ of groups defines a transitive action of $\mathrm{Aut}_{\mathbf{\Pi}(B)}(c)$ on $T(c)$.*

Theorem 2.0.7 (Classification of coverings)**.** *The functor $E(-)$ is fully faithful, and its essential image coincides with the full subcategory of transitive functors, $\mathbf{Fun}^{\mathrm{tr}}(\mathbf{\Pi}(B), \mathbf{Sets}) \subset \mathbf{Fun}(\mathbf{\Pi}(B), \mathbf{Sets})$. In other words, we have an equivalence of categories*

$$E(-)\colon \mathbf{Cov}(B) \overset{\sim}{\to} \mathbf{Fun}^{\mathrm{tr}}(\mathbf{\Pi}(B), \mathbf{Sets})$$

The above theorem might look abstract, but it actually classifies coverings in an explicit way. Indeed, let's fix $c \in B$ and let $G = \pi_1(B, c)$ be the corresponding fundamental group. Then the inclusion $c \hookrightarrow B$ given an equivalence of groupoids $\underline{G} \overset{\sim}{\to} \mathbf{\Pi}(B)$ since we assumed B is connected. Then,

$$\mathbf{Fun}(\mathbf{\Pi}(B), \mathbf{Sets}) \simeq \mathbf{Fun}(\underline{G}, \mathbf{Sets}) \simeq \mathbf{G\text{-}sets}$$

which implies that $\mathbf{Fun}^{\mathrm{tr}}(\mathbf{\Pi}(B), \mathbf{Sets})$ is isomorphic to the category $\mathbf{G\text{-}sets}^{\mathrm{tr}}$ of transitive G-sets, which is in turn isomorphic to the subgroup category $\mathcal{O}(G)$. The set of objects of the latter category consists of all subgroups $H \subset G$, and the set of morphisms between $H_1, H_2 \in \mathrm{Ob}(\mathcal{O}(G))$ is the set $\{g \in G \mid gH_1g^{-1} \subseteq H_2\}/N_G(H_1)$ where $N_G(H_1)$ is the normalizer of H_1 in G. The isomorphism $\mathcal{O}(G) \overset{\sim}{\to} \mathbf{G\text{-}sets}^{\mathrm{tr}}$ is defined by $H \mapsto G/H$.

Note that the connectivity assumption in the constructions above can be dropped. Then the analog of Theorem 2.0.7 for non-connected spaces is the following.

Theorem 2.0.8. *The fiber functor gives an equivalence of categories*

$$E(-)\colon \mathbf{Cov}(B) \overset{\sim}{\to} \mathbf{Fun}(\mathbf{\Pi}(B), \mathbf{Sets}) \text{ (\textit{no "tr" in the RHS !})}$$

3 Homotopy fibration sequences and homotopy fibers

Let $\mathbf{C}$ be a category, and $i, p \in \mathrm{Mor}(\mathbf{C})$ two morphisms in $\mathbf{C}$. Define a *lifting problem* for i and p as the family of all morphisms f, g making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

A solution to a lifting problem for i, p is an arrow $h\colon B \to X$ making all the diagrams above commutative, i.e. $p \circ h = g$ and $h \circ i = g$ for *all* f, g in the lifting problem. If such a solution h exists, we say that i has *left lifting property* (LLP for short) with respect to p, or, equivalently, that p has *right lifting property* (RLP) with respect to i. In this case we will write $i \boxslash p$. If $M \subseteq \mathrm{Mor}(\mathbf{C})$ is any class of morphisms, we define

$$M_{\boxslash} = \{p \in \mathrm{Mor}(\mathbf{C}) \mid i \boxslash p \quad \forall i \in M\}$$
$$_{\boxslash}M = \{i \in \mathrm{Mor}(\mathbf{C}) \mid i \boxslash p \quad \forall p \in M\}$$

Let's now take $\mathbf{C} = \mathbf{Spaces}$, and consider the family of maps

$$Disk = \{i_0\colon \mathbb{D}^n \hookrightarrow \mathbb{D}^n \times I\}_{n \geqslant 0}$$

where $\mathbb{D}^n$ denotes the n-dimensional cube $\mathbb{D}^n = I^n$, $I = [0,1]$ is the unit interval and the maps $i_0\colon \mathbb{D}^n \hookrightarrow \mathbb{D}^n \times I$ map $x \mapsto (x, 0)$.

Definition 3.0.1. *A surjective map $p\colon E \to B$ is called a* fibration *(in the sense of Serre) if $p \in Disk_{\boxslash}$.*

In other words, for any two maps $f\colon \mathbb{D}^n \to E$ and $g\colon \mathbb{D}^n \times I \to B$ satisfying $p \circ f = g \circ i_0$ there exists a map $h\colon \mathbb{D}^n \times I \to E$ making the following diagram commute:

$$\begin{tikzcd} \mathbb{D}^n \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ \mathbb{D}^n \times I \arrow[r, "g"'] \arrow[ur, dashed, "\exists h"] & B \end{tikzcd}$$

Lemma 3.0.2. *If $p\colon E \to B$ is a fibration, then pull-back $\widetilde{p}$ along any morphism $f\colon A \to B$ is again a fibration:*

$$\begin{tikzcd} A \times_f E \arrow[r] \arrow[d, "\widetilde{p}"'] & E \arrow[d, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Proof. Suppose we are given a map $g\colon \mathbb{D}^n \to A \times_f E$ and $h\colon \mathbb{D}^n \times I \to A$ making the following diagram commute:

$$\begin{tikzcd}
\mathbb{D}^n \arrow[r, "g"] \arrow[d, "i_0"'] & A \times_f E \arrow[r] \arrow[d, "\widetilde{p}"'] & E \arrow[d, "p"] \\
\mathbb{D}^n \times I \arrow[r, "h"'] & A \arrow[r, "f"'] & B
\end{tikzcd}$$

We would like to construct a map $\widetilde{h}\colon \mathbb{D}^n \times I \to A \times_f E$ lifting h.

Since $p\colon E \to B$ is a fibration, there exists a map $h'\colon \mathbb{D}^n \times I \to E$ lifting $f \circ h$. Now we have a map $h'\colon \mathbb{D}^n \times I \to E$ and $h\colon \mathbb{D}^n \times I \to A$ such that $p \circ h' = f \circ h$. Then by the universal property of pull-back, there exists unique map $\mathbb{D}^n \times I \to A \times_f E$. This map make the following diagram commute, and so it gives the required lifting $\widetilde{h}$ of h

$$\begin{tikzcd}
\mathbb{D}^n \arrow[r, "g"] \arrow[d, "i_0"'] & A \times_f E \arrow[r] \arrow[d, "\widetilde{p}"] & E \arrow[d, "p"] \\
\mathbb{D}^n \times I \arrow[r, "h"'] \arrow[ur, dashed, "\widetilde{h}"] \arrow[urr, dashed, "h'" near end] & A \arrow[r, "f"'] & B
\end{tikzcd}$$

$\square$

Remark 3.0.3. There is a more restrictive class of maps, called Hurewich fibrations, which consists of maps $p\colon E \to B$ having RLP with respect to maps $\{i_0\colon A \hookrightarrow A \times I\}$ for all spaces A.

Theorem 3.0.4 (Homotopy exact sequence)**.** *Let $(B,*)$ be a based space, and let $p\colon (E,*) \to (B,*)$ be a fibration. Denote the fiber $p^{-1}(*) \subset E$ by F, and fix a base point $* \in F$. Then there exists a long exact sequence of homotopy groups*

$$\begin{aligned} \dots \pi_{n+1}(B,*) \to \pi_n(F,*) \to \pi_n(E,*) \to \pi_n(B,*) \to \dots \\ \dots \to \pi_1(B,*) \to \pi_0(F,*) \to \pi_0(E,*) \to \pi_0(B,*) \end{aligned} \tag{A.3}$$

Example 3.0.5. Locally trivial fiber bundles $E \to B$ are Serre fibrations.

Example 3.0.6 (Path fibrations)**.** Let X be a path connected space. The *path space* of X is the space $X^I = \mathrm{Map}(I, X)$ of all continuous maps $\gamma\colon I \to X$ equipped with compact-open topology. That is, topology generated by open sets U^C of paths mapping a fixed compact subset $C \subset I$ into a fixed open subset $U \subset X$. If X has a base point $* \in X$ we define PX to be the space of paths based at $*$, i.e. $PX = \{\gamma\colon I \to X \mid \gamma(0) = *\}$. From now on we assume X to be pointed.

The *path fibration* over $(X,*)$ is the map $p_1\colon PX \to X$ mapping $\gamma \mapsto \gamma(1)$. The fiber $p_1^{-1}(*)$ over the base point is denoted by ΩX and is called the *space of based loops*. So

$$\Omega X = \{\gamma\colon I \to X \mid \gamma(0) = \gamma(1) = *\}$$

Note that both spaces PX and ΩX are naturally pointed, with the basepoint being the constant path $\gamma_*(t) \equiv *$.

Lemma 3.0.7. *(1) The space PX is contractible for any space X.*

(2) If X is homotopy equivalent to a CW complex, then so is ΩX.

Suppose $f\colon X \to Y$ is a map of topological spaces, and let's assume Y is path connected, so that its path space PY is connected. Define the *mapping path space* Nf by as the pull-back $Nf = X \times_f Y^I$ of the following diagram

$$\begin{tikzcd} Nf \arrow[r, "g"] \arrow[d, "\pi"'] & Y^I \arrow[d, "p_0"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

In other words, $Nf = \{(x, y) \mid x \in X, \gamma \in PY, \gamma(0) = f(x)\}$. We denote $p\colon Nf \to Y$ the composition $p = f \circ \pi = p_0 \circ g$ mapping $(x,\gamma) \mapsto p(x,\gamma) = \gamma(1)$. We can construct a section of the natural projection π as follows. Let $i\colon X \to Nf$ be the map that sends each point $x \in X$ to the pair $(x, \gamma_{f(x)}) \in Nf$ where $\gamma_{f(x)}$ is the constant path $\gamma_{f(x)}(t) = f(x)$.

Then $f\colon X \to Y$ can be decomposed as

$$X \xrightarrow{i} Nf \xrightarrow{p} Y$$

Note that $i\colon X \to Nf$ is a homotopy equivalence, with π giving its homotopy inverse. Indeed, $\pi \circ i = \mathrm{id}_X$, and $i \circ \pi \sim \mathrm{id}_{Nf}$ with homotopy $h\colon Nf \times I \to Nf$ given by $h_t(x,\gamma) = (x, \gamma_t)$, where $\gamma_t(s) = \gamma((1-t)s)$. Therefore, i is a homotopy equivalence.

On the other hand, $p\colon Nf \to Y$ is a fibration. To prove this, we need to show that for every diagram of the form

$$\begin{tikzcd} \mathbb{D}^n \arrow[r, "g"] \arrow[d, "i_0"'] & Nf \arrow[d, "p"] \\ \mathbb{D}^n \times I \arrow[r, "f"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

there exists a map h making the diagram commute. We write $g(a)$ as $g(a) = (g_1(a), g_2(a))$ and set $h(a,t) = (g_1(a), j(a,t))$ where

$$j(a,t)(s) = \begin{cases} g_2(a)(s + st), & \text{if } 0 \leqslant s \leqslant \frac{1}{1+t} \\ f(a, s + ts - 1), & \text{if } \frac{1}{1+t} \leqslant s \leqslant 1 \end{cases}$$

It is easy to see that this map is the lifting we want. Therefore, we have just proved the following

Proposition 3.0.8. *Any map $f\colon X \to Y$ of topological spaces can be written as a composition of a homotopy equivalence and a fibration.*

Definition 3.0.9. *The fiber $p^{-1}(*)$ of the map $p\colon Nf \to Y$ over the base point $*$ as called* the homotopy fiber *of f over $*$, and is denoted by Ff.*

In other words, the homotopy fiber is the pullback $Ff = Nf \times_Y \{*\}$, and so

$$\begin{aligned} Ff &= Nf \times_Y \{*\} \\ &= X \times_f Y^I \times_Y \{*\} \\ &= X \times_f PY \\ &= \{(x, \gamma) \mid x \in X, \gamma \in PY, \gamma(0) = * \text{ and } \gamma(1) = f(x)\}. \end{aligned}$$

Given any sequence of spaces $F \xrightarrow{j} X \xrightarrow{f} Y$ such that $f \circ j$ is a constant map, the universal property of Ff gives a canonical map $g\colon F \to Ff$ sending $x \mapsto (j(x), \gamma_{f(j(x))}) = (j(x), \gamma_*)$:

$$\begin{tikzcd} F \arrow[rr, "j"] \arrow[dr, dashed, "g"] \arrow[ddr] & & X \arrow[d, "i"] \arrow[dd, bend left, "f"] \\ & Ff \arrow[r] \arrow[d] & Nf \arrow[d, "p"] \\ & \{*\} \arrow[r] & Y \end{tikzcd} \tag{A.4}$$

Definition 3.0.10. *The sequence $F \xrightarrow{j} X \xrightarrow{f} Y$ is called a* homotopy fibration sequence *if the induced map $g\colon F \to Ff$ is a homotopy equivalence.*

Corollary 3.0.11. *If $F \to X \to Y$ is a homotopy fibration sequence, then there is a long exact sequence of homotopy groups*

$$\cdots \to \pi_{n+1}(Y) \to \pi_n(F) \to \pi_n(X) \to \pi_n(Y) \to \ldots$$

Remark 3.0.12. There is a dual construction. Namely, any map $f\colon X \to Y$ can be factored as a composition of a cofibration followed by a homotopy equivalence. This involves a construction called *the mapping cylinder* of f, and denoted by Mf. The mapping cylinder is defined by

$$Mf = Y \cup_f (X \times I) = Y \sqcup (X \times I) \Big/ f(x) \sim (x, 0)$$

Then the natural inclusion $j\colon X \hookrightarrow Mf$ sending $x \mapsto (x, 1) \in Mf$ is a cofibration, and the projection $p\colon Mf \twoheadrightarrow Y$ is a homotopy equivalence.

Definition 3.0.13. *The* homotopy cofiber *Cf of the map f is defined by $Cf = Mf/j(X)$. This is the* mapping cone *of the map $f\colon X \to Y$.*

For the details, see [May99, Chapters 6—8].

Bibliography

[BFR14] Yuri Berest, Giovanni Felder, and Ajay Ramadoss. Derived representation schemes and noncommutative geometry. In *Expository lectures on representation theory*, volume 607 of *Contemp. Math.*, pages 113–162. Amer. Math. Soc., Providence, RI, 2014. 109

[BK72] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin-New York, 1972. 11

[BKR13] Yuri Berest, George Khachatryan, and Ajay Ramadoss. Derived representation schemes and cyclic homology. *Adv. Math.*, 245:625–689, 2013. 132, 134

[BMP14] Yuri Berest, Daniel Miller, and Sasha Patotski. Lecture notes on homological algebra. 2014. 31, 33, 44, 84

[DK80] W. G. Dwyer and D. M. Kan. Calculating simplicial localizations. *J. Pure Appl. Algebra*, 18(1):17–35, 1980. 126

[DS95] W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995. 109

[Dup01] Johan L. Dupont. *Scissors congruences, group homology and characteristic classes*, volume 1 of *Nankai Tracts in Mathematics*. World Scientific Publishing Co., Inc., River Edge, NJ, 2001. 16

[FH13] Daniel S. Freed and Michael J. Hopkins. Chern-Weil forms and abstract homotopy theory. *Bull. Amer. Math. Soc. (N.S.)*, 50(3):431–468, 2013. 12

[FL91] Zbigniew Fiedorowicz and Jean-Louis Loday. Crossed simplicial groups and their associated homology. *Trans. Amer. Math. Soc.*, 326(1):57–87, 1991. 42

[GZ67] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967. 2

[Haz09] Michiel Hazewinkel. Witt vectors. I. In *Handbook of algebra. Vol. 6*, volume 6 of *Handb. Algebr.*, pages 319–472. Elsevier/North-Holland, Amsterdam, 2009. 94

[HH79] Jean-Claude Hausmann and Dale Husemoller. Acyclic maps. *Enseign. Math. (2)*, 25(1-2):53–75, 1979. 46

[Hil81] Howard L. Hiller. λ-rings and algebraic K-theory. *J. Pure Appl. Algebra*, 20(3):241–266, 1981. 89, 99, 100, 103

[Hir03] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003. 109, 110

[Hov99] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999. 109, 110

[KN96] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. I*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1996. Reprint of the 1963 original, A Wiley-Interscience Publication. 11

[KO74] Max-Albert Knus and Manuel Ojanguren. *Théorie de la descente et algèbres d'Azumaya*. Lecture Notes in Mathematics, Vol. 389. Springer-Verlag, Berlin-New York, 1974. 133

[Mal07] Georges Maltsiniotis. Le théorème de Quillen, d'adjonction des foncteurs dérivés, revisité. *C. R. Math. Acad. Sci. Paris*, 344(9):549–552, 2007. 128

[May99] J. P. May. *A concise course in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999. 29, 31, 58, 143

[Mil57] John Milnor. The geometric realization of a semi-simplicial complex. *Ann. of Math. (2)*, 65:357–362, 1957. 25

[Mil71] John Milnor. *Introduction to algebraic K-theory*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1971. Annals of Mathematics Studies, No. 72. 43

[MP12] J. P. May and K. Ponto. *More concise algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2012. Localization, completion, and model categories. 109, 111

[Qui67] Daniel G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967. 109

[Qui73] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories*, number 541 in Lecture Notes in Mathematics, pages 85–147. Springer, Berlin, 1973. 37, 41

[Seg74] Graeme Segal. Categories and cohomology theories. *Topology*, 13:293–312, 1974. 17

[SGA03] *Revêtements étales et groupe fondamental (SGA 1).* Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3. Société Mathématique de France, Paris, 2003. Séminaire de géométrie algébrique du Bois Marie 1960–61. [Algebraic Geometry Seminar of Bois Marie 1960-61], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; MR0354651 (50 #7129)]. 37

[Sta63] James Dillon Stasheff. Homotopy associativity of H-spaces. I, II. *Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid.*, 108:293–312, 1963. 27

[Ste67] N. E. Steenrod. A convenient category of topological spaces. *Michigan Math. J.*, 14:133–152, 1967. 12

[Str72] Arne Strøm. The homotopy category is a homotopy category. *Arch. Math. (Basel)*, 23:435–441, 1972. 111

[Str09] N. P. Strickland. The category of CGWH spaces. *preprint*, 2009. 12

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994. 47

[Whi78] George W. Whitehead. *Elements of homotopy theory*, volume 61 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978. 76, 81, 99