Homological Algebra

Yuri Berest[1]

Fall 2013 – Spring 2014

[1]Notes taken by Daniel Miller and Sasha Patotski at Cornell University

Contents

1 Standard complexes in Geometry **1**
1 Complexes and cohomology . . . 1
2 Simplicial sets and simplicial homology . . . 3
2.1 Motivation . . . 3
2.2 Definitions . . . 5
2.3 Geometric realization . . . 6
2.4 Homology and cohomology of simplicial sets . . . 8
2.5 Applications . . . 9
2.6 Homology and cohomology with local coefficients . . . 10
3 Sheaves and their cohomology . . . 12
3.1 Presheaves . . . 12
3.2 Definitions . . . 12
3.3 Kernels, images and cokernels . . . 13
3.4 Germs, stalks, and fibers . . . 14
3.5 Coherent and quasi-coherent sheaves . . . 16
3.6 Motivation for sheaf cohomology . . . 17
3.7 Sheaf cohomology . . . 19
3.8 Applications . . . 21
3.9 Čech cohomology . . . 23

2 Standard complexes in algebra **25**
1 Group cohomology . . . 25
1.1 Definitions and topological origin . . . 25
1.2 Interpretation of $\mathrm{H}^1(G, A)$ . . . 26
1.3 Interpretation of $\mathrm{H}^2(G, A)$ . . . 27
2 Hochschild (co)homology . . . 29
2.1 The Bar complex . . . 29
2.2 Differential graded algebras . . . 30
2.3 Why DG algebras? . . . 33
2.4 Interpretation of bar complex in terms of DG algebras . . . 34
2.5 Hochschild (co)homology: definitions . . . 35
2.6 Centers and Derivations . . . 37

2.7 Extensions of algebras . . . 37
2.8 Crossed bimodules . . . 38
2.9 The characteristic class of a DG algebra . . . 40
3 Deformation theory . . . 41
3.1 Motivation . . . 41
3.2 Formal deformations . . . 42
3.3 Deformation theory in general . . . 44
3.4 The Gerstenhaber bracket . . . 49
3.5 Stasheff construction . . . 50
3.6 Kontsevich Formality Theorem . . . 52
3.7 Deformation theory in algebraic number theory . . . 53

3 Category theory **57**
1 Basic category theory . . . 57
1.1 Definition of categories . . . 57
1.2 Functors and natural transformations . . . 59
1.3 Equivalences of categories . . . 61
1.4 Representable functors and the Yoneda lemma . . . 62
1.5 Adjoint functors . . . 66
1.6 Limits and colimits . . . 68
2 Special topics in category theory . . . 71
2.1 Brief introduction to additive categories . . . 71
2.2 Center of a category and Bernstein trace . . . 72
2.3 Morita theory . . . 76
2.4 Recollement (gluing) of abelian sheaves . . . 78
2.5 Kan extensions . . . 79

4 Classical homological algebra **83**
1 Abelian categories . . . 83
1.1 Additive categories . . . 83
1.2 Non-additive bimodules . . . 85
1.3 Abelian categories . . . 87
1.4 Complexes in abelian categories . . . 91
1.5 Exact functors . . . 95
1.6 Adjointness and exactness . . . 96
2 Finiteness conditions . . . 99
2.1 AB5 categories . . . 99
2.2 Grothendieck categories . . . 101
2.3 Inductive closure of an abelian category . . . 102
2.4 Finiteness conditions . . . 104
3 Classical derived functors . . . 106
3.1 Injectives and injective envelopes . . . 106
3.2 Canonical constructions on complexes . . . 108

3.3 "Classical" definition of classical derived functors . . . 111
3.4 δ-functors . . . 113
3.5 Main properties of resolutions . . . 115
3.6 Definition of classical derived functor via δ-functors . . . 117
3.7 Examples of derived functors . . . 121

5 Derived categories **125**
1 Localization of categories . . . 125
1.1 Motivation . . . 125
1.2 Definition of localization . . . 126
1.3 Calculus of fractions . . . 128
1.4 Ore localization . . . 130
2 Localization of abelian categories . . . 134
2.1 Serre quotients . . . 134
2.2 Injective envelopes . . . 137
2.3 Localizing subcategories . . . 138
3 Derived categories . . . 139
3.1 Definition and basic examples . . . 139
3.2 The homotopy category . . . 141
3.3 Verdier theorem . . . 143

6 Triangulated categories **145**
1 The basics . . . 145
1.1 Definitions . . . 145
1.2 Examples of triangulated categories . . . 146
1.3 Basic properties of triangulated categories . . . 148
2 Further properties . . . 152
2.1 Abstract cone and octahedron axiom . . . 152
2.2 The homotopy category is triangulated (need proofs!!) . . . 153
2.3 Localization of triangulated categories . . . 153
2.4 Exact functors . . . 154
2.5 Verdier quotients . . . 155
2.6 Exact categories . . . 156
3 t-structures and the recollement . . . 160
3.1 Motivation and definition . . . 160
3.2 The core of a t-structure and truncation functors . . . 161
3.3 Cohomological functors . . . 162
3.4 t-exact functors . . . 163
3.5 Derived category of t-structure . . . 164
3.6 Gluing t-structures . . . 165
3.7 Examples of gluing . . . 165
3.8 Recollement for triangulated categories . . . 167
3.9 Example: topological recollement . . . 168

3.10 Example: algebraic recollement 168

7 Applications of homological algebra 171
1 Perverse sheaves and quivers 171
1.1 Stratifications 171
1.2 Perversity function and perverse sheaves 172
1.3 Middle extension 172
1.4 Constructible complexes 174
1.5 Refining stratifications 175
1.6 Gluing 176
2 Matrix factorizations 177
2.1 Introduction 177
2.2 The classical exact equivalences 177
2.3 The stabilization of the residue field 183
2.4 An enlargement of the matrix factorization category 184
2.5 Homotopy theory of DG categories 186
2.6 Derived Morita Equivalence 189
2.7 Hochschild cohomology 191
3 Some physics 194
3.1 Physical motivation 195
3.2 Cobordism 196
3.3 Topological quantum field theories 197
3.4 Chern-Simons 198
3.5 Atiyah-Segal axioms 198
4 Perfectoid rings, almost mathematics, and the cotangent complex 199
4.1 Perfectoid fields 199
4.2 Almost mathematics 201
4.3 The cotangent complex 202
5 Nearby and Vanishing Cycle Functors 204
5.1 Construction 204
5.2 Monodromy 205
5.3 Gluing problems 206
5.4 Monadology 208
5.5 Reflection functors 210
6 Abstract Nearby Cycles functor 212
6.1 The setup 212
6.2 Multiplicative Preprojective Algebra Associated to a Quiver 214

A Miscellaneous topics 217
1 Characteristic classes of representations (after Quillen) 217
2 Generalized manifolds 220
3 Quivers and path algebras 222
3.1 Basic definitions 222

3.2 Path algebra of a quiver . . . 223
3.3 The structure of the path algebra . . . 224
3.4 Representations of quivers . . . 225

B Exercises **233**
1 Standard complexes in Algebra and Geometry . . . 233
2 Classical homological algebra . . . 233
3 Residues and Lie cohomology . . . 235
3.1 Commensurable subspaces . . . 235
3.2 Traces . . . 236
3.3 Residues . . . 237
3.4 Interpretation in terms of Lie algebra cohomology . . . 238
3.5 Adeles and residues on algebraic curves . . . 238

Chapter 1

Standard complexes in Geometry

In this chapter we introduce basic notions of homological algebra such as complexes and cohomology. Moreover, we give a lot of examples of complexes arising in different areas of mathematics giving different cohomology theories. For instance, we discuss simplicial (co)homology, cohomology of sheaves, group cohomology, Hochschild cohomology, differential graded (DG) algebras and deformation theory.

Throughout the chapter we will use language of category theory. All the necessary categorical definitions are reviewed in the first section of the Chapter 3.

1 Complexes and cohomology

Definition 1.0.1. *A* chain complex $C_\bullet$ *is a sequence of abelian groups together with group homomorphisms*

$$C_\bullet: \quad \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots$$

such that $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$.

Definition 1.0.2. *A* cochain complex *is a sequence*

$$C^\bullet: \quad \cdots \to C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} C^{n+2} \to \cdots$$

such that $d^{n+1} \circ d^n = 0$ for all n.

One usually calls d the *differential*, or a boundary operator in the case of a chain complex. Also, one often leaves out the subscripts / superscripts of d, writing $d^2 = 0$.

Remark 1.0.3. (C^n, d^n) is a cochain complex if and only if $(C_n = C^{-n}, d_n = d^{-n})$ is a chain complex.

Definition 1.0.4. *If (C_n, d_n) is a chain complex, then the n*-th homology of $C_\bullet$ *is*

$$\mathrm{H}_n(C) = \mathrm{Ker}(d_n)/\,\mathrm{Im}(d_{n+1})$$

If (C^n, d^n) *is a cochain complex, then the* n-th cohomology of $C^\bullet$ *is*

$$\mathrm{H}^n(C) = \mathrm{Ker}(d^{n+1}) / \mathrm{Im}(d^n)$$

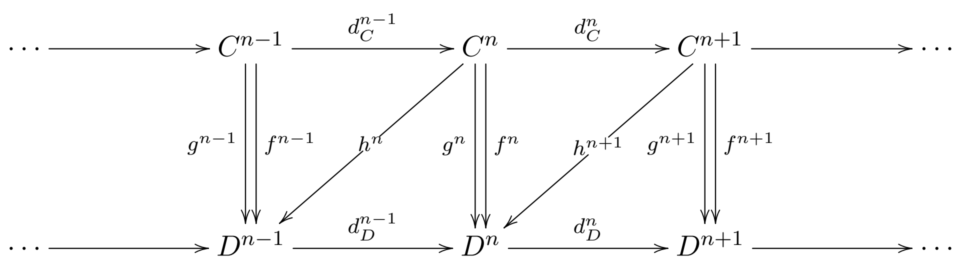
One writes $\mathrm{H}_\bullet(C) = \bigoplus_n \mathrm{H}_n(C)$ and $\mathrm{H}^\bullet(C) = \bigoplus_n \mathrm{H}^n(C)$. One calls $\mathrm{H}_\bullet(C)$ the *homology* and $\mathrm{H}^\bullet(C)$ the *cohomology* of the complex C. If $\mathrm{H}_\bullet(C) = 0$ or $\mathrm{H}^\bullet(C) = 0$, one says that C is acyclic (or an exact complex).

To define a morphism of complexes, we will work only with cochain complexes for simplicity. A morphism $f^\bullet \colon C^\bullet \to D^\bullet$ of complexes is a sequence $f^n \colon C^n \to D^n$ such that

$$\begin{array}{ccc} C^n & \xrightarrow{d_C^n} & C^{n+1} \\ \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} \\ D^n & \xrightarrow{d_D^n} & D^{n+1} \end{array}$$

commutes, that is $f^{n+1} \circ d_C^n = d_D^n \circ f^n$. Note that a morphism of complexes $f^\bullet \colon C^\bullet \to D^\bullet$ induces a morphism $\mathrm{H}^\bullet(f) \colon \mathrm{H}^\bullet(C) \to \mathrm{H}^\bullet(D)$ by letting $\mathrm{H}^n(f) \colon \mathrm{H}^n(C) \to \mathrm{H}^n(D)$ send the coset $[c]$ to $[f^n(c)]$. The definition of a morphism of complexes ensures that $\mathrm{H}^\bullet(f)$ is well-defined.

Definition 1.0.5. *Given* $f, g \colon C^\bullet \to D^\bullet$*, we say that* $f \sim g$ *(*f *and* g *are* homotopic*) if there is a sequence of homomorphisms* $\{h^n \colon C^n \to D^{n-1}\}_{n \in \mathbb{Z}}$ *such that* $f^n - g^n = d_D^{n-1} \circ h^n + h^{n+1} \circ d_C^n$ *for all* $n \in \mathbb{Z}$*:*



Lemma 1.0.6. *If* $f, g \colon C^\bullet \to D^\bullet$ *are homotopic, then* $\mathrm{H}^\bullet(f) = \mathrm{H}^\bullet(g)$*.*

Proof. Indeed, if $c \in \mathrm{Ker}(d_C^n)$, then $f^n(c) = g^n(c) + d(h(c))$. □

Corollary 1.0.7. *Suppose* $f \colon C^\bullet \to D^\bullet$ *and* $g \colon D^\bullet \to C^\bullet$ *are such that* $g^\bullet \circ f^\bullet \sim \mathrm{id}_C$ *and* $f^\bullet \circ g^\bullet \sim \mathrm{id}_D$*, then* f *and* g *induce mutually inverse isomorphisms between* $\mathrm{H}^\bullet(C)$ *and* $\mathrm{H}^\bullet(D)$*.*

Definition 1.0.8. *In this latter case we say that* $C^\bullet$ *is* homotopy equivalent *to* $D^\bullet$*. We call themaps* f *and* g homotopy equivalences*.*

Definition 1.0.9. *A morphism* $f \colon C^\bullet \to D^\bullet$ *is called a* quasi-isomorphism *if* $\mathrm{H}^n(f) \colon \mathrm{H}^n(C) \to \mathrm{H}^n(D)$ *is an isomorphism for each* n*.*

Example 1.0.10. Every homotopy equivalence is a quasi-isomorphism.

Remark 1.0.11. The converse is *not* true.

We are now ready to give a formal definition of a derived category.

Definition 1.0.12. *Let $\mathcal{A}$ be an abelian category (e.g. the category of abelian groups). Let* $\mathsf{Com}(\mathcal{A})$ *be the category of complexes over $\mathcal{A}$. The (unbounded)* derived category *of $\mathcal{A}$ is the (abstract) localisation of* $\mathsf{Com}(\mathcal{A})$ *at the class of all quasi-isomorphisms.*

2 Simplicial sets and simplicial homology

2.1 Motivation

There are a number of complexes that appear quite algebraic, but whose construction involves topology.

Definition 2.1.1. *The* geometric n-dimensional simplex *is the topological space*

$$\Delta_n = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \colon \sum_{i=0}^{n} x_i = 1, x_i \geqslant 0 \right\}$$

For example, Δ_0 is a point, Δ_1 is an interval, Δ_2 is an equilateral triangle, Δ_3 is a filled tetrahedron, etc. We will label the vertices of Δ_n as $e_0, \dots, e_n$.

Definition 2.1.2. *A* (geometric) complex $K = \{S_i\}_{i \in I}$ *is a union of geometric simplices S_i in $\mathbb{R}^N$ of varying dimensions such that the intersection $S_i \cup S_j$ of any two simplices is a face of each simplex.*

Definition 2.1.3. *A* polyhedron *is a space which is homeomorphic to a geometric complex.*

The choice of such a homeomorphism is usually called a *triangulation*. Clearly, triangulations are highly non-canonical. Both the n-sphere and the n-ball are polyhedra.

Remark 2.1.4. The triangulation of a space is a finitistic way of defining the space, similar to defining groups or algebras by a finite list of generators and relations.

Remark 2.1.5. In fact, one can “triangulate” groups, algebras, modules, and objects in any category.

If X is a geometric complex (or a polyhedron) we can associate to X the following chain complex:

$$C_n(X) = \bigoplus_{\substack{\sigma_i \in X \\ \dim(\sigma_i) = n}} \mathbb{Z}\sigma_i$$

$$d_n \colon C_n \to C_{n-1}, \sigma_i \mapsto \sum_{k=0}^{n} (-1)^k \varepsilon_k \sigma_i^k$$

In the above definition of d_n each simplex σ_i is equipped with an orientation (i.e. choice of an ordering of its vertices). Then σ_i^k denotes the simplex $\{e_0, \ldots, \hat{e_k}, \ldots, e_n\}$, and $\varepsilon_k = +1$ or -1 depending on the sign of the permutation that maps the sequence $\{e_0, \ldots, \hat{e_k}, \ldots, e_n\}$ to the sequence of vertices of σ_i^k determined by its orientation.

As an exercise, show that $d_n \circ d_{n-1} = 0$.

Theorem 2.1.6. *The homology groups* $\mathrm{H}_\bullet(X) = \bigoplus \mathrm{H}_n(X)$ *of the complex* $C_\bullet = (C_n, d_n)$ *are independent of the choice of triangulation and orientation of simplices.*

Proof. See any book on algebraic topology. □

It follows that the $\mathrm{H}_n(X)$ are invariants of X as a topological space.

Geometric intuition A homology cycle $c \in \mathrm{H}_n(X)$ can be viewed as n-dimensional chains (n-cycles) modulo the equivalence relation "$c \sim c'$ if there exists an $(n+1)$-cycle of which c and c' are the boundary."

Definition 2.1.7. 1. *A* simplicial set *is a family of sets* $X_\bullet = \{X_n\}_{n \geqslant 0}$ *and a family of maps* $\{X(f) \colon X_n \to X_m\}$, *one for each non-decreasing function* $f \colon [m] \to [n]$, *where* $[n] = \{0, \ldots, n\}$, *satisfying*

- $X(\mathrm{id}) = \mathrm{id}$
- $X(f \circ g) = X(g) \circ X(f)$

2. *A map of simplicial sets* $\varphi \colon X \to Y$ *is a family of maps* $\{\varphi_n \colon X_n \to Y_n\}_{n \geqslant 0}$ *such that for all* $f \colon [m] \to [n]$:

$$\begin{array}{ccc} X_n & \xrightarrow{\varphi_n} & Y_n \\ {\scriptstyle X(f)}\downarrow & & \downarrow{\scriptstyle Y(f)} \\ X_m & \xrightarrow{\varphi_m} & Y_m \end{array}$$

commutes.

Remark 2.1.8. A simplicial set is just a contravariant functor from the simplicial category Δ to the category of sets Set, and a map of simplicial sets is just a natural transformation of functors. The simplicial category has finite sets $[n]$ as objects, and non-decreasing functions as morphisms. We denote the category ofsimplicial sets by $\Delta^\circ\mathsf{Set}$.

Definition 2.1.9. *Let X be a simplicial set. The* geometric realization *of X is*

$$|X| = \coprod_{n=0}^{\infty} (\Delta_n \times X_n) \Big/ \sim$$

where the equivalence relation is defined by $(s, x) \sim (t, y)$ *if, for* $(s, x) \in \Delta_n \times X_n$ *and* $(t, y) \in \Delta_m \times X_m$, *there exists* $f \colon [m] \to [n]$ *non-decreasing such that* $y = X(f)x$ *and* $t = \Delta_f s$. *We give* $|X|$ *the weakest topology such that* $\coprod (\Delta_n \times X_n) \twoheadrightarrow |X|$ *is continuous.*

2.2 Definitions

Recall that we defined simplicial set as a family $X_\bullet = \{X_n\}_{n \geqslant 0}$ of sets and a family of maps $X(f): X_n \to X_m$, one for each non-decreasing map $f: [m] \to [n]$, such that $X(\mathrm{id}) = \mathrm{id}$ and $X(f \circ g) = X(g)X(f)$ when the compositions are defined. This can be rephrased more conceptually using the simplicial category.

Definition 2.2.1. *The* simplicial category Δ *has as objects all finite well ordered sets. That is,* $\operatorname{Ob}\Delta = \{[n] = \{0 < 1 < \cdots < n\}\}$*. Morphisms are order-preserving maps (i.e.* $i \leqslant j \Rightarrow f(i) \leqslant f(j)$*).*

A simplicial set is just a contravariant functor $X: \Delta \to \mathsf{Set}$. Thus the category of simplicial sets is just the category $\Delta^\circ\mathsf{Set} = \mathsf{Set}^{\Delta^\circ}$. There are two distinguished classes of maps in Δ:

$$\begin{aligned} &\delta_i^n: [n] \hookrightarrow [n+1] && 0 \leqslant i \leqslant n \\ &\sigma_j^i: [n+1] \twoheadrightarrow [n] && 0 \leqslant j \leqslant n+1 \end{aligned}$$

called the *face maps* δ_i^n and *degeneracy maps* σ_j^n. They are defined by

$$\delta_i^n(k) = \begin{cases} k & \text{if } k < i \\ \text{k+1} & \text{if } k \geqslant i \end{cases} \qquad \sigma_j^n(k) = \begin{cases} k & \text{if } k \leqslant j \\ k-1 & \text{if } k > j \end{cases}$$

Theorem 2.2.2. *Any morphism* $f \in \operatorname{Hom}_\Delta([n],[m])$ *can be decomposed in a unique way as*

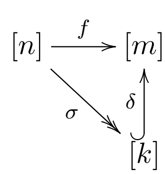
$$f = \delta_{i_1}\delta_{i_2}\cdots\delta_{i_r}\sigma_{j_1}\cdots\sigma_{j_s}$$

such that $m = n - s + r$ *and* $i_1 \leqslant \cdots \leqslant i_r$ *and* $j_1 \leqslant \cdots \leqslant j_s$*.*

The proof of this theorem is a little technical, but a few examples make it clear what is going on.

Example 2.2.3. Let $f: [3] \to [1]$ be $\{0,1 \mapsto 0; 2,3 \mapsto 1\}$. One can easily check that $f = \sigma_1^1 \circ \sigma_2^2$.

Corollary 2.2.4. *For any* $f \in \operatorname{Hom}_\Delta([n],[m])$*, there is a unique factorization*



Corollary 2.2.5. *The category Δ can be presented by $\{\delta_i\}$ and $\{\sigma_j\}$ as generators with the following relations:*

$$\begin{aligned}
&\delta_j\delta_i = \delta_i\delta_j && i < j \\
&\sigma_j\sigma_i = \sigma_i\sigma_{j+1} && i \leqslant j \\
&\sigma_j\delta_i = \begin{cases} \delta_i\sigma_{j-1} & \textit{if } i < j \\ id & \textit{if } i = j \textit{ or } i = j+1 \\ \delta_{i-1}\sigma_j & \textit{if } i \geqslant j+1 \end{cases}
\end{aligned} \tag{1.1}$$

Corollary 2.2.6. *Giving a simplicial set $X_\bullet = \{X_n\}_{n\geqslant 0}$ is equivalent to giving a family of sets $\{X_n\}$ equipped with morphisms $\partial_i^n\colon X_n \to X_{n-1}$ and $s_i^n\colon X_n \to X_{n+1}$ satisfying*

$$\begin{aligned}
&\partial_i\partial_j = \partial_j\partial_i && i < j \\
&s_i s_j = s_{j+1}s_i && i \leqslant j \\
&\partial_j s_i = \begin{cases} s_{j-1}\partial_i & \textit{if } i < j \\ \mathrm{id} & \textit{if } i = j \textit{ or } i = j+1 \\ s_j\partial_{i-1} & \textit{if } i \geqslant j+1 \end{cases}
\end{aligned} \tag{1.2}$$

The relation between (1.1) *and* (1.2) *is given by $\partial_i^n = X(\delta_i^{n-1})$ and $s_i^n = X(\sigma_i^n)$.*

Consider the n-dimensional geometric simplex $\Delta_n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}_{\geqslant 0}\colon \sum x_i = 1\}$. For a non-empty subset $I \subset [n]$, define the "I-th face" of Δ_n by $e_I = \{(x_0, \ldots, x_n) \in \Delta_n\colon \sum_{i\in I} x_i = 1\}$. In particular, if $I = \{i\}$, then the I-th face of Δ_n is just the i-th vertex $e_i = (0, \ldots, 1, \ldots, 0)$.

It is more convenient to parametrize faces by maps $f\colon [m] \to [n]$ for $m \leqslant n$ with $\mathrm{Im}(f) = I$.

Example 2.2.7. Let $I = \{0, 1, 3\} \subset [3]$. The corresponding map $f\colon [2] \to [3]$ is just $\{0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 3\} = \delta_2^3$.

In general, given $f\colon [m] \to [n]$, the corresponding $\Delta_f\colon \Delta_m \to \Delta_n$ is defined to be the restriction of the linear map $\mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ sending e_i to $e_{f(i)}$.

2.3 Geometric realization

Recall that given any simplicial set $X = \{X_n\}$, we defined the geometric realization of X as

$$|X| = \coprod_{n=0}^{\infty} (\Delta_n \times X_n) \Big/ \sim$$

where $(s, x) \in \Delta_m \times X_m$ is equivalent to $(y, t) \in \Delta_n \times X_n$ if there is $f\colon [m] \to [n]$ such that $y = X(f)(x)$ and $t = \Delta_f(s)$.

To any triangulated space, we can associate a simplicial set. Let $\mathbb{X}$ be a triangulated space. We define the *gluing data* of $\mathbb{X}$ as follows:

1. Let $\mathbb{X}_{(n)}$ be the set of all n-simplices in $\mathbb{X}$;
2. For each $f\colon [m] \to [n]$ define the "gluing" maps $\mathbb{X}(f)\colon \mathbb{X}_{(n)} \to \mathbb{X}_{(m)}$ so that the fibre of $\mathbb{X}(f)$ over an m-simplex $x \in \mathbb{X}_{(m)}$ consists of exactly all n-simplices in $\mathbb{X}_{(n)}$ which have x as a common face in $\mathbb{X}$.

Now, with this gluing data we can associate the simplicial set

$$X_\bullet := \{X_m, X(f)\} \in \Delta^\circ \mathsf{Set}$$

defined as follows. First we can define X_m by

$$X_m := \big\{(x, g) \mid x \in \mathbb{X}_{(k)}, g \in \mathrm{Surj}_\Delta([m], [k])\big\}$$

Suppose $f \in \mathrm{Hom}_\Delta([n], [m])$. For $(x, g) \in X_m$ consider the composition $g \circ f\colon [n] \to [m] \to [k]$. By Corollary 2.2.4 we can factorize $g \circ f$ as $g \circ f = \delta \circ \sigma$ with $\sigma \in \mathrm{Surj}_\Delta([n], [l])$ and $\mathrm{Inj}_\Delta([l], [k])$. Then define $X(f)$ to be

$$X(f)(x, g) = (X(\delta)x, \sigma) \in X_n$$

It is straightforward to check that $X(\mathrm{id}) = \mathrm{id}$ and $X(f' \circ f) = X(f) \circ X(f')$.

Theorem 2.3.1. *The geometric realization $|X_\bullet|$ is homotopically equivalent to $\mathbb{X}$.*

Proof. (sketch) By definition, X_n consists of all pairs $\tilde{x} = (x, g)$ with $x \in \mathbb{X}_{(m)}$ and $g\colon [n] \twoheadrightarrow [m] \in \mathrm{Mor}(\Delta)$. Define

$$\varphi\colon \coprod_{n=0}^{\infty} \Delta_n \times X_n \to \coprod_{m=0}^{\infty} \Delta_m \times \mathbb{X}_{(m)}$$

by $(s, \tilde{x}) \mapsto (\Delta_g(s), x) \in \Delta_m \times \mathbb{X}_{(m)}$. Clearly if $(s, \tilde{x}) \sim (s', \tilde{x}')$ then $\varphi(s, \tilde{x}) = \varphi(s', \tilde{x}')$. Hence φ induces a continuous map $\tilde{\varphi}\colon |X| \to \mathbb{X}$.The homotopically inverse map is induced by the map

$$\psi\colon \coprod_{m=0}^{\infty} \Delta_m \times \mathbb{X}_{(m)} \to \coprod_{n=0}^{\infty} \Delta_n \times X_n$$

defined by $(s, x) \mapsto (s, (x, \mathrm{id}_{[n]}))$. □

Example 2.3.2 (Simplicial model of the circle S^1)**.** The simplest simplicial model for the circle S^1 is a simplicial set $S^1_\bullet$ which is generated by two non-trivial cells: one in dimension 0 (the basepoint $*$) and one in dimension 1 which we will denote α. The face maps on α are given by $d_0(\alpha) = d_1(\alpha) = *$. But we also need to introduce an element $s_0(*)$ in S^1_1. Similarly, at the level n, the set S^1_n has $n+1$ elements:

$$S^1_n = \{s_0^n(*), s_{n-1}s_{n-2}\ldots\hat{s}_{i-1}\ldots s_0(\alpha),\ i = 1, 2, \ldots, n\}$$

This is enough because of the relations between d_i and S_j. Elements in S^1_n are in natural bijection with the (additive) group $\mathbb{Z}/(n+1)\mathbb{Z}$:

$$S^1_0 = \{*\},\ S^1_1 = \{s_0(*), \alpha\},\ S^1_2 = \{s_0^2(*), s_1(\alpha), s_0(\alpha)\},\ \ldots$$

Simplicial set S^1 is a special kind of simplicial set, called a *cyclic set* (A.Connes). Such sets give rise to cyclic homology. We will discuss this type of homology later in the course.

2.4 Homology and cohomology of simplicial sets

Let $X = \{X_n\}$ be a simplicial set. Recall that for each $n \in \mathbb{Z}$ and for each fixed abelian group A, we defined

$$C_n(X, A) = \begin{cases} 0 & \text{if } n < 0 \\ AX_n = A \otimes \mathbb{Z}X_n & \text{otherwise} \end{cases}.$$

The differential $d_n \colon C_n \to C_{n-1}$ is defined by

$$\begin{aligned} \sum_{x \in X_n} a(x) \cdot x \mapsto \sum_{x \in X_n} a(x) \sum_{i=0}^{n} (-1)^i X(\delta_i^{n-1}) x \\ = \sum_{x \in X_n} a(x) \sum_{i=0}^{n} (-1)^i \partial_i^n x \end{aligned}$$

In other words, we define differential $d_n \colon C_n \to C_{n-1}$ to be $d_n = \sum_{i=0}^{n} (-1)^i \partial_i$.

Dually, we define $C^n(X, A) = \{\text{functions } X_n \to A\}$, and the differential $d^n \colon C^n \to C^{n+1}$ by

$$\begin{aligned} (d^n f)(x) &= \sum_{i=0}^{n+1} (-1)^i f\left(X(\delta_i^n) x\right) \\ &= \sum_{i=0}^{n+1} (-1)^i f(\partial_i^{n+1} x) \end{aligned}$$

Theorem 2.4.1. *The objects $C_n(X, A)$ and $C^n(X, A)$ are actually complexes, i.e. $d_{n-1} \circ d_n = 0$ and $d^{n+1} d^n = 0$.*

Proof. Let's check that $d_{n-1} \circ d_n = 0$. We have $d_{n-1} \circ d_n = d_{n-1} \left[\sum_{j=0}^{n} (-1)^j \partial_j \right] = \sum_{j=0}^{n} \sum_{i=0}^{n-1} (-1)^{i+j} \partial_i \partial_j$. Then we can split this sum into two parts and use the relations (1.2) (actually, only the first one of these relations) to get

$$\begin{aligned} d_{n-1} \circ d_n &= \sum_{j=0}^{n} \sum_{i=0}^{n-1} (-1)^{i+j} \partial_i \partial_j \\ &= \sum_{i<j} (-1)^{i+j} \partial_i \partial_j + \sum_{i \geq j} (-1)^{i+j} \partial_i \partial_j \\ &= \sum_{i \leq j-1} (-1)^{i+j} \partial_{j-1} \partial_i + \sum_{i \geq j} (-1)^{i+j} \partial_i \partial_j \\ &= \sum_{i' \geq j'} (-1)^{i'+j'+1} \partial_{i'} \partial_{j'} + \sum_{i \geq j} (-1)^{i+j} \partial_i \partial_j \\ &= 0 \end{aligned}$$

□

Remark 2.4.2. For any category C we can define *simplicial objects in* C as functors $\Delta^{op} \to \mathsf{C}$. If category C is abelian (for example abelian groups Ab, or vector spaces Vect, or modules $\mathsf{Mod(R)}$ over some algebra), then we can define *homology* of a simplicial object $X \in \Delta^{op}\mathsf{C}$ as the homology of *complex* X, where the differential $d_n\colon X_n \to X_{n-1}$ is defined by the same formula $d_n = \sum(-1)^i X(\partial_i)$ as before.

Then the above definition of homology of a simplicial set coincides with homology of the simplicial abelian group $S\colon \Delta^{op} \to \mathsf{Ab}$ defined by the composition $\Delta^{op} \to \mathsf{Sets} \xrightarrow{\text{free}_A} \mathsf{Ab}$, where free_A sends a set X to the abelian group $A \otimes \mathbb{Z}X$.

Dually we can define *cosimplicial objects in* C as functors $\Delta \to \mathsf{C}$. Again, if C is abelian, then we can define *cohomology* of a cosimplicial object Y in C as the cohomology of a complex Y where the differential is defined by the formula $d^n = \sum(-1)^i Y(\delta_i)$.

2.5 Applications

Example 2.5.1 (Singular (co)homology)**.** Let X be a topological space. A (singular) n-simplex of X is a continuous map $\varphi\colon \Delta_n \to X$. Put, for $n \geqslant 0$, $\mathbb{X}_n = \mathrm{Hom}_{\mathsf{Top}}(\Delta_n, X) = \{\text{singular } n\text{-simplices in } X\}$. For $f \in \mathrm{Hom}_\Delta([m],[n])$, define $\mathbb{X}(f)\colon \mathbb{X}_m \to \mathbb{X}_n$ by $\mathbb{X}(f)(\varphi) = \varphi \circ \Delta_f$.

If A is an abelian group, then we can define singular homology and cohomology of X by $\mathrm{H}^{\text{sing}}_\bullet(X,A) = \mathrm{H}_\bullet(\mathbb{X},A)$ and $\mathrm{H}^\bullet_{\text{sing}}(X,A) = \mathrm{H}^\bullet(\mathbb{X},A)$ respectively.

Note that if X has some extra structure (e.g. is a C^∞-manifold or a complex manifold) then it is often convenient to take simplices compatible with that structure.

Example 2.5.2 (Nerve of a covering)**.** Let X be a topological space, $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ a covering of X. Define

$$\mathbb{X}_n = \{(\alpha_0,\dots,\alpha_n) \in I^{n+1}\colon U_{\alpha_0} \cap \dots \cap U_{\alpha_n} \neq \varnothing\}.$$

For $f \in \mathrm{Hom}_\Delta([m],[n])$, the morphism $\mathbb{X}(f)\colon \mathbb{X}_n \to \mathbb{X}_m$ is given by $(\alpha_0,\dots,\alpha_n) \mapsto (\alpha_{f(0)},\dots,\alpha_{f(n)})$. The Čech (co)homology of X with respect to $\mathcal{U}$ is

$$\check{\mathrm{H}}_\bullet(\mathcal{U},A) = \mathrm{H}_\bullet(\mathbb{X},A)$$
$$\check{\mathrm{H}}^\bullet(\mathcal{U},A) = \mathrm{H}^\bullet(\mathbb{X},A)$$

Example 2.5.3 (Classifying space of a group)**.** Let G be a group. Define a simplicial set BG by $(BG)_n = G^n = G \times \cdots \times G$ (n-fold product). For $f \in \mathrm{Hom}_\Delta([m],[n])$, define $BG(f)\colon G^n \to G^m$ by $(g_1,\dots,g_n) \mapsto (h_1,\dots,h_m)$ where

$$h_i = \begin{cases} \prod\limits_{f(i-1)<j\leqslant f(i)} g_j & \text{if } f(i-1) \neq f(i) \\ e_G & \text{otherwise} \end{cases}$$

We call $|BG|$ the *classifying space* of G. Group (co)homology is defined as $\mathrm{H}_\bullet(G,A) = \mathrm{H}^{\text{sing}}_\bullet(BG,A)$, $\mathrm{H}^\bullet(G,A) = \mathrm{H}^\bullet_{\text{sing}}(BG,A)$.

Example 2.5.4 (Hochschild homology)**.** Let A be an algebra over a field k and M be A-bimodule. Then we can form a simplicial module $C_\bullet(A, M)$ by setting $C_n(A, M) = M \otimes A^{\otimes n}$ and defining face maps and degeneracy maps as follows:

$$\begin{aligned} d_0(m, a_1, \ldots, a_n) &= (ma_1, a_2, \ldots, a_n) \\ d_i(m, a_1, \ldots, a_n) &= (m, a_1, \ldots, a_i a_{i+1}, \ldots, a_n),\ i = 1, \ldots, n-1 \\ d_n(m, a_1, \ldots, a_n) &= (a_n m, a_1, \ldots, a_{n-1}) \\ s_j(m, \ldots, a_n) &= (m, a_0, \ldots, a_j, 1, a_{j+1}, \ldots, a_n) \end{aligned}$$

Homology $\mathrm{HH}_\bullet(A, M) := \mathrm{H}(C_\bullet(A, M))$ of this simplicial module is called *Hochschild homology* of A with coefficients in bimodule M. We will consider this cohomology theory in more details in section 2.

What is a (co)homology theory? A (co)homology theory should be a "function" of two arguments: $\mathrm{H}(X, \mathscr{A})$, where X is a "nonabelian" argument, and $\mathscr{A}$ is an object in some abelian category. For example, X could be a topological space, algebra, group etc. and usually $\mathscr{A}$ will be a sheaf, (bi)module, representation etc. The modern perspective is that we should "fix X" and think of $\mathrm{H}(X, -)$ as a functor from some abelian category to abelian groups. More formally, we have some non-abelian (that is, arbitrary) category $\mathcal{C}$, and an additive category $\mathcal{A}$ over $\mathcal{C}$ fibred in abelian categories. For example, we can consider $\mathcal{C} = \mathsf{Top}$, and the fiber of $\mathcal{A}$ over Top being $\mathsf{Sh}(X)$.

2.6 Homology and cohomology with local coefficients

Recall that given a simplicial set $X_\bullet$ we defined $C_n(X, A) = \bigoplus_{x \in X_n} Ax$. That is, elements $a \in C_n(X, A)$ are of the form $\sum_{x \in X} a(x) \cdot x$ with $a(x) \in A$. What if we allowed the $a(x)$ to live in different abelian groups? That is exactly what we will try to do!

Definition 2.6.1. *A* homological system of coefficients *for X consists of*

1. *a family of abelian groups $\{\mathscr{A}_x\}_{x \in X_n}$ one for each simplex $x \in X_n$*
2. *a family of group homomorphisms $\{\mathscr{A}(f, x) \colon \mathscr{A}_x \to \mathscr{A}_{X(f)x}\}_{x \in X_n, f \colon [m] \to [n]}$*

satisfying

1. *$\mathscr{A}(\mathrm{id}, x) = \mathrm{id}_{\mathscr{A}_x}$ for all $x \in X_n$*
2. *the following diagram commutes:*

$$\begin{array}{ccc} \mathscr{A}_x & \xrightarrow{\mathscr{A}(f,x)} & \mathscr{A}_{X(f)x} \\ {\scriptstyle \mathscr{A}(fg,x)}\downarrow & & \downarrow{\scriptstyle \mathscr{A}(g,X(f)x)} \\ \mathscr{A}_{X(fg)x} & = & \mathscr{A}_{X(g)X(f)x} \end{array}$$

that is, $\mathscr{A}(fg, x) = \mathscr{A}(g, X(f)x)\mathscr{A}(f, x)$. (this is a cocycle condition).

Definition 2.6.2. *Given* $(X_\bullet, \mathscr{A})$*, define*

$$C_n(X,\mathscr{A}) = \bigoplus_{x\in X_n} \mathscr{A}_x \cdot x$$

We define a differential $d_n\colon C_n(X,\mathscr{A}) \to C_{n-1}(X,\mathscr{A})$ *by*

$$d_n\left(\sum_{x\in X_n} a(x)\cdot x\right) = \sum_{x\in X_n}\sum_{i=0}^{n+1}(-1)^i \mathscr{A}(\delta_i^{n-1}, x)\,(a(x))\,\partial_i^n x$$

One can check that the cocycle condition forces $d^2 = 0$.

Definition 2.6.3. *A* cohomological system of coefficients *is*

1. *a family of abelian groups* $\{\mathscr{B}_x\}_{x\in X_n}$
2. *a family of group homomorphisms* $\{\mathscr{B}(f,x)\colon \mathscr{B}_{X(f)x} \to \mathscr{B}_x\}_{x\in X_n, f\colon [m]\to[n]}$

satisfying

1. $\mathscr{B}(\mathrm{id}, x) = \mathrm{id}_{\mathscr{B}_x}$
2. $\mathscr{B}(fg,x) = \mathscr{B}(fx)\mathscr{B}(g, X(f)x)$

Definition 2.6.4. *Given a cohomological system* $(X_\bullet, \mathscr{B})$*, define*

$$C^n(X,\mathscr{B}) = \left\{ \textit{functions } f\colon X_n \to \coprod_{x\in X_n} \mathscr{B}_x \right\}$$

We define $d^n\colon C^n(X,\mathscr{B}) \to C^{n+1}(X,\mathscr{B})$ *by*

$$(d^n f)(x) = \sum (-1)^i \mathscr{B}(\delta_i^n, x)\left(f(\partial_i^{n+1} x)\right)$$

Example 2.6.5. The system of *constant coefficients* is $\mathscr{A}_x = A$ for all $x \in X_n$, with $\mathscr{A}(f,x) = \mathrm{id}_A$. One can verify that $C_\bullet(X,\mathscr{A}) = C_\bullet(X,A)$.

Remark 2.6.6. The notion of a system of coefficients can be defined much more succinctly. The category of simplicial sets is, as noted, just $\Delta^\circ\mathsf{Set}$, i.e. the category $\mathsf{Psh}(\Delta)$ of presheaves on Δ. For $X \in \mathsf{Psh}(\Delta)$, consider the category of "elements over X," $\int_\Delta X$. Objects of $\int_\Delta X$ are pairs (n,x) where $x \in X_n$, and a morphism $(n,x) \to (m,y)$ is just a nondecreasing map $f\colon [n] \to [m]$ such that $X(f)(y) = x$. One can readily check that the category of coefficient systems on X is $\mathsf{AbPsh}\left(\int_\Delta X\right)$.

3 Sheaves and their cohomology

3.1 Presheaves

Sheaves were originally considered by J. Leray.

Let X be a topological space. Define $\mathsf{Open}(X)$ to be the category of open sets in X. That is, $\mathrm{Ob}\left(\mathsf{Open}(X)\right) = \{\text{open sets in } X\}$, and

$$\mathrm{Hom}_{\mathsf{Open}(X)}(U,V) = \begin{cases} \varnothing & \text{if } U \not\subset V \\ U \hookrightarrow V & \text{if } U \subset V \end{cases}$$

Here and elsewhere, $U \subset V$ means that U is *not necessarily a proper* subset of V. We will use notation $U \subsetneq V$ if U is proper subset of V.

Definition 3.1.1. *Let X be a topological space. A* presheaf *on X with values in a category $\mathcal{C}$ is a contravariant functor $\mathscr{F} : \mathsf{Open}(X) \to \mathcal{C}$.*

Common categories are $\mathcal{C} = \mathsf{Set}, \mathsf{Grp}, \mathsf{Ring}, \dots$. Elements $s \in \mathscr{F}(U)$ are called *sections* of $\mathscr{F}$ over U, and $\mathscr{F}(X)$ is the set of *global sections*. One often writes $\Gamma(U, \mathscr{F})$ instead of $\mathscr{F}(U)$, and thinks of $\Gamma(U, -)$ as a functor on $\mathscr{F}$. From the definition, we see that for $U \subset V$ we have maps $\rho^V_U : \mathscr{F}(V) \to \mathscr{F}(U)$; these are called the *restriction* maps from V to U. Since $\mathscr{F}$ is a functor, these satisfy:

1. $\rho^U_U = \mathrm{id}$ for all open U.
2. if $U \subset V \subset W$, then $\rho^W_U = \rho^V_U \circ \rho^W_V$.

There is an obvious notion of a morphism between presheaves – namely a morphism $\varphi : \mathscr{F} \to \mathscr{G}$ is just a natural transformation from $\mathscr{F}$ to $\mathscr{G}$. That is, $\varphi = \{\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)\}$, and for $U \subset V$ open, the following diagram commutes:

$$\begin{array}{ccc} \mathscr{F}(V) & \xrightarrow{\varphi(V)} & \mathscr{G}(V) \\ \downarrow{\scriptstyle \rho^V_U} & & \downarrow{\scriptstyle \rho^V_U} \\ \mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{G}(U) \end{array}$$

3.2 Definitions

Definition 3.2.1. *A presheaf $\mathscr{F}$ is a* sheaf *if given any open $U \subset X$, any open cover $U = \bigcup_\alpha U_\alpha$, and $\{s_\alpha \in \mathscr{F}(U_\alpha)\}$ such that $\rho^{U_\alpha}_{U_\alpha \cap U_\beta}(s_\alpha) = \rho^{U_\beta}_{U_\alpha \cap U_\beta}(s_\beta)$, there exists a unique $s \in \mathscr{F}(U)$ such that $\rho^U_{U_\alpha}(s) = s_\alpha$ for all α.*

Remark 3.2.2. Sheaves are *continuous* functors, in the sense that they map colimits (in $\mathsf{Open}(X)$) to limits in Set. That is, the diagram

$$\mathscr{F}(U \cup V) \longrightarrow \mathscr{F}(U) \times \mathscr{F}(V) \rightrightarrows \mathscr{F}(U \cap V) \tag{1.3}$$

is exact, i.e. is an equalizer.

We will mostly deal with *abelian sheaves*, that is sheaves of abelian groups. The following are all examples of sheaves.

Example 3.2.3. Let X be a topological space. Set $\mathscr{O}_X^{\mathrm{c}}(U) = \mathrm{Hom}_{\mathsf{Top}}(U, \mathbb{C})$.

If X is a differentiable manifold, we can define $\mathscr{O}_X^{\mathrm{diff}} \subset \mathscr{O}_X^{\mathrm{c}}$, the "sheaf of differentiable functions" by letting $\mathscr{O}_X^{\mathrm{diff}}(U)$ be the ring of C^∞-functions $U \to \mathbb{C}$.

If X is a complex analytic manifold, e.g. $X = \mathbb{P}^1(\mathbb{C})$, then we can define $\mathscr{O}_X^{\mathrm{an}} \subset \mathscr{O}_X^{\mathrm{diff}}$ to be the "sheaf of holomorphic functions."

If we go even further and stipulate that X is an algebraic variety over $\mathbb{C}$, then we can define $\mathscr{O}_X^{\mathrm{alg}}$ to be the "sheaf of regular functions."

All the above sheaves are often called "structure sheaves." Indeed, smooth manifolds, analytic manifolds... can be *defined* to be topological spaces along with a sheaf of rings satisfying certain properties.

For abelian sheaves, (1.3) is exact in the usual sense:

$$0 \longrightarrow \mathscr{F}(U \cup V) \longrightarrow \mathscr{F}(U) \times \mathscr{F}(V) \longrightarrow \mathscr{F}(U \cap V)$$

where the first map is $\rho_U^{U\cup V} \times \rho_V^{U \cup V}$ and the second is $(s,t) \mapsto \rho_{U\cap V}^U(s) - \rho_{U \cap V}^V(t)$.

Two basic problems in sheaf theory are the following:

Example 3.2.4 (Extensions of sections)**.** Given a sheaf $\mathscr{F}$ and a section $s \in \mathscr{F}(U)$ for some open U, does there exist $\tilde{s} \in \mathscr{F}(V)$ for some $V \supset U$, such that $\rho_U^V(\tilde{s}) = s$? For example, let $\mathscr{F} = \mathscr{O}_{\mathbb{C}P^1}^{\mathrm{an}}$, $f \in \mathscr{F}(U)$ a holomorphic function. Then for every $z_0 \in U$, there exists a neighborhood $U_{z_0} \subset U$ such that $f(z) = \sum_{n \geqslant 0} a_n (z - z_0)^n$ converges for all $z \in U_{z_0}$. For example, we could take the Riemann zeta-function defined by $\zeta(s) = \sum_{n \geqslant 1} n^{-s}$ on $U = \{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$. One can prove that ζ can be extended analytically to all of $\mathbb{C} \setminus \{1\}$. There is a large class of similar functions (for example, Artin L-functions and the L-functions of more general Galois representations) for which existence of analytic extensions to all of $\mathbb{C}$ is an open problem.

Example 3.2.5 (Riemann-Roch)**.** Compute $\Gamma(X, \mathscr{F})$ for a given $\mathscr{F}$. An important example is when X is a compact Riemann surface and $\mathscr{F} = \Omega^1$ is the sheaf of 1-forms. In this case, $\dim_{\mathbb{C}} \mathrm{H}^0(X, \Omega^1) = g$, the *genus* of X. More generally, the Riemann-Roch theorem says that if $\mathscr{L}$ is an invertible sheaf on X, then $\dim \mathrm{H}^0(\mathscr{L}) - \dim \mathrm{H}^0(\Omega^1 \otimes \mathscr{L}^{-1}) = \deg \mathscr{L} - g + 1$. Since $\mathrm{H}^0(\Omega \otimes \mathscr{L}^{-1}) = \mathrm{H}^1(\mathscr{L})$ by Serre duality, we can write this as $\chi(\mathscr{L}) = \deg \mathscr{L} - g + 1$.

3.3 Kernels, images and cokernels

Definition 3.3.1. *If $\mathscr{F}$ and $\mathscr{G}$ are sheaves, a* morphism *of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ is just a morphism of their underlying presheaves.*

Consider the category $\mathsf{AbSh}(X)$ of abelian sheaves. This is clearly an additive category, so the notions of kernel / image make sense. We can define, for $\varphi : \mathscr{F} \to \mathscr{G}$, two presheaves

$$\mathscr{K}(U) = \mathrm{Ker}\,(\varphi(U))$$
$$\mathscr{I}'(U) = \mathrm{Im}\,(\varphi(U))$$

The hope would be that $\mathscr{K}$ and $\mathscr{I}'$ are the category-theoretic kernel / image of φ.

Lemma 3.3.2. *With the above notation, $\mathscr{K}$ is a sheaf, but $\mathscr{I}'$ is not a sheaf in general.*

Proof. Given open $U \subset X$, $U = \bigcup_\alpha U_\alpha$ an open cover, and $s_\alpha \in \mathscr{K}(U_\alpha)$ such that $\rho^{U_\alpha}_{U_\alpha \cap U_\beta}(s_\alpha) = \rho^{U_\beta}_{U_\alpha \cap U_\beta}(s_\beta)$, since $\mathscr{F}$ is a sheaf, there exists a unique $s \in \mathscr{F}(U)$ such that $\rho^U_{U_\alpha}(s) = s_\alpha$ for all α. We need to show that s is actually in $\mathscr{K}$. Since φ is a morphism, for any α we have

$$\rho^U_{U_\alpha}(\varphi(U)s) = \varphi(U_\alpha)(\rho^U_{U_\alpha}(s)) = \varphi(U_\alpha)(s_\alpha) = 0$$

since each $s_\alpha \in \mathscr{K}(U_\alpha)$. Since $\mathscr{G}$ is also a sheaf, this force $\varphi(U)(s) = 0$, hence $s \in \mathscr{K}(U)$.

To show that $\mathscr{I}'$ is not in general a sheaf, we give a counterexample. Let $X = \mathbb{C}^\times = \mathbb{C}\backslash\{0\}$ with the analytic topology. Consider $\varphi : \mathscr{O}^{\mathrm{an}}_X \to \mathscr{O}^{\mathrm{an}}_X$ given by $f \mapsto f' = \frac{df}{dz}$. One can check that for all $x \in X$, there exists a neighborhood $U_x \subset X$ with $x \in U_x$ such that $\varphi(U_x) : \mathscr{O}^{\mathrm{an}}_X(U_x) \to \mathscr{O}^{\mathrm{an}}_X(U_x)$ is surjective. But, the equation $\frac{df}{dz} = g$ for $g = \sum_{n=-\infty}^{+\infty} a_n z^n \in \Gamma(X, \mathscr{O}^{\mathrm{an}}_X)$, has a global solution if and only if $a_{-1} = 0$, which implies that the function $\frac{1}{z}$ is not in the image $\mathrm{Im}\,(\varphi(X))$ although $\frac{1}{z} \in \mathrm{Im}\,(\varphi(U_x))$ for any $x \in X$. This violates the sheaf axiom. □

The moral of this is that $\mathscr{I}'$ needs to be redefined in order to be a sheaf.

Definition 3.3.3. *Given a morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$, define* $\mathrm{Im}(\varphi)$ *by*

$$\mathrm{Im}(\varphi)(U) = \{s \in \mathscr{G}(U) : \forall x \in U, \exists U_x \subset U : \rho^U_{U_x}(s) \in \mathrm{Im}\,\varphi(U_x)\}$$

It is a good exercise to show that $\mathrm{Im}(\varphi)$ actually is a sheaf, and is moreover the category-theoretic image of φ. If one is more ambitious, it is not especially difficult to show that $\mathsf{AbSh}(X)$ is an abelian category.

3.4 Germs, stalks, and fibers

Let X be a topological space, $\mathscr{F}$ an (abelian) presheaf on X.

Definition 3.4.1. *Let $x \in X$. A* germ *of sections of $\mathscr{F}$ at x is an equivalence class of pairs (s, U), where $U \subset X$ is an open neighborhood of x and $s \in \mathscr{F}(U)$. We say that (s, U) and (t, V) are equivalent if there exists $W \subset U \cap V$ such that $\rho^U_W(s) = \rho^V_W(t)$.*

One checks easily that what we have defined actually is an equivalence relation. The *stalk* (or fiber) of $\mathscr{F}$ at $x \in X$ is the set $\mathscr{F}_x$ of all equivalence classes of pairs (s, U) with $x \in U$, $s \in \mathscr{F}(U)$. More formally,

$$\mathscr{F}_x = \varinjlim_{U \ni x} \mathscr{F}(U)$$

Note that for any $x \in U$, there is a canonical map $\rho^U_x : \mathscr{F}(U) \to \mathscr{F}_x$ given y $s \mapsto s_x = [(s, U)]$.

Remark 3.4.2. There is an even more abstract characterization of $\mathscr{F}_x$. Let $\mathcal{E} = \mathsf{Sh}(X)$, the category of sheaves of sets on X. This is a topos, i.e. it has all finite limits and colimits, exponentials, and a subobject classifier. See [MLM92] for the definition of exponentials and subobject classifiers. For arbitrary topoi $\mathcal{X}, \mathcal{S}$, one says that a *geometric morphism* $x : \mathcal{X} \to \mathcal{S}$ is a pair (x^*, x_*) where $x^* : \mathcal{S} \to \mathcal{X}$ and $x_* : \mathcal{X} \to \mathcal{S}$ form an adjoint pair (with $x^* \dashv x_*$) and x^* commutes with finite limits. (Since x^* is a left-adjoint, it already commutes with all colimits.)

For $\mathcal{X}$ an arbitrary topos, call a *geometric point* a geometric morphism $x : \mathsf{Set} \to \mathcal{X}$. Let $|\mathcal{X}|$ denote the class of geometric points of $\mathcal{X}$. If X is a sober topological space (every irreducible closed subset has a unique generic point) then there is a natural bijection $X \to |\,\mathsf{Sh}(X)|$ that sends $x \in X$ to the pair (x^*, x_*) where $x^* \mathscr{F} = \mathscr{F}_x$ and

$$x_* S = S_x : U \mapsto \begin{cases} S & \text{if } x \in U \\ \varnothing & \text{otherwise} \end{cases}$$

Topoi of the form $\mathcal{X} = \mathsf{Sh}(X)$ have one very nice property: a morphism $f : F \to G$ in $\mathcal{X}$ is an isomorphism in $\mathcal{X}$ if and only if $x^* f$ is an isomorphism for all $x \in |\mathcal{X}|$. Such topoi are said to have *enough points*.

It is possible to give $|\mathcal{X}|$ a topology in a canonical way. The functor $\mathcal{X} \mapsto \mathsf{Sh}\left(|\mathcal{X}|\right)$ can be characterized as an adjoint – for details, see [Hak72]

Definition 3.4.3. *Let $\mathscr{F}$ be a presheaf on X. The* total space *of $\mathscr{F}$ is* $\mathrm{Et}(\mathscr{F}) = \coprod_{x \in X} \mathscr{F}_x$. *For $s \in \mathscr{F}(U)$, define* $\mathrm{Et}(\mathscr{F})(s) = \{s_x\}_{x \in U} \subset \mathbb{F}$. *We put the coarsest topology on* $\mathrm{Et}(\mathscr{F})$ *such that each* $\mathrm{Et}(\mathscr{F})(s)$ *is open.*

It is an easy consequence of the definitions that the projection map $\pi : \mathrm{Et}(\mathscr{F}) \twoheadrightarrow X$ is continuous (in fact, it is a local homeomorphism).

Example 3.4.4. Let $f : Y \to X$ be any continuous map of topological spaces. We can define the sheaf Γ_f on X of continuous local sections of f, i.e.

$$X \supset U \mapsto \Gamma_f(U) = \{s \in \mathrm{Hom}_{\mathsf{Top}}(U, Y) : f \circ s = \mathrm{id}_U\}$$

It is easy to check that Γ_f is a sheaf (without any hypotheses on f).

Definition 3.4.5. *Let $\mathscr{F}$ be a presheaf on X. Define the* sheafification *of $\mathscr{F}$ as $\mathscr{F}^+ = \Gamma_\pi$, where $\pi : \mathrm{Et}(\mathscr{F}) \to X$ is the canonical projection.*

There is a canonical morphism of presheaves $\varphi_{\mathscr{F}} : \mathscr{F} \to \mathscr{F}^+$, called the *sheafification map*. One defines $\varphi_{\mathscr{F}}(U) : \mathscr{F}(U) \to \mathscr{F}^+(U)$ by $s \mapsto (u \mapsto s_u)$. If $\mathscr{F}$ is already a sheaf, then $\varphi : \mathscr{F} \to \mathscr{F}^+$ is an isomorphism.

Remark 3.4.6. Let $\mathsf{Sh}(X)$ and $\mathsf{PSh}(X)$ be the categories of sheaves and presheaves on X. Let $\iota : \mathsf{Sh}(X) \to \mathsf{PSh}(X)$ be the natural inclusion. One can characterize sheafification by saying that it is the left-adjoint to ι. In diagrams:

$$(-)^+ : \mathsf{PSh}(X) \rightleftarrows \mathsf{Sh}(X) : \iota$$

that is, $\mathrm{Hom}_{\mathsf{Sh}(X)}(\mathscr{F}^+,\mathscr{G}) \simeq \mathrm{Hom}_{\mathsf{PSh}(X)}(\mathscr{F},\iota\mathscr{G})$. The map $\varphi_{\mathscr{F}}$ is the one induced by $\mathrm{id}_{\mathscr{F}^+}$, when we set $\mathscr{G} = \mathscr{F}^+$.

Definition 3.4.7. *For sheaves $\mathscr{F}$ and $\mathscr{G}$ with an embedding $\mathscr{G} \hookrightarrow \mathscr{F}$ define the* quotient *$\mathscr{F}/\mathscr{G}$ as sheafification of the presheaf $U \mapsto \mathscr{F}(U)/\mathscr{G}(U)$.*

3.5 Coherent and quasi-coherent sheaves

In this section we will introduce some basic types of sheaves (finitely generated, coherent,etc.) that we will widely use later.

Let $(X, \mathscr{O})$ be a topological space with the structure sheaf $\mathscr{O}$ of "continuous functions" on it.

Definition 3.5.1. *A sheaf $\mathscr{F}$ on X of $\mathscr{O}$-modules is called* finitely generated *(or of* finite type*) if every point $x \in X$ has an open neighbourhood U such that there is a surjective morphism of restricted sheaves*

$$\mathscr{O}^{\oplus n}|_U \twoheadrightarrow \mathscr{F}|_U,\ n \in \mathbb{N}$$

In other words, locally such sheaf is generated by finite number of sections. That is, for any $x \in X$ and small enough open $U \ni x$, for any $V \subset U$ the abelian group $\mathscr{F}(V)$ is finitely generated as a module over $\mathscr{O}(V)$.

Example 3.5.2. The structure sheaf $\mathscr{O}$ itself is of finite type, as well as $\mathscr{O}^{\oplus n}$.

Example 3.5.3. If $\mathscr{F}$ is finitely generated, then any quotient $\mathscr{F}/\mathscr{G}$ and any inverse image $\varphi^{-1}(\mathscr{F})$ will be finitely generated.

Proposition 3.5.4. *Suppose $\mathscr{F}$ is finitely generated. Suppose for some point $x \in X$ and open $U \ni x$ the images of sections $s_1, \dots, s_n \in \mathscr{F}(U)$ in $\mathscr{F}_x$ generate the stalk $\mathscr{F}_x$. Then there exists an open subset $V \subset U$ s.t. the images of $s_1, \dots, s_n \in \mathscr{F}(U)$ in $\mathscr{F}_y$ generate $\mathscr{F}_y$ for all $y \in V$.*

Proof. Since $\mathscr{F}$ is finitely generated, there is some $V' \subset U$ and $t_1, \dots, t_m \in \mathscr{F}(V')$ such that $t_1, \dots, t_m$ generate $\mathscr{F}_y$ for any $y \in V'$. Since $\mathscr{F}_x$ is also generated by $s_1, \dots, s_n$, we can express t_i in terms of s_j: $t_i = \sum a_{ij} s_j$, where $a_{ij} \in \mathscr{O}_x$. There are finitely many a_{ij}, and since they are germs, there is a small open neighbourhood V'' s.t. a_{ij} are actually restrictions of some $\tilde{a}_{ij} \in \mathscr{F}(V'')$. If we now take $V = V' \cap V''$, sections s_i generate $\mathscr{F}_y$ for any $y \in V$. □

Corollary 3.5.5. *If $\mathscr{F}$ is of finite type and $\mathscr{F}_x = 0$ for some $x \in X$, then $\mathscr{F}|_V = 0$ for some small open neighbourhood V of x.*

Definition 3.5.6. *A sheaf $\mathscr{F}$ is called* quasi-coherent *if it is locally presentable, i.e. for every $x \in X$ there is an open $U \subset X$ containing x s.t. there exist an exact sequence*

$$\mathscr{O}^{\oplus I}|_U \to \mathscr{O}^{\oplus J}|_U \to \mathscr{F}|_U \to 0,$$

where I and J may be infinite, i.e. if $\mathscr{F}$ is locally the cokernel of free modules. If both I and J can be chosen to be finite then $\mathscr{F}$ is called finitely presented*.*

Definition 3.5.7. *A sheaf $\mathscr{F}$ is called* coherent *if it is finitely generated and for every open $U \subset X$ and every finite $n \in \mathbb{N}$, every morphism $\mathscr{O}^{\oplus n}|_U \to \mathscr{F}|_U$ of $\mathscr{O}|_U$-modules has a finitely generated kernel.*

Example 3.5.8. If X is Noetherian topological space, i.e. such that any chain $V_1 \subset V_2 \subset \dots$ of closed subspaces stabilizes, then the structure sheaf $\mathscr{O}_X$ is coherent.

Example 3.5.9. The sheaf of complex analytic functions on a complex manifold is coherent. This is a hard theorem due to Oka, see [Oka50].

Example 3.5.10. The sheaf of sections of a vector bundle on a scheme or a complex analytic space is coherent.

Example 3.5.11. If Z is a closed subscheme of a scheme X, the sheaf $\mathscr{I}_Z$ of all regular functions vanishing on Z is coherent.

Lemma 3.5.12. *If X is Noetherian then $\mathscr{F}$ is of finite type if and only if $\mathscr{F}$ is finitely presented, if and only if $\mathscr{F}$ is coherent.*

Lemma 3.5.13. *For coherent and quasi-coherent sheaves the "two out of three" property holds. Namely, if there is a short exact sequence*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

and two out of three sheaves $\mathscr{F}, \mathscr{G}, \mathscr{H}$ are coherent (resp. quasi-coherent), then the third one is also coherent (resp. quasi-coherent).

Theorem 3.5.14. *On affine variety (or better affine scheme) X with affine algebra of functions $A = \mathscr{O}_X(X)$ the global section functor Γ gives equivalence of categories $\mathsf{Qcoh}(X) \to \mathsf{Mod}(A)$. Moreover, restriction of Γ to $\mathsf{coh}(X) \subset \mathsf{Qcoh}(X)$ gives equivalence of categories $\mathsf{coh}(X) \to \mathsf{fgMod}(A)$, where $\mathsf{fgMod}(A) \subset \mathsf{Mod}(A)$ is a full subcategory of finitely generated A-modules.*

Proof. The inverse functor is given by tilde-construction. For details see, for example, [EH00]. □

3.6 Motivation for sheaf cohomology

Let X be a topological space. Recall that a presheaf (of abelian groups) on X is a contravariant functor $\mathscr{F} : \mathsf{Open}(X)^\circ \to \mathsf{Ab}$. The presheaf $\mathscr{F}$ is a sheaf if whenever $U = \bigcup U_i$ is an open cover, the sequence

$$0 \longrightarrow \mathscr{F}(U) \longrightarrow \prod_i \mathscr{F}(U_i) \longrightarrow \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$

where the first map is $s \mapsto (\rho^U_{U_i}(s))_i$ and the second is $(s_i)_i \mapsto \left(\rho^{U_i}_{U_i \cap U_j}(s) - \rho^{U_j}_{U_i \cap U_j}(t)\right)_{i,j}$.

Warning. Some textbooks only require that this sequence be exact for *finite* open covers. This does not yield the same notion of a sheaf. For example, let $X = \mathbb{C}^n$ and $\mathscr{F}$ be the sheaf of bounded continuous $\mathbb{C}$-valued functions. Then $\mathscr{F}$ satisfies the sheaf axiom for all finite covers, but it is easily seen that $\mathscr{F}$ is not a sheaf.

In the previous section, we defined kernels and images of sheaves. This enables us to define an exact sequence of sheaves. In particular, we can consider short exact sequences

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow 0 \tag{1.4}$$

The global section functor $\Gamma(X, -) : \mathsf{Sh}(X) \to \mathsf{Ab}$ is left exact. That is, if we apply $\Gamma(X, -)$ to the sequence (1.4), then

$$0 \longrightarrow \mathscr{K}(X) \longrightarrow \mathscr{F}(X) \longrightarrow \mathscr{G}(X)$$

is exact. It can be proved directly.

Remark 3.6.1. Another way to prove that $\Gamma(X, -) : \mathsf{Sh}(X) \to \mathsf{Ab}$ is left exact is to notice that $\Gamma(X, -)$ is right adjoint to the functor $F : \mathsf{Ab} \to \mathsf{Sh}(X)$ that sends any abelian group G to the constant G-valued sheaf on X. Then we can use the general fact that right adjoint functor is left exact.

However, the morphism $\mathscr{F}(X) \to \mathscr{G}(X)$ on the right may not be surjective (i.e. Γ may not be right exact).

Example 3.6.2. Let X be any compact connected Riemann surface (e.g. $\mathbb{P}^1(\mathbb{C})$). Let $\mathscr{F} = \mathscr{O}_X^{\mathrm{an}}$, the sheaf of holomorphic functions defined earlier. For $\Phi = \{x_1, \ldots, x_n\} \subset X$, we define a sheaf $\mathscr{G}$ by

$$\mathscr{G}(U) = \bigoplus_{x_i \in U \cap \Phi} \mathbb{C} \cdot [x_i]$$

where $[x_i]$ is a formal basis element. Define $\varphi : \mathscr{F} \to \mathscr{G}$ by $\varphi(U) f = \sum_{x \in U \cap \Phi} f(x_i) \cdot [x_i]$. Set $\mathscr{K} = \mathrm{Ker}(\varphi)$. Taking global sections, we obtain $\Gamma(\mathscr{G}) = \mathbb{C}^{\oplus k}$, and, by Louiville's theorem, $\Gamma(\mathscr{F}) = \Gamma(X, \mathscr{O}_X^{\mathrm{an}}) = \mathbb{C}$. If $k > 1$, then it is certainly not possible for $\Gamma(\mathscr{F}) \to \Gamma(\mathscr{G})$ to be surjective, even though it is trivial to check that φ is surjective (at the level of sheaves).

Given an exact sequence $0 \to \mathscr{K} \to \mathscr{F} \to \mathscr{G} \to 0$, we can apply the global sections functor Γ to obtain an exact sequence $0 \to \Gamma(X, \mathscr{K}) \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{G})$. The main idea of classical homological algebra is to *canonically* construct groups $\mathrm{H}^i(X, \mathscr{F})$ that extend the exact sequence on the right:

$$0 \to \Gamma(\mathscr{K}) \to \Gamma(\mathscr{F}) \to \Gamma(\mathscr{G}) \to \mathrm{H}^1(\mathscr{K}) \to \mathrm{H}^1(\mathscr{F}) \to \mathrm{H}^1(\mathscr{G}) \to \mathrm{H}^2(\mathscr{K}) \to \cdots$$

Here, and elsewhere, we will write $\Gamma(\mathscr{F})$ and $\mathrm{H}^i(\mathscr{F})$ for $\Gamma(X, \mathscr{F})$ and $\mathrm{H}^i(X, \mathscr{F})$ when X is clear from the context.

3.7 Sheaf cohomology

In this section we will define sheaf cohomology using the classical Godement resolution, and compare it to Čech cohomology.

Definition 3.7.1. *A sheaf $\mathscr{F}$ is called* flabby *if for all $U \subset X$, the restriction map $\rho^X_U : \mathscr{F}(X) \twoheadrightarrow \mathscr{F}(U)$ is surjective.*

Note that if $\mathscr{F}$ is flabby, then the sequence

$$0 \longrightarrow \mathscr{F}(U \cup U') \longrightarrow \mathscr{F}(U) \oplus \mathscr{F}(U') \longrightarrow \mathscr{F}(U \cap U') \longrightarrow 0$$

is exact on the right. Indeed, given any $r \in \mathscr{F}(U \cap U')$, we can choose $\tilde{r} \in \mathscr{F}(X)$ with $\rho_{U\cap U'}(\tilde{r}) = r$. Then $(\rho^X_U(\tilde{r}), 0)$ maps to r.

Lemma 3.7.2. *Let $0 \to \mathscr{K} \to \mathscr{F} \to \mathscr{G} \to 0$ be an exact sequence of sheaves, and assume $\mathscr{K}$ is flabby. Then $0 \to \mathscr{K}(X) \to \mathscr{F}(X) \to \mathscr{G}(X) \to 0$ is also exact.*

Proof. Recall that the surjectivity of $\varphi : \mathscr{F} \to \mathscr{G}$ implies that $\varphi(X) : \mathscr{F}(X) \to \mathscr{G}(X)$ is "locally surjective." That is, for all $t \in \mathscr{G}(X)$ and for all $x \in X$, there exists an open neighborhood U of x and $s \in \mathscr{F}(U)$ such that $\rho^X_U(t) = \varphi(U)(s)$. Assume now that a given $t \in \mathscr{G}(X)$ lifts to local sections $s \in \mathscr{F}(U)$ and $s' \in \mathscr{F}(U')$. Put $\varphi(U)s = \rho^X_U(t)$ and $\varphi(U')s' = \rho^X_{U'}(t)$. If it happens that s and s' agree over $U \cap U'$, i.e. $\rho^U_{U\cap U'}(s) = \rho^{U'}_{U\cap U'}(s')$, then we can glue s and s' along $U \cup U'$. Unfortunately s and s' do not always agree over $U \cap U'$. However, if we let $r = \rho^U_{U\cap U'}(s) - \rho^{U'}_{U\cap U'}(s')$, then

$$\begin{aligned}
\varphi(U \cap U')(r) &= \rho^U_{U\cap U'}(\varphi(U)(s)) - \rho^{U'}_{U\cap U'}(\varphi(U')(s)) \\
&= \rho^U_{U\cap U'}\rho^X_U(t) - \rho^{U'}_{U\cap U'}\rho^X_{U'}(t) \\
&= 0
\end{aligned}$$

In other words, $r \in \mathscr{K}(U \cap U')$. Since $\mathscr{K}$ is flabby, there exists $\tilde{r} \in \mathscr{K}(X)$ such that $\rho^X_{U\cap U'}(\tilde{r}) = r$. Using $\tilde{r}$, we can "correct" s' by replacing it with $s'' = s' + \rho^X_U(\tilde{r}) \in \mathscr{F}(U')$. Now

$$\begin{aligned}
\rho^{U'}_{U\cap U'}(s'') &= \rho^U_{U\cap U'}(s') + \rho^{U'}_{U\cap U'}\rho^X_{U'}(\tilde{r}) \\
&= \rho^{U'}_{U\cap U'}(s') + \rho^X_{U\cap U'}(\tilde{r}) \\
&= \rho^U_{U\cap U'}(s)
\end{aligned}$$

thus there exists $\tilde{s} \in \mathscr{F}(U \cup U')$ such that $\rho^{U\cup U'}_U = s$ and $\rho^{U\cup U'}_{U'}(\tilde{s}) = s''$. By (transfinite) induction, the result follows. □

Lemma 3.7.3. *There are "enough" flabby sheaves. More precisely, for any abelian sheaf $\mathscr{F}$, there is a (functorial) embedding $\epsilon : \mathscr{F} \to C^0(\mathscr{F})$, where $C^0(\mathscr{F})$ is flabby.*

Proof. Define $C^0(\mathscr{F})(U) = \prod_{x\in U} \mathscr{F}_x$ and $\epsilon(U)(s) = (s_x)_{x\in U}$, where $s_x = \rho^U_x(s)$ is the germ of s at x. It is not difficult to check that $C^0(\mathscr{F})$ is actually a flabby sheaf. □

The assignment $\mathscr{F} \mapsto C^0(\mathscr{F})$ is actually an exact functor $C^0 : \mathsf{Sh}(X) \to \mathsf{Sh}(X)$. Let $C^1(\mathscr{F}) = C^0(\operatorname{Coker}\epsilon) = C^0(C^0(\mathscr{F})/\epsilon\mathscr{F})$. There is a canonical map $d^0 : C^0(\mathscr{F}) \to C^0(\mathscr{F})/\epsilon\mathscr{F} \hookrightarrow C^0(C^0(\mathscr{F})/\epsilon\mathscr{F}) = C^1(\mathscr{F})$. This procedure can be iterated: assume we have defined $(C^k(\mathscr{F}), d^{k-1})$ for all $k \leqslant n$. Then set

$$C^{n+1}(\mathscr{F}) = C^0\left(C^n\mathscr{F}/d^{n-1}C^{n-1}\mathscr{F}\right)$$
$$d^n = C^n \twoheadrightarrow C^n\mathscr{F}/d^{n-1}C^{n-1}\mathscr{F} \hookrightarrow C^0(C^n/d^{n-1}C^{n-1}) = C^{n+1}$$

Definition 3.7.4. *The complex*

$$C^\bullet(\mathscr{F}) : \cdots \to 0 \to 0 \to C^0(\mathscr{F}) \xrightarrow{d^0} C^1(\mathscr{F}) \xrightarrow{d^1} \cdots$$

together with the morphism of complexes $\epsilon : \mathscr{F} \to C^\bullet(\mathscr{F})$, where we regard $\mathscr{F}$ as the complex $\cdots \to 0 \to 0 \to \mathscr{F} \to 0 \to \cdots$, is called the Godement resolution *of $\mathscr{F}$.*

Note that by construction, $\mathrm{H}^n(C^\bullet(\mathscr{F})) = 0$ for all $n > 0$, and $\mathrm{H}^0(C^\bullet(\mathscr{F})) \simeq \mathscr{F}$. Equivalently, we can say that ϵ is a quasi-isomorphism. Applying $\Gamma(X, -)$ termwise to $C^\bullet(\mathscr{F})$, we get a new complex $\Gamma(X, C^\bullet(\mathscr{F}))$ which may not be acyclic.

Definition 3.7.5. *The* cohomology *of X with coefficients in $\mathscr{F}$ is* $\mathrm{H}^n(X, \mathscr{F}) = \mathrm{H}^n\left(C^\bullet(\mathscr{F})(X)\right)$.

It follows immediately from the definition that $\mathrm{H}^n(X, \mathscr{F}) = 0$ if $n < 0$, and that $\mathrm{H}^0(X, \mathscr{F}) \simeq \Gamma(X, \mathscr{F}) = \mathscr{F}(X)$ canonically.

Theorem 3.7.6. *Given any short exact sequence of sheaves*

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow 0 \tag{1.5}$$

there is a long exact sequence, functorial in (1.5)

$$\cdots \longrightarrow \mathrm{H}^n(X, \mathscr{K}) \longrightarrow \mathrm{H}^n(X, \mathscr{F}) \longrightarrow \mathrm{H}^n(X, \mathscr{G}) \xrightarrow{\delta} \mathrm{H}^{n+1}(X, \mathscr{K}) \longrightarrow \cdots$$

Proof. We know that $0 \to \mathscr{K} \to \mathscr{F} \to \mathscr{G} \to 0$ is exact. It is not hard to show that $C^\bullet$ is an exact functor, so $0 \to^\bullet (\mathscr{K}) \to C^\bullet(\mathscr{F}) \to C^\bullet(\mathscr{G}) \to 0$ is also exact. Since $C^\bullet$ takes sheaves to complexes of flabby sheaves, Lemma 3.7.2 shows that $0 \to C^\bullet(\mathscr{K})(X) \to C^\bullet(\mathscr{F})(X) \to C^\bullet(\mathscr{G})(X) \to 0$ is exact. By Theorem 3.7.7 it follows that $\cdots \to \mathrm{H}^n(X, \mathscr{K}) \to \mathrm{H}^n(X, \mathscr{F}) \to \mathrm{H}^n(X, \mathscr{G}) \to \mathrm{H}^{n+1}(X, \mathscr{K}) \to \cdots$ is exact. □

Theorem 3.7.7 ("main theorem" of homological algebra)**.** *If $0 \to K^\bullet \to F^\bullet \to G^\bullet \to 0$ is a termwise exact sequence of complexes of abelian groups, there is a natural exact sequence in cohomology*

$$\cdots \longrightarrow \mathrm{H}^n(K^\bullet) \longrightarrow \mathrm{H}^n(F^\bullet) \longrightarrow \mathrm{H}^n(G^\bullet) \xrightarrow{\delta} \mathrm{H}^{n+1}(K^\bullet) \longrightarrow \cdots$$

where all the maps but δ are the obvious induced ones, and δ is canonically constructed.

Definition 3.7.8. *A sheaf* $\mathscr{F}$ *is* acyclic *if* $\mathrm{H}^n(X, \mathscr{F}) = 0$ *for all* $n > 0$.

By Lemma 3.7.2, flabby sheaves are acyclic. It is a good exercise to show that if X is an irreducible topological space, then any constant sheaf on X is flabby, hence acyclic. Recall that X is *reducible* if there is some decomposition $X = X_1 \cup X_2$, where the X_i are nonempty proper closed subsets.

Remark 3.7.9. In classical topology, many interesting invariants of a space X appear as $\mathrm{H}^\bullet(X, \mathbb{Z})$, where here $\mathbb{Z}$ represents the constant sheaf. In algebraic geometry, algebraic varieties (equipped with the Zariski topology) are generally irreducible, so this construction is completely useless. There are two ways to fix this. One is to use the étale topology (or some other Grothendieck topology). Alternatively, one can replace constant sheaves by (quasi-)coherent sheaves. The latter idea is due to Serre.

3.8 Applications

Recall that a sheaf homomorphism $\varphi : \mathscr{F} \to \mathscr{G}$ is an *epimorphism* if we can locally lift sections of $\mathscr{G}$ to $\mathscr{F}$. The obstructions to lifting global sections "live in" the first sheaf cohomology group $\mathrm{H}^1(X, \mathscr{F})$.

Example 3.8.1 (M. Noether's "AF+BG" Theorem)**.** Consider the projective space $\mathbb{P}^n_k$ over some field k. Recall that $\mathbb{P}^n_k = \left(\mathbb{A}^{n+1}_k \setminus \{0\}\right)/k^\times$ as a set. Explicitly, elements of $\mathbb{P}^n_k$ are equivalence classes of tuples $(x_0, \dots, x_n) \neq 0$, where $(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n)$ for all $\lambda \in k^\times$. We will write $(x_0 : \cdots : x_n)$ for the equivalence class of $(x_0, \dots, x_n)$ in $\mathbb{P}^n_k$. Let $\pi : \mathbb{A}^{n+1}_k \setminus \{0\} \twoheadrightarrow \mathbb{P}^n_k$ be the canonical projection. For any integer $m \in \mathbb{Z}$, define a sheaf $\mathscr{O}_{\mathbb{P}^n}(m)$ by

$$\mathscr{O}_{\mathbb{P}^n}(m)(U) = \{\text{regular functions on } \pi^{-1}(U) \text{ that are homogeneous of degree } m\}$$

Note that $\mathscr{O}_{\mathbb{P}^n}(0) = \mathscr{O}_{\mathbb{P}^n}$, the structure sheaf of $\mathbb{P}^n$. For each m, $\mathscr{O}_{\mathbb{P}^n}(m)$ is a sheaf of $\mathscr{O}_{\mathbb{P}^n}$-modules. For any sheaf $\mathscr{F}$ of $\mathscr{O}_{\mathbb{P}^n}$-modules, we can define $\mathscr{F}(m) = \mathscr{F} \otimes_{\mathscr{O}_{\mathbb{P}^n}} \mathscr{O}_{\mathbb{P}^n}(m)$. It is easy to check that multiplication induces an isomorphism $\mathscr{O}_{\mathbb{P}^n}(m) \otimes_{\mathscr{O}_{\mathbb{P}^n}} \mathscr{O}_{\mathbb{P}^n}(m') \to \mathscr{O}_{\mathbb{P}^n}(m + m')$. As an application, let C_1 and C_2 be curves in $\mathbb{P}^2_k$ given by

$$C_i = V(F_i) = \left\{(x_0 : x_1 : x_2) \in \mathbb{P}^2 : F_i(x_0 : x_1 : x_2) = 0\right\}$$

where the F_i are homogeneous polynomials of degrees, say, $\deg F_1 = m$ and $\deg F_2 = n$. For simplicity, we will assume that C_1 and C_2 intersect transversely. Let C be another curve that passes through all intersection points of the curves C_1 and C_2. We don't assume that the C_i are smooth. Write $C = V(F)$ for some homogeneous polynomial F. Then Max Noether proved that $F = A_1 F_1 + A_2 F_2$ for some homogeneous polynomials A_i. We will prove this using sheaves.

Proof. Let $X = \mathbb{P}^2$, and let $\mathscr{I}$ be the ideal sheaf of $C_1 \cap C_2$. One has $\mathscr{I}(U) = \{a \in \mathscr{O}_X(U) : a(c) = 0 \forall c \in C_1 \cap C_2\}$. We can define $\mathscr{I}(k) = \mathscr{I} \otimes_{\mathscr{O}_{\mathbb{P}^2}} \mathscr{O}_{\mathbb{P}^2}(k)$ for any $k \in \mathbb{Z}$. So $\Gamma\left(X, \mathscr{I}(k)\right)$

is the set of homogeneous forms of degree k that vanish on $C_1 \cap C_2$. There is the following exact sequence, which is actually a locally free resolution of $\mathscr{I}(k)$.

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(k-m-n) \xrightarrow{\alpha} \mathscr{O}_{\mathbb{P}^2}(k-m) \oplus \mathscr{O}_{\mathbb{P}^2}(k-n) \xrightarrow{\beta} \mathscr{I}(k) \longrightarrow 0$$

Here, $\alpha(c) = (F_2 c, -F_1 c)$ and $\beta(a,b) = F_1 a + F_2 b$. The theorem we are trying to prove simply asserts the surjectivity of β on global sections. Taking global sections, we get

$$0 \to \Gamma\left(\mathscr{O}(k-m-n)\right) \to \Gamma\left(\mathscr{O}(k-m)\right) \oplus \Gamma\left(\mathscr{O}(k-n)\right) \to \Gamma\left(\mathscr{I}(k)\right) \to \mathrm{H}^1\left(\mathscr{O}(k-m-n)\right) \to \cdots$$

We will see that $\mathrm{H}^1\left(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(k)\right) = 0$ for all $n \geqslant 1$ and all $k \in \mathbb{Z}$. This yields the result. □

Example 3.8.2 (Exponential Sequence)**.** Let X be a complex analytic manifold, for example a Riemann surface. Let $\mathscr{O}_X$ be the structure sheaf of X. Let $\mathscr{O}_X^\times$ be the sheaf of holomorphic functions $X \to \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, with the natural multiplicative structure. Then there is an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^\times \longrightarrow 1$$

where $\mathbb{Z}$ is the constant sheaf and $\exp(s) = e^{2\pi i s}$. The obstructions to lifting sections from $\mathscr{O}_X^\times$ to $\mathscr{O}_X$ lie in the first cohomology group $\mathrm{H}^1(X, \mathbb{Z})$, which does not vanish in general.

In fact, in cohomology we get an exact sequence

$$\ldots \longrightarrow \mathrm{H}^1(X,\mathbb{Z}) \longrightarrow \mathrm{H}^1(X,\mathscr{O}_X) \longrightarrow \mathrm{H}^1(X,\mathscr{O}_X^\times) \xrightarrow{\delta} \mathrm{H}^2(X,\mathbb{Z}) \longrightarrow \mathrm{H}^2(X,\mathscr{O}_X) \longrightarrow \ldots \tag{1.6}$$

If X is complete (i.e. compact in this case), then the global sections $\mathrm{H}^0(X, \mathscr{O}_X^\times)$ is $\mathbb{C}^\times$, and the map $\mathrm{H}^0(X, \mathscr{O}_X^\times) \to \mathrm{H}^1(X, \mathbb{Z})$ will actually be zero, so we can put 0 at the beginning of (1.6).

From the geometric point of view the most important is the term in the middle which is denoted by $\mathrm{Pic}(X) = \mathrm{H}^1(X, \mathscr{O}_X^\times)$. The elements of this group classify (up to isomorphism) all invertible sheaves, i.e. locally free sheaves of rank = 1 on X. All such sheaves are actually sheaves of sections of holomorphic line bundles on X. By definition, for a line bundle $[\mathcal{L}] \in \mathrm{Pic}(X)$ we call the element $c_1(\mathcal{L}) = \delta([\mathcal{L}]) \in \mathrm{H}^2(X, \mathbb{Z})$ the *Chern class* of $\mathcal{L}$.

Assume now that X is complete. Then right hand side of (1.6) gives the famous Hodge-Lefschetz Theorem which asserts that an integral cohomology class $c \in \mathrm{H}^2(X, \mathbb{Z})$ represents Chern class $c_1(\mathcal{L})$ of some line bundle $\mathcal{L}$ if and only if c vanishes in $\mathrm{H}^2(X, \mathscr{O}_X)$.

Next, let $\mathrm{Pic}^0(X) = \mathrm{Ker}(\delta) \subset \mathrm{Pic}(X)$. From (1.6) we see that $\mathrm{Pic}^0(X) \simeq \mathrm{H}^1(X, \mathscr{O}_X)/\,\mathrm{H}^1(X, \mathbb{Z})$ which is the quotient of finite dimensional vector space over $\mathbb{C}$ modulo lattice of finite rank. The natural complex structure on $\mathrm{H}^1(X, \mathscr{O}_X)$ descends to $\mathrm{Pic}^0(X)$ making it an analytic variety called the Picard variety of X. A deeper fact is that if X is an *analytification* of a projective algebraic variety then so is $\mathrm{Pic}^0(X)$.

The quotient group $\mathrm{NS}(X) := \mathrm{Pic}(X)/\,\mathrm{Pic}^0(X)$ is called the *Neron-Severi group*. Since $\mathrm{NS}(X)$ embeds into $\mathrm{H}^2(X, \mathbb{Z})$ via δ, it is finitely generated.

3.9 Čech cohomology

Recall that if X is a topological space, $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ an open covering, we defined a simplicial set (the *nerve* of $\mathcal{U}$) by

$$X_n = \left\{ (\alpha_0, \dots, \alpha_n) \in I^{n+1} : U_{\alpha_0} \cap \dots \cap U_{\alpha_n} \neq \varnothing \right\}$$

and for $f \in \mathrm{Hom}_\Delta([m],[n])$:

$$X(f) : X_n \to X_m, \quad (\alpha_0, \dots, \alpha_n) \mapsto (\alpha_{f(0)}, \dots, \alpha_{f(m)})$$

Given any abelian sheaf $\mathscr{F}$, we define a cohomological system of coefficients for $X_\bullet$, by

$$\mathscr{B}_x = \mathscr{F}_{\alpha_0 \dots \alpha_n} = \mathscr{F}(U_{\alpha_0} \cap \dots \cap U_{\alpha_n})$$

and $\mathscr{B}(f,x) : \mathscr{B}_{X(f)x} \to \mathscr{B}_x$ as restriction maps

$$\mathscr{F}(f, (\alpha_0, \dots, \alpha_n)) : \mathscr{F}(U_{\alpha_{f(0)}} \cap \dots \cap U_{\alpha_{f(m)}}) \xrightarrow{\rho^{U_{\alpha_{f(0)}} \cap \dots \cap U_{\alpha_{f(m)}}}_{U_{\alpha_0} \cap \dots \cap U_{\alpha_n}}} \mathscr{F}(U_{\alpha_0} \cap \dots \cap U_{\alpha_n})$$

Definition 3.9.1. *The* Čech cohomology *of $\mathcal{U}$ with coefficients in $\mathscr{F}$ is*

$$\check{\mathrm{H}}(\mathcal{U}, \mathscr{F}) := \mathrm{H}^\bullet(X_\bullet, \mathscr{B})$$

Definition 3.9.2. *A covering $\mathcal{U}$ is called $\mathscr{F}$-*acyclic *if $\mathrm{H}^i(U_{\alpha_0} \cap \dots \cap U_{\alpha_n}, \mathscr{F}) = 0$ for all $\alpha_0, \dots, \alpha_n \in I$ and $i > 0$.*

The following result allows one to establish acyclicity of some coverings.

Theorem 3.9.3 (H.Cartan's criterion)**.** *Let $\mathcal{A}$ be a class of open subsets of a topological space X such that*

(a) $\mathcal{A}$ is closed under finite intersections, i.e.

$$\forall U_1, \dots, U_n \in \mathcal{A} \Rightarrow U_1 \cap \dots \cap U_n \in \mathcal{A}$$

(b) $\mathcal{A}$ contains arbitrary small open subsets, i.e. for any open U there is $V \subsetneq U$ such that $V \in \mathcal{A}$.

Suppose that for any $U \in \mathcal{A}$ and $\mathcal{A}$-covering $\mathcal{U} = \{U_i\}$ of U, $\mathrm{H}^i(\mathcal{U}, \mathscr{F})$ for all $i > 0$. Then any $\mathcal{A}$-covering is $\mathscr{F}$-acyclic. In particular, for any $\mathcal{A}$-covering of the space X there is isomorphism

$$\check{\mathrm{H}}(\mathcal{U}, \mathscr{F}) \simeq \mathrm{H}^\bullet(X, \mathscr{F}).$$

Chapter 2

Standard complexes in algebra

1 Group cohomology

1.1 Definitions and topological origin

Recall that given a (discrete) group G, we define $(BG)_n = G^n$, the n-fold cartesian product of G with itself. For $f : [m] \to [n]$, we define $BG(f) : G^n \to G^m$ by $(g_1, \ldots, g_n) \mapsto (h_1, \ldots, h_m)$, where

$$h_i = \begin{cases} \prod_{f(i-1)<j\leqslant f(i)} g_j & \text{if } f(i-1) \neq f(i) \\ 1 & \text{otherwise} \end{cases}$$

This is a simplicial set. Next, given a representation of G in an abelian group A (i.e. a homomorphism $G \to \operatorname{Aut}(A)$), define a cohomological system of coefficients:

$$\begin{aligned} \mathscr{B}_x &= A && \text{for all } x \in BG \\ \mathscr{B}(f,x)(a) &= ha \end{aligned}$$

where for $x = (g_1, \ldots, g_n) \in G^n$, we set $h = \prod_{j=1}^{f(0)} g_j$ if $f(0) \neq 0$, and $h = 1$ otherwise.

Definition 1.1.1. *With the above notation, we define $C^\bullet(G,A) = C^\bullet(BG,\mathscr{B})$, and define the* cohomology of G with coefficients in A *to be* $\mathrm{H}^\bullet(G,A) = \mathrm{H}^\bullet[C(G,A)]$.

Explicitly, $C^\bullet(G,A)$ has $C^0(G,A) = A$, and for $n \geqslant 1$, $C^n(G,A) = \operatorname{Hom}_{\mathsf{Set}}(G^n, A)$, with $d^n : C^n \to C^{n+1}$ defined by

$$(df)(g_1, \ldots, g_{n+1}) = g_1 \cdot f(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^{i+1} f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} f(g_1, \ldots, g_n)$$

There is a topological interpretation of $\mathrm{H}^\bullet(G,A)$ if A has trivial G-action. Suppose G acts continuously on a topological space X. Let $Y = G\backslash X$ be the orbit space, with the quotient topology, and let $\pi : X \to Y$ be the projection map.

Theorem 1.1.2. *If X is a contractible space, G acts freely on X (so π is a principal G-bundle over Y), then $\mathrm{H}^n(G, A) = \mathrm{H}^n(Y, A)$, where in the second term A is viewed as the constant sheaf on Y.*

Proof (sketch). Under our assumptions, we have $\pi_1(Y) \simeq G$ and $\pi_i(Y) = 0$ for $i \geqslant 2$. Moreover, $X \to Y$ is a universal cover for Y. From topology, we know that all spaces with $\pi_1(Y) = G$ and $\pi_{>1}(Y) = 0$ are homotopy equivalent. They are often denoted by $K(G, 1)$, and called the *first Eilenberg-Mac Lane space of* G. We know that $|BG|$ satisfies the conclusions of the theorem, so the proof is complete. □

Remark 1.1.3. For the data (G, A) we can define a homological system of coefficients $\mathscr{A}$ with $\mathscr{A}_x = A$ and $\mathscr{A}(f, x) : A \to A$ by $a \mapsto h^{-1}a$, where $x = (g_1, \dots, g_n)$ and h is the same as above.

Definition 1.1.4. *Let the notation be as above. The* homology of G with coefficients in A *is* $\mathrm{H}_\bullet(G, A) = \mathrm{H}_\bullet(C(BG, \mathscr{A}))$.

1.2 Interpretation of $\mathrm{H}^1(G, A)$

We would like to interpret $\mathrm{H}^1(G, A)$ and $\mathrm{H}^2(G, A)$ in terms of more familiar objects. Recall that an *extension of G by N* is an exact sequence of groups:

$$1 \longrightarrow N \longrightarrow E \longrightarrow G \longrightarrow 1$$

We say that two extensions are equivalent if there is a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \| & & \downarrow f & & \| & & \\ 1 & \longrightarrow & N & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

An extension $1 \to N \xrightarrow{i} E \xrightarrow{\pi} G \to 1$ is *split* if it splits on the right, i.e. there is a homomorphism $s : G \to E$ such that $\pi s = \mathrm{id}_G$.

Lemma 1.2.1. *Let A be an abelian group with is also a G-module. Then any split extension of G by A is equivalent to the canonical one:*

$$0 \longrightarrow A \xrightarrow{\iota} A \rtimes G \xrightarrow{\pi} G \longrightarrow 1.$$

Recall that $A \rtimes G$, the *semidirect product* of G and A, is $A \times G$ as a set, with $(a, g) \cdot (b, h) = (a + gb, gh)$. The first cohomology group $\mathrm{H}^1(G, A)$ classifies splittings up to A-conjugacy. Every splitting $s : G \to A \rtimes G$ is of the form $g \mapsto (dg, g)$, where d is some map $G \to A$. The fact that s is a group homomorphism forces $(dg, g) \cdot (dh, h) = (dg + gdh, gh) = (d(gh), gh)$. Thus we need $d(gh) = dg + gdh$. It would be natural to write "$dg \cdot h + g \cdot dh$," but A is not a G-bimodule. If it were, then this condition would require $d : G \to A$ to be a derivation.

Definition 1.2.2. *A map* $d : G \to A$ *is called a* derivation *on* G *with coefficiens in* A *if* $d(gh) = dg + gdh$ *for all* $g, h \in G$.

Again, if A is a G-bimodule, then we require the Leibniz rule to hold, i.e. $d(gh) = dg \cdot h + g \cdot dh$.

Definition 1.2.3. *Two sections* $s_1, s_2 : G \to A \rtimes G$ *are said to be* A-conjugate *if there is an* $a \in A$ *such that*

$$s_2(g) = \iota(a)s_1(g)\iota(a)^{-1}$$

for all $g \in G$.

It is a good exercise to check that $Z^1(G, A) = \{f \in C^1(G, A) : d^1 f = 0\}$ is the set of derivations $d : G \to A$, i.e. $Z^1(G, A) = \mathrm{Der}(G, A)$. If we write $s_1 g = (d_1 g, g)$, $s_2 g = (d_2 g, g)$, then the definition of A-conjugacy means that for some a, we have $d_2 g - d_1 g = g \cdot a - a$ for all g. Once again, if A were a G-bimodule, we would want $d_2 g - d_1 g = g \cdot a - a \cdot g$, i.e. $d_2 - d_1 = [-, a]$.

Definition 1.2.4. *A derivation* $d : G \to A$ *is called* inner *if* $dg = ag - g$ *for some* $a \in A$.

It is a good exercise to check that if $B^1(G, A) := \mathrm{Im}(d^0)$, then $B^1(G, A)$ is exactly the set of inner derivations.

Theorem 1.2.5. *The set of splittings of the canonical extension of* G *by* A *up to* A*-conjugacy is in natural bijection with* $\mathrm{H}^1(G, A)$.

1.3 Interpretation of $\mathrm{H}^2(G, A)$

Theorem 1.3.1. *The set of equivalence classes of extensions of* G *by* A *is in natural bijection with* $\mathrm{H}^2(G, A)$.

Proof. Recall that $\mathrm{H}^2(G, A) = Z^2(G, A)/B^2(G, A)$, where

$$Z^2(G, A) = \{f : G^2 \to A : g_1 f(g_2, g_3) + f(g_1, g_2 g_3) = f(g_1, g_2) + f(g_1 g_2, g_3)\}$$

We will interpret this as a kind of "associativity condition." Given a cocycle $f \in Z^2(G, A)$, we define an extension of G by A explicitly as follows. We have

$$0 \longrightarrow A \rtimes_f G \longrightarrow G \longrightarrow 0$$

where $A \rtimes_f G = A \times G$ as a set, and $(a_1, g_1) \cdot (a_2, g_2) = (a_1 + g_1 a_2 + f(g_1, g_2), g_2 g_2)$. One can check that the associativity

$$(a_1, g_1) \cdot ((a_2, g_2) \cdot (a_3, g_3)) = ((a_1, g_1) \cdot (a_2, g_2)) \cdot (a_3, g_3)$$

is equivalent to $g_1 f(g_2, g_3) + f(g_1, g_2 g_3) = f(g_1, g_2) + f(g_1 g_2, g_3)$. In addition, for $(0, 1)$ to be the identity element in $A \rtimes_f G$, we need to impose the normalisation condition $f(g, 1) = 0 = f(1, g)$ for all $g \in G$. Thus we have a map

$$\{\text{normalized 2-cocycles}\} \longrightarrow \left\{ \begin{array}{c} \text{extensions } 0 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1 \\ \text{with a (set-theoretic) normalized} \\ \text{section } s\colon G \to E \text{ s.t. } s(1) = 1 \end{array} \right\}$$

where a normalized 2-cocycle f maps to $A \rtimes_f G$ along with the section $s : G \to A \rtimes_f G$ given by $g \mapsto (0, g)$. The inverse of this map associates to an extension E with normalized section s the map

$$f(g_1, g_2) = i^{-1}\left(s(g_1)s(g_2)s(g_1 g_2)^{-1}\right)$$

As an exercise, check that choosing a different section s corresponds to changing f by a 2-boundary. □

Example 1.3.2 (Cyclic groups)**.** Let $G = \mathbb{Z}/2$, and let G act on $X = S^\infty = \bigcup_{n \geqslant 1} S^n$ by reflection. Then $X/G = Y = \mathbb{R}P^\infty$, and from topology we know that $\pi_1(Y) = \mathbb{Z}/2$ and $\pi_i(Y) = 0$ for $i \geqslant 2$. Thus $\mathbb{R}P^\infty = K(\mathbb{Z}/2, 1)$, and thus

$$\mathrm{H}_p(\mathbb{Z}/2, \mathbb{Z}) = \mathrm{H}_p(\mathbb{R}P^\infty, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ 0 & \text{if } p \equiv 0 \pmod 2 \text{ and } p \geqslant 2 \\ \mathbb{Z}/2 & \text{otherwise} \end{cases}$$

On the other hand, $\mathrm{H}_2(\mathbb{Z}/2, \mathbb{Q}) = 0$ for all $p \geqslant 1$.

Algebraically, let $G = \mathbb{Z}/n$, and consider the complex of $\mathbb{Z}[G]$-modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{1-t} \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

where $G = \langle t \rangle$ and $N(1) = \sum_{i=0}^{n-1} t^i$. This gives us an infinite resolution

$$\cdots \longrightarrow \mathbb{Z}[G] \xrightarrow{\bar{N}} \mathbb{Z}[G] \xrightarrow{\bar{N}} \mathbb{Z}[G] \xrightarrow{1-t} \mathbb{Z} \longrightarrow 0$$

One can check that this is a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-modules, and it yields

$$\mathrm{H}_p(\mathbb{Z}/n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ 0 & \text{if } p \equiv 0 \pmod 2 \text{ and } p \geqslant 2 \\ \mathbb{Z}/n & \text{otherwise} \end{cases}$$

2 Hochschild (co)homology

2.1 The Bar complex

For the rest of this section, let k be a field, and A be a associative, unital k-algebra. Also, let M be an A-bimodule (also called a two-sided module), i.e. we have $(am)b = a(mb)$. Define *enveloping algebra* of A by $A^e = A \otimes_k A^o$, where A° denotes the opposite algebra of A. It is easy to see that the category of A-bidmodules is equivalent to the categories of left and right A^e-modules. Indeed, we define

$$(a \otimes b^\circ)m = amb$$
$$m(a \otimes b^\circ) = bma$$

Example 2.1.1. Consider $M = A^e$ as a module over itself. It is naturally a A^e-bimodule in two different (commuting) ways. We can compute explicitly:

$$(a \otimes b^\circ)(x \otimes y) = ax \otimes b^\circ y = ax \otimes yb \qquad \text{“outer structure”}$$
$$(x \otimes y)(a \otimes b^\circ) = xa \otimes yb^\circ = xa \otimes by \qquad \text{“inner structure”}$$

Consider the multiplication map $m : A \otimes A \to A$. Define $\widetilde{B}_\bullet A$ to be the complex

$$\widetilde{B}_\bullet A := \quad [\quad \cdots \xrightarrow{b} A^{\otimes 3} \xrightarrow{b} A^{\otimes 2} \xrightarrow{m} A \longrightarrow 0 \quad]$$

where $b : A^{\otimes(n+1)} \to A^{\otimes n}$ is given by

$$b(a_0, \ldots, a_n) = \sum_{i=1}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$

It is an easy exercise to check that $b^2 = 0$.

Definition 2.1.2. *Let $B_n A = A \otimes A^{\otimes n} \otimes A$. The* bar complex *of A is the complex of A-bimodules (with outer structure):*

$$B_\bullet A = \quad [\quad \cdots \longrightarrow B_2 A \xrightarrow{b} B_1 A \longrightarrow B_0 A \longrightarrow 0 \quad]$$

Write $m : B_\bullet A \to A$, where A is regarded as a complex supported in degree zero. This is actually a morphism of complexes because $m \circ b' = 0$ by associativity. We call $B_\bullet \to A$ the *bar resolution* of A as a A-bimodule.

Lemma 2.1.3. *The morphism $m : B_\bullet A \to A$ is a quasi-isomorphism.*

Proof. It is equivalent to say that $\widetilde{B}_\bullet A$ is exact. We use the fact that if the identity on $\widetilde{B}_\bullet A$ is homotopic to zero, then $\widetilde{B}_\bullet A$ is quasi-isomorphic to A. So we want to construct maps $h_n : A^{\otimes n} \to A^{\otimes(n+1)}$ such that $\mathrm{id} = b' \circ h + h \circ b'$. Define

$$h_n(a_1 \otimes \cdots \otimes a_n) = 1 \otimes a_1 \otimes \cdots \otimes a_n$$

We now compute

$$(h \circ b')(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i 1 \otimes a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$

$$(b' \circ h)(a_0 \otimes \cdots \otimes a_n) = 1 \cdot a_0 \otimes a_1 \otimes \cdots \otimes a_n + \sum_{i=0}^{n-1} (-1)^{i+1} \cdot 1 \otimes a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$

It follows that $(h \circ b' + b' \circ h)(a_0 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_n$, as desired. □

2.2 Differential graded algebras

Definition 2.2.1. *A chain differential graded (DG) algebra (resp. cochain DG algebra) over a field k is a $\mathbb{Z}$-graded k-algebra, equipped with a k-linear map $d : A_\bullet \to A_{\bullet-1}$ (resp. $d : A^\bullet \to A^{\bullet+1}$) such that*

1. $d^2 = 0$
2. $d(ab) = (da)b + (-1)^{|a|} a db$ *for all* $a, b \in A_\bullet$ *with* a *homogeneous.*

Here $|a|$ denotes the degree of a in $A_\bullet$. We call the second requirement the graded Leibniz rule*. (Recall that a graded algebra is a direct sum $A_\bullet = \bigoplus_{i \in \mathbb{Z}} A_i$ such that $1 \in A_0$ and $A_i \cdot A_j \subset A_{i+j}$.)*

A DG algebra $A_\bullet$ is called *non-negatively graded* if $A_i = 0$ for all $i < 0$. In addition, if $A_0 = k$ then $A_\bullet$ is called *connected*. We let DGA_k denote the category of all DG k-algebras, and DGA_k^+ denote the full subcategory of DGA_k consisting of non-negatively graded DG algebras.

Example 2.2.2 (Trivial DG algebra)**.** An ordinary associative algebra A can be viewed as DG algebra with differential $d = 0$ and grading $A_0 = 0$, $A_i = 0$ for $i \neq 0$. Hence the category Alg_k of associative algebras over k can be identified with a full subcategory of DGA_k.

Example 2.2.3 (Differential forms)**.** Let A be a *commutative* k-algebra. The *de Rham algebra* of A is a non-negatively graded commutative DG algebra $\Omega^\bullet(A) = \bigoplus_{n \geq 0} \Omega^n(A)$ defined as follows. First, we set $\Omega^0(A) = A$ and take $\Omega^1(A)$ to be the A-module of *Kähler differentials*. By definition, $\Omega^1(A)$ is generated by k-linear symbols da for all $a \in A$ (so $d(\lambda a + \mu b) = \lambda da + \mu db$ for $\lambda, \mu \in k$) with the relation

$$d(ab) = a(db) + b(da), \ \forall a, b \in A.$$

It is easy to show that $\Omega^1(A)$ is isomorphic (as an A-module) to the quotient of $A \otimes A$ modulo the relations $ab \otimes c - a \otimes bc + ca \otimes b = 0$ for all $a, b, c \in A$. Then we define $\Omega^n(A)$ using the exterior product *over* A by

$$\Omega^n(A) := \textstyle\bigwedge_A^n \Omega^1(A)$$

Thus $\Omega^n(A)$ is spanned by the elements of the form $a_0 da_1 \wedge \cdots \wedge da_n$, which are often denoted simply by $a_0 da_1 \dots da_n$ and called differential forms of degree n.

The differential $d\colon \Omega^n \to \Omega^{n+1}$ is defined by

$$d(a_0 da_1 \dots da_n) := da_0 da_1 \dots da_n$$

The product $\wedge$ on the space $\Omega^\bullet(A)$ is given by the formula

$$(a_0 da_1 \dots a_n) \wedge (b_0 db_1 \dots b_m) = a_0 b_0 da_1 \dots da_n db_1 \dots db_m$$

This makes $\Omega^\bullet(A)$ a differential graded algebra over k. If X is a complex variety and $A = \mathscr{O}(X)$ is the algebra of regular functions, then $\Omega^\bullet(A) = \Omega^\bullet(X)$, where $\Omega^\bullet(X)$ is the algebra of regular differential forms on X. However, if M is a smooth manifold and $A = C^\infty(M)$, then the natural map $\Omega^\bullet(A) \to \Omega^\bullet(M)$ is *not* an isomorphism. Indeed, David Speyer pointed out that if f, g are algebraically independent in A, then df and dg are linearly independent in $\Omega^\bullet(A)$. (see the discussion before Theorem 26.5 in [Mat89]). Since e^x and 1 are algebraically independent, $d(e^x)$ and $d(1) = dx$ are linearly independant over $A = C^\infty(\mathbb{R})$ in $\Omega^1(A)$. But certainly $d(e^x) = e^x \cdot d(1)$ in $\Omega^1(\mathbb{R})$.

Example 2.2.4 (Noncommutative differential forms)**.** The previous example can be generalized to all associative (not necessarily commutative) algebras.

Suppose A is an (associative) algebra over a field k. First we define *noncommutative Kähler differentials* $\Omega^1_{\mathrm{nc}}(A)$ as the kernel of multiplication map $m\colon A \otimes A \to A$:

$$0 \longrightarrow \Omega^1_{\mathrm{nc}}(A) \longrightarrow A \otimes A \xrightarrow{m} A \longrightarrow 0$$

So $\Omega^1_{\mathrm{nc}}(A)$ is naturally an A-bimodule. Then we can define DG algebra of *noncommutative differential forms* $\Omega^\bullet_{\mathrm{nc}}(A)$ as the tensor algebra $T(\Omega^1_{\mathrm{nc}}(A))$:

$$\Omega^\bullet_{\mathrm{nc}}(A) := T(\Omega^1_{\mathrm{nc}}(A)) = A \,\oplus\, \Omega^1_{\mathrm{nc}}(A) \,\oplus\, \Omega^1_{\mathrm{nc}}(A)^{\otimes 2} \,\oplus \dots$$

Differential d on $\Omega^\bullet_{\mathrm{nc}}(A)$ is completely defined by the derivation

$$d'\colon A \to \Omega^1_{\mathrm{nc}}(A) \quad d'(a) = a \otimes 1 - 1 \otimes a \in \mathrm{Ker}(m) = \Omega^1_{\mathrm{nc}}(A)$$

Indeed, there exists unique differential $d\colon \Omega^\bullet_{\mathrm{nc}}(A) \to \Omega^{\bullet+1}_{\mathrm{nc}}(A)$ of degree 1 that lifts d'. Explicitly it can be defined by the following formula:

$$d(a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n) = 1 \otimes \bar{a}_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n$$

Here for $a \in A$ we denote by $\bar{a}$ the element $d'(a) = a \otimes 1 - 1 \otimes a \in \Omega^1_{\mathrm{nc}}(A)$.

Actually, there is more conceptual way of defining Kähler differential and noncommutative forms. Consider the functor $\mathrm{Der}(A, -)\colon A\text{-}\mathsf{bimod} \to \mathsf{Sets}$ associating to a bimodule M the set of all derivations $\mathrm{Der}(A, M)$. This functor is representable. Precisely, we have the following

Proposition 2.2.5. *For every A-bimodule M there exists canonical isomorphism*

$$\mathrm{Der}(A, M) \simeq \mathrm{Hom}_{A\text{-bimod}}(\Omega^1_{\mathrm{nc}}(A), M)$$

For any DG algebra B let B_0 be it's 0-component. This is just an ordinary algebra. Then noncommutative differential forms can be also described by the following universal property.

Proposition 2.2.6. *For any associative k-algebra A and any k-algebra B there is natural isomorphism*

$$\mathrm{Hom}_{\mathsf{DGA}^+_k}(\Omega^\bullet_{\mathrm{nc}}(A), B) \simeq \mathrm{Hom}_{\mathsf{Alg}_k}(A, B_0).$$

This proposition says essentially that the functor $\Omega^\bullet_{\mathrm{nc}}(-)\colon \mathsf{Alg}_k \to \mathsf{DGA}^+_k$ is left adjoint to the forgetful functor $(-)_0\colon \mathsf{DGA}^+_k \to \mathsf{Alg}_k$. For more details on noncommutative differential forms see paper by Cuntz-Quillen [CQ95], or lecture notes by Ginzburg [Gin05].

Definition 2.2.7. *If A is any graded algebra and $d : A \to A$ is a derivation of A we say that d is even or odd if one of the following holds:*

$$\begin{aligned} d(ab) &= (da)b + a(db) && \textit{(even)} \\ d(ab) &= (da)b + (-1)^{|a|} adb && \textit{(odd)} \end{aligned}$$

Lemma 2.2.8. *Any derivation d (even or odd) is uniquely determined by its values on the generators of A as a k-algebra. In other words, if $S \subset A$ is a generating set and $d_1(s) = d_2(s)$ for all $s \in S$, then $d_1 = d_2$.*

Proof. Apply iteratively the Leibniz rule. □

Corollary 2.2.9. *If $d : A \to A$ is an odd derivation and $d^2(s) = 0$ for all s in some generating set of A, then $d^2 = 0$ on all of A.*

Proof. If d is an odd derivation, then d^2 is an even derivation. Indeed, $d^2 = \frac{1}{2}[d, d]_+$, or explicitly

$$\begin{aligned} d^2(ab) &= d((da)b + (-1)^{|a|} adb) \\ &= (d^2a)b + (-1)^{|da|} dadb + (-1)^{|a|} dadb + (-1)^{|a|+|a|} ad^2b \\ &= (d^2a)b + a(d^2b) \end{aligned}$$

The result follows now from the previous lemma. □

Definition 2.2.10. *If $(A_\bullet, d)$ is a DG algebra define the set of* cycles *in A to be*

$$Z_\bullet(A, d) := \{a \in A : da = 0\}$$

Notice that $Z_\bullet(A, d)$ is a graded subalgebra of A.

Moreover, define the set of boundaries $B_\bullet(A,d)$ *to be*

$$B_\bullet(A,d) = \{b \in A : b = da \text{ for some } a \in A\}$$

Then $B_\bullet(A,d)$ *is a two-sided graded ideal in* $Z_\bullet(A)$*. Thus the quotient*

$$\mathrm{H}_\bullet(A) = Z_\bullet(A)/B_\bullet(A)$$

is a graded algebra, called the homology algebra *of* A*. For trivial reasons, the differential is zero on* $\mathrm{H}_\bullet(A)$*. This gives us a functor from the category of DG algebras to the category of graded algebras.*

2.3 Why DG algebras?

Let A be a k-vector space equipped with an (arbitrary) bilinear product $A \times A \to A$, or equivalently a linear map $\mu : A \otimes A \to A$, $(x, y) \mapsto xy$. Assume that $\dim_k A < \infty$. Then, we have the commutative diagram

$$\begin{array}{ccccc} A^* & \xrightarrow{\mu^*} & (A \otimes A)^* & \xleftarrow{\sim} & A^* \otimes A^* \\ \downarrow{\scriptstyle \text{can}} & & & & \downarrow{\scriptstyle \text{can}} \\ T_k(A^*) & & \xrightarrow{\exists!\, d} & & T_k(A^*) \end{array} \tag{2.1}$$

In this diagram, $\mu^* : A^* \to (A \otimes A)^*$ is the linear map dual to μ, the map $A^* \otimes A^* \to (A \otimes A)^*$ is given by $f \otimes g \mapsto [x \otimes y \mapsto f(x)g(y)]$ and it is an isomorphism because A is finite-dimensional. By Lemma 2.2.8, any linear map $A^* \to T_k(A^*)$ determines a derivation $d\colon T(A^*) \to T(A^*)$: precisely, there is a unique $d\colon T(A^*) \to T(A^*)$ such that

(1) $d|_{A^*} = \mu^*$

(2) $\deg(d) = +1$

(3) d satisfies the graded Leibniz rule

Conversely, if $d\colon T(A^*) \to T(A^*)$ satisfies (2) – (3), then restricting $d|_{A^*}\colon A^* \to A^* \otimes A^*$ and dualizing $d^*\colon [A^* \otimes A^*]^* \simeq A \otimes A \to A$ we get a linear mapping $A \otimes a \to A$.

Thus, if A is finite-dimensional, giving a bilinear map $A \times A \to A$ is equivalent to giving a derivation of degree 1 on $T(A^*)$.

Remark 2.3.1. For notational reasons, one usually takes $\delta = -d$, so that $\delta\colon A^* \to (A \otimes A)^*$ is given by $\delta(\omega)(x \otimes y) = -\omega(xy)$, $\omega \in A^*$, $x, y \in A$.

Lemma 2.3.2. *The map* $\mu\colon A \otimes A \to A$ *is associative if and only if* $\delta^2 = 0$ *on* $T(A^*)$*.*

Proof. Take any $\omega \in A^*$ and $x, y, z \in A$. Then we have

$$\begin{aligned}\delta\left(\delta(\omega)\right)(x \otimes y \otimes z) &= -\delta(\omega)(xy \otimes z) - (-1)\delta(\omega)(x \otimes yz) \\ &= \delta(\omega)(x \otimes yz) - \delta(\omega)(xy \otimes z) \\ &= -\omega(x(yz)) + \omega((xy)z) \\ &= \omega\left((xy)z - x(yz)\right).\end{aligned}$$

Hence μ is associative iff $\delta^2 = 0$ on A^* iff $\delta^2 = 0$ on $T(A^*)$. To get the last "iff" we used Corollary 2.2.9. $\square$

To sum up, giving a finite-dimensional associative k-algebra is equivalent to giving a free connected DG algebra which is generated by finitely many elements in degree 1. It is therefore natural (and for many purposes, useful) to think of *all* finitely generated free DG algebras, including the ones having generators in degree ≥ 1, as a 'categorical closure' of the finite-dimensional associative algebras.

2.4 Interpretation of bar complex in terms of DG algebras

Example 2.4.1. Define $A\langle\varepsilon\rangle = A *_k k[\varepsilon]$, where ε is an indeterminate. Here $A *_k B$ denotes the coproduct in the category of (not necessarily commutative) k-algebras, which is given by the free product of algebras. Assume $|a| = 0$ for all $a \in A$, and suppose $|\varepsilon| = 1$. This makes $A\langle\varepsilon\rangle$ a graded algebra whose elements look like

$$a_1\varepsilon^{n_1}a_2\varepsilon^{n_2}\cdots a_k$$

Since $\varepsilon^n = \varepsilon 1\varepsilon 1\varepsilon 1 \ldots 1\varepsilon$, any element in $A\langle\varepsilon\rangle$ can be written as $a_1\varepsilon a_2\varepsilon\cdots\varepsilon a_k$, i.e. ε is a separator (or "bar," if we write $a\varepsilon b$ as $a \mid b$). We can identify $A\langle\varepsilon\rangle$ with $\widetilde{B}_\bullet A$ via

$$\psi a_1\varepsilon a_2\cdots\varepsilon a_n \mapsto a_1 \otimes \cdots \otimes a_n$$

This actually is degree-preserving because $\widetilde{B}_{n-1}A = A^{\otimes n}$ and $a_1\varepsilon a_2\cdots\varepsilon a_n$ also has degree $n-1$. Define the differential on $A\langle\varepsilon\rangle$ by

$$\begin{aligned}da &= 0, \quad \forall a \in A \\ d\varepsilon &= 1\end{aligned}$$

This makes $A\langle\varepsilon\rangle$ a DG algebra. Notice, that since $d(a) = 0$ and $|a| = 0$ for $\forall a \in A$, then d is A-linear. DG algebra $A\langle\varepsilon\rangle$ is isomorphic as a complex to $(\widetilde{B}_\bullet A, b')$. Indeed, we have

$$\begin{aligned}d(a_0\varepsilon a_2\ldots\varepsilon a_n) &= a_0d(\varepsilon)a_1\varepsilon a_2\ldots a_n - a_0\varepsilon d(a_1\varepsilon\ldots a_n) \\ &= a_0a_1\varepsilon a_2\ldots a_n - a_0\varepsilon a_1d(\varepsilon)a_2\varepsilon\ldots a_n + a_0\varepsilon a_1\varepsilon d(a_2\varepsilon\ldots a_n) = \\ &= a_0a_1\varepsilon a_2\ldots a_n - a_0\varepsilon a_1a_2\varepsilon\ldots a_n + a_0\varepsilon a_1\varepsilon d(a_2\varepsilon\ldots a_n) \\ &= \ldots \\ &= \sum_{i=0}^{n}(-1)^i a_0\varepsilon\ldots\varepsilon a_ia_{i+1}\varepsilon\ldots a_n,\end{aligned}$$

which exactly maps to the differential $b'(a_0 \otimes a_2 \cdots \otimes a_n)$ via the identification map ψ.

Notice that $1 = d\varepsilon$, so $1 = 0$ in $\mathrm{H}_\bullet(A\langle\varepsilon\rangle)$, hence $\mathrm{H}_\bullet(A\langle\varepsilon\rangle) = 0$. Under our identification of $\widetilde{B}_\bullet A$ with $A\langle\varepsilon\rangle$, the homotopy h is just $u \mapsto \varepsilon u$. Indeed, we can check that for all $u \in A\langle\varepsilon\rangle$,

$$(dh + hd)(u) = (d\varepsilon u + (-1)^1 \varepsilon du) + \varepsilon du = 1 \cdot u - \varepsilon du = u$$

Given an A-bimodule M, define $M \otimes_{A^e} B_\bullet A$ to be the complex

$$\cdots \to M \otimes_{A^e} A^{\otimes(n+2)} \to \cdots$$

Note that $M \otimes_{A^e} A^{\otimes(n+2)} \simeq M \otimes_{A^e} A^e \otimes A^{\otimes n} \simeq M \otimes_k A^n$ via the map

$$m \otimes_{A^e} (a_0 \otimes \cdots \otimes a_{n+1}) \mapsto a_{n+1} m a_0 \otimes (a_1 \otimes \cdots \otimes a_n)$$

The induced differential $b : M \otimes_k A^n \to M \otimes_k A^{n-1}$ turns out to be

$$m \otimes (a_1 \otimes \cdots \otimes a_n) \mapsto m a_1 \otimes a_2 \otimes \cdots \otimes a_n + \sum_{i=1}^{m-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}$$

2.5 Hochschild (co)homology: definitions

Definition 2.5.1. *The* Hochschild homology *of A with coefficients in M is*

$$\mathrm{HH}_\bullet(A, M) = \mathrm{H}_\bullet(M \otimes_{A^e} B_\bullet A)$$

To define Hochschild cohomology we need the notion of the *morphism complex.*

Definition 2.5.2. *Let A be a ring (or k-algebra), and $(M_\bullet, d_M)$, $(N_\bullet, d_N)$ two complexes of left A-modules. Set*

$$\underline{\mathrm{Hom}}_A(M, N) = \bigoplus_{n \in \mathbb{Z}} \underline{\mathrm{Hom}}_A(M, N)_n,$$

where

$$\underline{\mathrm{Hom}}_A(M, N)_n = \{f \in \mathrm{Hom}_A(M, N) : f(M_i) \subset N_{i+n} \text{ for all } i \in \mathbb{Z}\}$$

is the set of A-module homomorphisms $M \to N$ of degree *n.*

Warning In general, $\underline{\mathrm{Hom}}_A(M, N) \neq \mathrm{Hom}_A(M, N)$, i.e. not every A-module map $f : M \to N$ can be written as a sum of homogeneous maps.

Example 2.5.3. Let $A = k$ be a field, $N = k$ and $M = V = \bigoplus_{n \in \mathbb{Z}} V_n$ a graded k-vector space such that $\dim V_n \geqslant 1$ for all n. Let $f : V \to k$ be such that $f(V_n) \neq 0$ for infinitely many n'. Then $f \notin \underline{\mathrm{Hom}}_k(V, k)$. Why?

Exercise Prove that if M is a finitely generated (as A-module) then $\underline{\mathrm{Hom}}_A(M,N) = \mathrm{Hom}_A(M,N)$.

Definition 2.5.4. *For A a ring and M, N chain complexes over A, define*

$$d_{\mathrm{Hom}} : \underline{\mathrm{Hom}}_A(M,N)_n \to \underline{\mathrm{Hom}}_A(M,N)_{n-1}$$

by

$$f \mapsto d_N \circ f - (-1)^n f \circ d_M$$

Note that this is well-defined because $(d_N \circ f)(M_i) \subset d_N(N_{i+n}) \subset N_{i+n-1}$ and $(f \circ d_M)(M_i) \subset f(M_{i-1}) \subset N_{i-1+n}$. We claim that $d_{\mathrm{Hom}}^2 = 0$. Indeed, we have

$$\begin{aligned} d_{\mathrm{Hom}}^2(f) &= d_N\left(d_N f - (-1)^n f d_M\right) - (-1)^{n-1}\left(d_N f - (-1)^n f d_M\right) d_M \\ &= d_N^2 f - (-1)^n d_N f d_M + (-1)^n_N f d_M + (-1)^{2n-1} f d_M^2 \\ &= 0 \end{aligned}$$

Definition 2.5.5. *Let A be a k-algebra, M an A-bimodule. The* Hochschild cochain complex *of M is*

$$C^n(A,M) = \underline{\mathrm{Hom}}_{A^e}(B_\bullet A, M)_{-n}$$

where M is viewed as a left A^e-module via $(a \otimes b^\circ)m = amb$, and we view M as a complex concentrated in degree zero.

Explicitly, we have

$$\begin{aligned} C^n(A,M) &= \mathrm{Hom}_{A^e}(A^{\otimes(n+2)}, M) \\ &= \mathrm{Hom}_{A^e}(A^{n+2}, M) \\ &= \mathrm{Hom}_k(A^n, M) \end{aligned}$$

A map $f \in \mathrm{Hom}_k(A^n, M)$ is identified with $\varphi : A^{\otimes(n+2)} \to M$, where $\varphi(a_0 \otimes \cdots \otimes a_{n+1}) = a_0 f(a_1, \ldots, a_n) a_{n+1}$. The differential is $d_{\mathrm{Hom}}^n \varphi = (-1)^{n+1} \varphi \circ b'$, or in terms of $f : A^n \to M$,

$$(d^n f)(a_1, \ldots, a_{n+1}) = a_1 f(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) + (-1)^{n-1} f(a_1, \ldots, a_n) a_{n+1}$$

There are many different interpretations of Hochschild cohomology – we will concentrate on extensions of algebras and deformation theory. Hochschild homology is related to de Rham algebras, and can be used to compute the cohomology of free loop spaces. It is also useful in studying the representation theory of preprojective algebras of graphs.

2.6 Centers and Derivations

Example 2.6.1 (Center)**.** Unpacking the definition, we get $C^0(A,M) = \mathrm{Hom}_k(A^{\otimes 0}, M) = \mathrm{Hom}_k(k,M) = M$. The differential $d^0 : M \to C^1(A,M) = \mathrm{Hom}_k(A,M)$ sends m to the function

$$d^0(m)(a) = am - am$$

We have $\mathrm{HH}^0(A,M) = \mathrm{Ker}(d^0) = \{m \in M : am - ma = 0\} = Z(M)$, the *center* of the bimodule M.

Example 2.6.2 (Derivations)**.** We have $d^1 : C^1(A,M) \to C^2(A,M)$, defined by

$$(d^1 f)(a_1 \otimes a_2) = af(a_2) - f(a_1 a_2) + f(a_1)a_2$$

One checks that $\mathrm{Ker}(d^1) = \mathrm{Der}_k(A,M) = \{f \in \mathrm{Hom}_k(A,B) : f(ab) = af(b) + f(a)b\}$. The map $d^0 : M \to C^1(A,M)$ sends m to the inner derivation $\mathrm{ad}_m : a \mapsto [a,m]$. Thus we have an exact sequence

$$0 \longrightarrow Z(M) \longrightarrow M \xrightarrow{\mathrm{ad}} \mathrm{Der}_k(A,M) \longrightarrow \mathrm{HH}^1(A,M) \longrightarrow 0$$

In other words, we have

$$\mathrm{HH}^1(A,M) = \mathrm{Der}_k(A,M)/\,\mathrm{InnDer}_k(A,M).$$

2.7 Extensions of algebras

Let k be a field, and let A be a k-algebra.

Definition 2.7.1. *An* extension *of A is just a surjective k-algebra homomorphism $\pi : R \twoheadrightarrow A$. Equivalently, we can write a short exact sequence*

$$0 \longrightarrow M \longrightarrow R \xrightarrow{\pi} A \longrightarrow 0$$

where $M = \mathrm{Ker}(\pi)$ is a two-sided ideal in R. We call the extension $R \twoheadrightarrow A$ a nilpotent extension *if M is a nilpotent ideal of degree $n \geqslant 1$, i.e. $M^n = 0$ in R. An* abelian extension *of A is a nilpotent extension of A of degree 2.*

Lemma 2.7.2. *If $\pi : R \twoheadrightarrow A$ is an abelian extension with $M = \mathrm{Ker}(\pi)$, then M is canonically an A-bimodule.*

Proof. Choose a k-linear section $s : A \to R$ of π. We can do this because k is a field. We then define a map $A \otimes M \otimes A \to M$ by

$$a \otimes m \otimes b \mapsto s(a)ms(b) = a \cdot m \cdot b$$

To see that this map is well-defined, first let's check that $(a_1 a_2) \cdot m = a_1 \cdot (a_2 \cdot m)$ for all $a_1, a_2 \in A$. Indeed, we have $\pi(s(a_1 a_2) - s(a_1)s(a_2)) = a_1 a_2 - \pi s(a_1) \pi s(a_2) = 0$. So $s(a_1 a_2) - s(a_1)s(a_2) \in \mathrm{Ker}(\pi) = M$. Since $M^2 = 0$,

$$(a_1 a_2) \cdot m - a_1 \cdot (a_2 \cdot m) = (s(a_1 a_2) - s(a1)s(a_2))\, m = 0$$

Finally, if $s' : A \to R$ is another section of π, the fact that $\pi(s - s') = 0$ implies $s(a) - s'(a) \in M$ for all a, whence $(s(a) - s'(a))\, m = 0$, i.e. $s(a)m = s'(a)m$.

Note that we could have defined $A \otimes M \to M$ by $a \otimes m \mapsto rm$ for any r with $\pi(r) = a$, without using the existence of a section. Thus, the lemma will be true if we replace k by any commutative ring. $\square$

Reversing the logic, we fix A and M.

Definition 2.7.3. *An* abelian extension *of A by M is an extension $\pi : R \twoheadrightarrow A$ where* $\operatorname{Ker}(\pi) \simeq M$ *as an A-bimodule.*

A trivial example is $M \rtimes A$, which is $M \oplus A$ as a k-vector space, and which has multiplication

$$(m_1, a_1) \cdot (m_2, a_2) = (m_1 a_2 + a_1 m_2, a_1 a_2)$$

We say that two extensions E, E' of A by M are equivalent if there is a commutative diagram (as in the case of group cohomology):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow\wr & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Theorem 2.7.4. *There is a natural bijection*

$$\mathrm{HH}^2(A, M) \simeq \left\{ \begin{array}{c} \textit{equivalence classes of abelian} \\ \textit{extensions of } A \textit{ by } M \end{array} \right\}$$

Proof. Essentially as in the group case, the bijection is induced by the map $C^2(A, M) \to \mathscr{E}xt(A, M)$ assigning to a 2-cochain $f : A \otimes A \to M$ a k-algebra $M \rtimes_f A$ of the form $M \oplus A$ with multiplication defined by

$$(m_1, a_1) \cdot_f (m_2, a_2) = (m_1 a_2 + a_1 m_2 + f(a_1, a_2), a_1 a_2)$$

The key point is that the product $\cdot_f$ is associative if and only if f is a Hochschild 2-cocycle, i.e. $d^2 f = 0$. Moreover, two algebras $M \rtimes_f A$ and $M \rtimes_g A$ give equivalent extensions of A if and only if $f - g$ is a Hochschild coboundary. (Check this!) $\square$

2.8 Crossed bimodules

Definition 2.8.1. *A* crossed bimodule *is a DG algebra $C_\bullet$ with $C_n = 0$ for all $n \neq 0, 1$.*

So as a complex, $C_\bullet$ is

$$\cdots \longrightarrow 0 \longrightarrow C_1 \xrightarrow{\partial} C_0 \longrightarrow 0 \longrightarrow \cdots$$

Explicitly, C_0 is an algebra, C_1 is a bimodule over C_0, and $C_1^2 = 0$. The Leibniz rule implies that for all $a \in C_0$, $b \in C_1$, we have $\partial(ab) = a(\partial b)$ and $\partial(ba) = (\partial b)a$, i.e. $\partial : C_1 \to C_0$ is a homomorphism of C_0-bimodules. For any $b_1, b_2 \in C_1$, because $b_1 b_2 = 0$, the Leibniz rule implies $(\partial b_1) b_2 = b_1(\partial b_2)$. So we could have defined a crossed bimodule to be a bimodule C_1 over C_0 together with a C_0-bimodule map $\partial : C_1 \to C_0$ satisfying $(\partial b_1) b_2 = b_1(\partial b_2)$.

Remark 2.8.2. If $C_\bullet$ is a crossed bimodule, we can define the structure of an algebra on C_1 by $b_1 * b_2 = (\partial b_1) \cdot b_2$, and with this structure ∂ is an algebra homomorphism.

Remark 2.8.3. Let XBimod be the category of crossed bimodules, which is a full subcategory of DGA_k^+. The inclusion functor $i \colon \mathsf{XBimod} \to \mathsf{DGA}_k^+$ has left adjoint functor $\chi \colon \mathsf{DGA}_k^+ \to \mathsf{XBimod}$ that assignes to any DG algebra

$$C_\bullet = [\, \cdots \longrightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \longrightarrow 0 \,]$$

correspondent cross-bimodule defined by

$$\chi\,(C_\bullet) = [\, 0 \longrightarrow \operatorname{coker}(d_2) \xrightarrow{d_1} C_0 \longrightarrow 0 \,]$$

Lemma 2.8.4. *Let $C_\bullet = (C_1 \xrightarrow{\partial} C_0)$ be a crossed bimodule, and let $A = \mathrm{H}_0(C) = \operatorname{Coker}(\partial)$, $M = \mathrm{H}_1(C) = \operatorname{Ker}(\partial)$. Then A is a k-algebra and M is canonically an A-bimodule.*

We have the exact sequence

$$0 \longrightarrow M \xrightarrow{i} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\pi} A \longrightarrow 0$$

Definition 2.8.5. *A* crossed extension *of A by M is a crossed bimodule $C_\bullet$ with $\mathrm{H}_0(C) = A$ and $\mathrm{H}_1(C) = M$.*

We say that two crossed extensions $C_\bullet, C'_\bullet$ are equivalent if there is an isomorphism of DG algebras $\varphi : C_\bullet \to C'_\bullet$ inducing the identity on A and M.

Let $\mathcal{XE}xt(A, M)$ denote the set of equivalence classes of crossed extensions of A by M.

Theorem 2.8.6. *Let k be a field, A a k-algebra, and M an A-bimodule. Then there is a natural bijection $\mathcal{XE}xt(A, M) \simeq \mathrm{HH}^3(A, M)$.*

Proof. We will define the map $\theta \colon \mathcal{XE}xt(A, M) \to \mathrm{HH}^3(A, M)$. Given an extension

$$\mathcal{E} = [\, 0 \longrightarrow M \xrightarrow{i} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\pi} A \longrightarrow 0 \,]$$

choose splittings $s : A \to C_0$ and $q : \operatorname{Im}(\partial) \to C_1$ of π and ∂. Define $g : A \otimes A \to C_0$ by $g(a \otimes b) = q(s(ab) - s(a)s(b))$. Since π is a morphism of algebras, $\pi \circ g = 0$ implies $s(ab) - s(a)s(b) \in \operatorname{Ker}(\partial)$, so g is well-defined. We can define $\Theta_{\mathcal{E}} : A^{\otimes 3} \to C_1$ by

$$\Theta_{\mathcal{E}}(a_1 \otimes a_2 \otimes a_3) = s(a_1) g(a_2 \otimes g_3) - g(a_1 a_2 \otimes a_3) + g(a_1 \otimes a_2 a_3) - g(a_1 \otimes a_2) s(a_3)$$

Note that $\partial \circ \Theta_{\mathcal{E}} = 0$. Since ∂ is a bimodule map over C_0 and $\partial q = 1$, the image of $\Theta_{\mathcal{E}}$ is contained in $\operatorname{Ker}(\partial) = \operatorname{Im}(i)$. We leave it as an exercise to show that $i^{-1} \circ \Theta_{\mathcal{E}}$ is a Hochschild 3-cocycle whose class in $\mathrm{HH}^3(A, M)$ is independent of the choice of s and q. □

2.9 The characteristic class of a DG algebra

Let $A_\bullet = (\bigoplus_{p\geq 0} A_p, d)$ be a DG algebra. Consider the graded vector spaces

$$C_1 := \underline{\mathrm{Coker}}(d)_{\geq 0}[-1] \quad C_0 := \underline{\mathrm{Ker}}(d)_\bullet$$

Note that C_0 is a graded subalgebra of $A_\bullet$ while C_1 is a graded C_0-bimodule. The differential d on A induces a graded map

$$\partial \colon C_1 \to C_0 \tag{2.2}$$

which makes (2.2) a graded cross-bimodule. The cokernel of ∂ is the algebra $\mathrm{H}_\bullet(A)$, while the kernel of ∂ is the $\mathrm{H}_\bullet(A)$-bimodule whose underlying (graded) vector space is $\mathrm{H}_{\geq 1}(A)[-1]$. The right multiplication on $\mathrm{H}_{\geq 1}(A)[-1]$ is given by the usual multiplication in $\mathrm{H}_\bullet(A)$, while the left multiplication is twisted by a sign:

$$\bar{a} \cdot s(\bar{x}) = (-1)^{|\bar{a}|} s(\bar{a}\bar{x}),$$

where $\bar{a} \in \mathrm{H}_\bullet(A)$ is homogeneous, $\bar{x} \in \mathrm{H}_{\geq 1}(A)[-1]$.

Definition 2.9.1. *By Theorem 2.8.6 the crossed bimodule*

$$0 \longrightarrow \mathrm{H}_{\geq 1}(A)[-1] \xrightarrow{\partial} \mathrm{H}_\bullet(A) \longrightarrow 0$$

represents an element $\gamma_A \in \mathrm{HH}^3(\mathrm{H}_\bullet(A), \mathrm{H}_{\geq 1}(A)[-1])$, which is called the characteristic class *of A.*

This class is secondary (co)homological invariant of A. It is naturally related to Massey triple products. In more detail, let

$$\mathcal{E} := [\, 0 \longrightarrow M \xrightarrow{i} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\pi} B \longrightarrow 0 \,]$$

be a crossed extension of an algebra B by M. Given $a, b, c \in B$ such that $ab = bc = 0$ we define *Massey triple product* $\langle a, b, c\rangle \in M/(aM + Mc)$ as follows. Choose a k-linear section $s\colon b \to C_0$ so that $\pi s = \mathrm{id}_B$, and let $q\colon \mathrm{Im}(\partial) \to C_1$ be a section of ∂ so that $\partial q = \mathrm{id}_{\mathrm{Im}(\partial)}$. Since $ab = 0$ we have $s(a)\cdot s(b) \in Ker\pi$ so we can take $q(s(a)\cdot s(b)) \in C_1$. Similarly, since $bc = 0$ we may define $q(s(b)s(c)) \in C_1$. Now, consider the element

$$\{a, b, c\} := s(a)q(s(b)\cdot s(c)) - q(s(a)\cdot s(b))s(c) \in C_1$$

Since $\partial\{a, b, c\} = 0$ we see that $\{a, b, c\} \in M$. We define

$$\langle a, b, c\rangle := \overline{\{a, b, c\}} \in M/aM + Mc,$$

where $\overline{\{\dots\}}$ denotes the residue class modulo $aM + Mc$. The class $\langle a, b, c\rangle$ is independent of the choice of sections s and q. It only depends on the class of $(C_\bullet, \partial)$ in $\mathrm{HH}^3(B, M)$ and the elements $a, b, c \in B$. In fact, $\langle a, b, c\rangle$ can be computed from $\mathrm{HH}^3(B, M)$ by

$$\langle a, b, c\rangle = \overline{\Theta_{\mathcal{E}}(a, b, c)},$$

where $\Theta_{\mathcal{E}}$ is the Hochschild 3-cocycle associated to the crossed extension $\mathcal{E}$ in the proof of Theorem 2.8.6.

Finally, if the crossed extension $\mathcal{E}$ comes from a DG algebra $A_\bullet$, i.e. $\gamma_A = [\mathcal{E}]$ in $\mathrm{HH}^3(\mathrm{H}_\bullet(A), \mathrm{H}_{\geq 1}(A)[-1])$, then we recover the classical definition of triple Massey products for homology classes $a, b, c \in \mathrm{H}_\bullet(A)$ of a GD algebra (see [GM03] for details).

Remark 2.9.2. One useful application of characteristic classes of DG algebras is concerned with *realizability of modules* in homology:

Given a DG algebra $A_\bullet$ with homology $\mathrm{H}_\bullet(A)$ and a graded $\mathrm{H}_\bullet(A)$-module $\bar{M}$, we say that $\bar{M}$ is *realizable* if there is a *DG module* M over A such that $\mathrm{H}_\bullet(M) \simeq \bar{M}$. Here, by DG module we mean a graded module $M_\bullet$ over the DG algebra $A_\bullet$ endowed with a differential $d_M : M_\bullet \to M_{\bullet-1}$ satisfying $d_M^2 = 0$ and $d_M(am) = d_A(a)\, m + (-1)^{|a|} a\, d_M(m)$.

It turns out that the characteristic class γ_A of the DG algebra $A_\bullet$ provides a single obstruction to realizability of $\bar{M}$. In particular, if $\gamma_A = 0$, then *any* graded $\mathrm{H}_\bullet(A)$-module is realizable. For details, see [BKS03].

3 Deformation theory

The main reference for the main part of this section is the survey by Bertrand Keller [Kel03].

3.1 Motivation

In classical mechanics, one starts with the phase space, which is a symplectic manifold (e.g. the cotangent bundle T^*X). The ring of smooth functions $C^\infty(M)$ has extra structure: the Poisson bracket $\{-,-\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, and the Hamiltonian $H \in C^\infty(M)$. Locally, the equations of motion (i.e. the Hamilton equations) are, for coordinates $p_i, q_i \in M$:

$$\begin{aligned} \dot{p}_i &= \{H, p_i\} \\ \dot{q}_i &= \{H, q_i\} \end{aligned}$$

where $\dot{f} = \frac{df}{dt}$.

Example 3.1.1. Let $X = \mathbb{R}^n$, $M = T^*X = \mathbb{R}^{2n}$, with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$, where we think of the q_i as space coordinates and the p_i as momentum coordinates. Let $F, G \in C^\infty(\mathbb{R}^n)$. Then the Poisson bracket is

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i} \frac{\partial F}{\partial q_i} \right)$$

Note that $\{p_i, p_j\} = 0$, $\{q_i, q_i\} = 0$, and $\{p_i, q_j\} = \delta_{ij}$. In this context, the Hamilton equations are

$$\begin{aligned} \dot{p}_i &= \frac{\partial H}{\partial q_i} \\ \dot{q}_i &= -\frac{\partial H}{\partial p_i} \end{aligned}$$

One has to choose an H; one common example represents n harmonic oscillators:

$$H = \sum_{i=1}^{n} \left(\frac{1}{2} p_i^2 + \omega_i^2 q_i^2 \right)$$

In quantum mechanics, we replace the Hamiltonian function H with a differential operator. In this context, instead of $C^\infty(M)$ we have a non-commutative algebra and we replace $\{-,-\}$ with the commutator $[-,-]$.

For example, we have the Heisenberg relation $[\hat{p}, q] = i\hbar\delta_{ij}$, where $\hat{p}_i = i\hbar\frac{\partial}{\partial q_i}$, and this relation lives in the ring $C^\infty(\mathbb{R}^n)[\frac{\partial}{\partial q_i}, \ldots, \frac{\partial}{\partial q_n}]$.

(**needs more details on quantum mechanics**)

3.2 Formal deformations

Let k be a commutative ring (later a field of characteristic zero). Let A be a fixed unital associative k-algebra. Write $k_t = k[\![t]\!]$ for the ring of formal power series in t, and let $A_t = A[\![t]\!] = A \otimes_k k[\![t]\!]$. Elements of A_t look like

$$u = \sum_{n \geqslant 0} a_n t^n \qquad , a_n \in A.$$

Let $m : A \otimes A \to A$ be the multiplication map $a \otimes b \mapsto ab$.

Definition 3.2.1. *Let k be a commutative ring, A an associative unital k-algebra. A* formal deformation *(or star product) on A is a continuous k_t-linear map $* : A_t \hat{\otimes}_{k_t} A_t \to A_t$ such that the following diagram commutes:*

$$\begin{array}{ccc} A_t \times A_t & \xrightarrow{\ *\ } & A_t \\ \downarrow{\scriptstyle t \mapsto 0} & & \downarrow{\scriptstyle t \mapsto 0} \\ A \times A & \xrightarrow{\ m\ } & A \end{array}$$

Here, $\hat{\otimes}$ denotes completed tensor product, i.e. $A_t \hat{\otimes}_{k_t} A_t = \varprojlim A[t]/t^n \otimes_{k[t]} A[t]/t^m$. Basically, we are thinking of A_t as a topological k_t-algebra, and taking tensor product in the category of topological k_t-algebras. By continuity, $*$ is determined uniquely by its restriction to $A \times A$. For all $a, b \in A$, we can write $a * b = ab + B_1(a,b)t + B_2(a,b)t^2 + \cdots + B_n(a,b)t^n + \cdots$, where the $B_i : A \otimes A \to A$ are bilinear maps and $B_0 = m$.

Let $G_t = \mathrm{Aut}^\circ_{k_t}(A_t)$ be the group of all k_t-linear automorphisms σ of A_t such that such that $\sigma : A_t \to A_t$ satisfies $\sigma(a) \equiv a \pmod{t}$ for all $a \in A$. In other words, we have $\sigma(a) = a + \sigma_1(a)t + \sigma_2(a)t^2 + \cdots$ where the σ_i are k-linear.

Definition 3.2.2. *Two products $*$ and $*'$ are* equivalent *if there exists $\sigma \in G_t$ such that $\sigma(u * v) = \sigma(u) *' \sigma(v)$ for all $u, v \in A_t$.*

Definition 3.2.3. *A* Poisson bracket *on A is a k-bilinear map $\{-,-\} : A \times A \to A$ such that for al $a, b, c \in A$*

1. $\{a,b\} = -\{b,a\}$ *(skew-symmetry)*

2. $\{\{a,b\},c\} + \{\{b,c\},a\} + \{\{c,a\},b\} = 0$ *(Jacobi identity)*

3. $\{a,bc\} = b\{a,c\} + \{a,b\}c$ *(Leibniz rule)*

Lemma 3.2.4. *Let A be a commutative k-algebra, and let $*$ be an associative (not necessarily commutative) formal deformation of (A,m). Write $* = \sum_{n\geqslant 0} B_n \cdot t^n$. Define $\{-,-\} : A\times A \to A$ by $\{a,b\} = B_1(a,b) - B_1(b,a)$. Then*

1. *$\{-,-\}$ is a Poisson bracket on A*

2. *$\{-,-\}$ depends only on the equivalence class of $*$.*

Proof. We can define $\langle -,-\rangle : A_t \times A_t \to A_t$ by $\langle u,v\rangle = \frac{1}{t}[u,v]_* = \frac{u*v-u*v}{t} \pmod t$. This induces a bracket $\{-,-\}_*$ on A, and one can check that $\{a,b\}_* = B_1(a,b) - b_1(b,a)$ is indeed a Poisson bracket.

Suppose $* \sim *'$ via some σ, i.e. $\sigma(a*b) = \sigma(a) *' \sigma(b)$ for all $a,b\in A$. Then

$$\sigma(a*b) = ab + (\sigma_1(a,b) + B_1(a,b))t + O(t^2)$$
$$\sigma(a) *' \sigma(b) = ab + (a\sigma_1(b) + \sigma_1(a)b + B_1'(a,b)) + O(t^2)$$

The equality $\sigma(a*b) = \sigma(a)*'\sigma(b)$ yields $B_1(a,b) + \sigma_1(a,b) = B_1'(a,b) + a\sigma_1(b) + \sigma_1(a)b$. This implies $B_1(a,b) - B_1(b,a) = B_1'(a,b) - B_1'(b,a)$, i.e. $\{-,-\}_*$ is independent of the equivalence class of $*$. □

Remark 3.2.5. Our definition of $\{-,-\}$ for a noncommutative ring is a bit superfluous. It turns out that any Poisson bracket on any (possibly noncommutative) prime ring A is just $\{-,-\} = \lambda[-,-]$ for some $\lambda \in \mathrm{Frac}\,(Z(A))$. This is a theorem of Farkas-Letzter, see [FL98].

Exercise Show that any associative star product on A_t is unital, and that for any $*$, there exists $*' \sim *$ such that $1_{*'} = 1_A$.

For any associative k-algebra A, define a map

$$\Theta_A : \{\text{star products on } A\}/\sim \longrightarrow \{\text{Poisson brackets on } A\}$$

by $\Theta_A(*) = \{-,-\}_*$. A major question is: "is Θ_A surjective"?

Theorem 3.2.6 (Kontsevich 1997)**.** *Let M be a smooth manifold over $k = \mathbb{R}$. Let $A = C^\infty(M)$ be the ring of smooth functions on M. Then Θ_A is surjective. More precisely, for any M, there exists a canonical (up to equivalence) section ψ_A to Θ_A.*

Example 3.2.7 (Moyal-Weyl)**.** Let $M = \mathbb{R}^2$ with the standard bracket

$$\{F,G\} = \frac{\partial F}{\partial x}\frac{\partial G}{\partial y} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial x}.$$

Then ψ_A is given by

$$F *_t G = \sum_{n=0}^{\infty} \frac{\partial^n F}{\partial x^n} \frac{\partial^n G}{\partial y^n} \frac{t^n}{n!}$$

It is not obvious that $*_t$ is associative.

If $M = U \subset \mathbb{R}^n$ is open and $A = C^\infty(M)$, then any Poisson bracket on A is of the following form:

$$\{F, G\} = \sum_{i<j} \alpha^{ij}(x_1, \ldots, x_n) \left(\frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j} - \frac{\partial G}{\partial x_i} \frac{\partial F}{\partial x_j} \right)$$

It means that there are (unique) smooth functions $\alpha^{ij} \in C^\infty(M)$, $1 \leq i < j \leq n$ such that

$$\{-, -\} = \sum_{i<j} \alpha^{ij} \left(\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \right)$$

The functions (α^{ij}) are actually components of a tensor field of type $(2, 0)$ on M which is called *Poisson bivector*.

3.3 Deformation theory in general

The general idea is "any deformation problem is controlled by a dg Lie algebra". A bit more precisely, suppose we hav a category $\mathcal{A}$ and an object $A \in \operatorname{Ob} \mathcal{A}$. We can define a deformation functor $\operatorname{Def}_{\mathcal{A}}(A, -) : \mathcal{R} \to \mathsf{Set}$, where $\mathcal{R}$ is some category of "test" commutative algebras (or cocommutative coalgebras). For example, $\mathcal{R}$ could be the category of artinian local algebras over a field (which can be of characteristic $p > 0$). The functor $\operatorname{Def}_{\mathcal{A}}(A, -)$ assigns to some R the set of all deformations of A parameterized by $\operatorname{Spec}(R)$, modulo equivalence. We say that a dgla $\mathfrak{L}_A$ controls this deformation problem if there is a natural isomorphism

$$\operatorname{Def}_{\mathcal{A}}(A, R) \simeq \operatorname{MC}(\mathfrak{L}_A, R)$$

where $\operatorname{MC}(\mathfrak{L}_A, R)$ is the set of *Maurer-Cartan* elements in $\mathfrak{L}_A \otimes R$.

Assume from now on that k is a field of characteristic zero. Let $\mathcal{A} = \mathsf{Alg}_k$ be the category of associative unital k-algebras. Let $\mathcal{R} = \mathsf{Art}_k$ be the category of local Artinian k-algebras with residue field k. That is, $R \in \operatorname{Ob} \mathcal{R}$ if and only if R is a local commutative k-algebra with finite-dimensional maximal ideal $\mathfrak{m} \subset R$, such that $R/\mathfrak{m} = k$. This clearly implies $\mathfrak{m}^n = 0$ for all $n \gg 0$. A good example is $R = k[t]/(t^n)$.

Given $R \in \mathcal{R}$, write A_R for $A \otimes_k R$.

Definition 3.3.1. *An R-*deformation *of A is an associative R-linear map $* : A_R \otimes_R A_R \to A_R$ such that the following diagram commutes:*

$$\begin{array}{ccc} A_R \otimes_R A_R & \xrightarrow{\;*\;} & A_R \\ \big\downarrow{\scriptstyle \widetilde{\pi}_R \otimes \widetilde{\pi}_R} & & \big\downarrow{\scriptstyle \widetilde{\pi}_R} \\ A \otimes A & \xrightarrow{\;m\;} & A \end{array}$$

where $\widetilde{\pi}_R = 1 \otimes \pi_R$ *and* $\pi_R : R \to R/\mathfrak{m}$ *is the canonical projection.*

We say that two R-deformations $*$ and $*'$ are equivalence if there is an R-module isomorphism $g : A_R \to A_R$ such that

$$\begin{array}{ccc} A_R & \xrightarrow{g} & A_R \\ \downarrow{\scriptstyle \pi_R} & & \downarrow \\ A & = & A \end{array}$$

commutes, and such that $g(u * v) = g(u) *' g(v)$ for all $u, v \in A_R$. Note that by R-linearity, $*$ is determined by its restriction to $A \otimes A$. Moreover, $*$ is determined by $\widetilde{*} : A \otimes A \to A \otimes \mathfrak{m}$ because $* = m + \widetilde{*}$.

Definition 3.3.2. *The deformation functor* $\mathrm{Def}(A, -) : \mathcal{R} \to \mathsf{Set}$ *is given by*

$$\mathrm{Def}(A, R) = \{R\text{-}deformations\ of\ A\}/equivalence.$$

Definition 3.3.3. *If* $R = k[t]/t$*, then* R*-deformations of* A *are called* infinitesimal.

Lemma 3.3.4. *There is a natural bijection* $\mathrm{Def}(A, k[t]/t^2) = \mathrm{HH}^2(A, A)$*.*

Proof. By definition, for $R = k[t]/t^2$, an R-deformation is determined by

$$* : A \otimes A \to A[t]/t^2 = A \otimes k[t]/t^2$$

which will be of the form $a \otimes b \mapsto ab + B_1(a, b)t$, where $B_1 : A \otimes A \to A$ is some k-linear map. The associativity of $*$ is equivalent to B_1 being a Hochshield 2-cocycle. Indeed,

$$\begin{aligned} (a * b) * c &= (ab)c + (B_1(a, b)c + B_1(ab, c))t \\ a * (b * c) &= a(bc) + (aB_1(b, c) + B_1(a, bc))t \end{aligned}$$

Since A is associative, the two are equal exactly when

$$(d^2_{\mathrm{Hoch}} B_1)(a, b, c) = aB_1(b, c) - B_1(ab, c) + B_1(a, bc) - B_1(a, b)c = 0$$

Moreover, if $*, *'$ are equivalent, then there is $g : A \to A[t]/t^2$ such that $g(a * b) = g(a) *' g(b)$, which is equivalent to

$$B_1'(a, b) - B_1(a, b) = ag_1(b) - g(ab) + g(a)b$$

Thus $* \sim *'$ if and only if $B_1' - B_1 = dg_1$. □

Remark 3.3.5. One might hope that $\mathrm{Def}_{\mathsf{Alg}_k}(A, k[t]/t^n) \simeq \mathrm{HH}^n(A, A)$. Unfortunately, this is not true in general if $n \geqslant 3$. However, it is true that $\mathrm{Def}_{\mathsf{Alg}_\infty(k)}(A, k[t]/t^n) = \mathrm{HH}^n(A, A)$, where $|t| = 2 - n$, and $\mathsf{Alg}_\infty(k)$ is the category of A_∞-algebras (also called strongly homotopy-associative algebras) over k. Note that A_∞-algebras are *not* in general associative.

Definition 3.3.6. *Let k be a field. A* differential graded Lie algebra *over k is a $\mathbb{Z}$-graded vector space $\mathfrak{L}^\bullet = \bigoplus_{n\in\mathbb{Z}} \mathfrak{L}^n$ with a bracket $[-,-] : \mathfrak{L}\otimes\mathfrak{L} \to \mathfrak{L}$ that is homogeneous of degree zero (that is, $[\mathfrak{L}^p, \mathfrak{L}^q] \subset \mathfrak{L}^{p+q}$ for all p, q) such that*

1. $[x, y] = -(-1)^{|x|\cdot|y|}[y, x]$ *for all homogeneous x, y*
2. $[x, [y, z]] = [[x, y], z] + (-1)^{|x|\cdot|y|}[y, [x, z]]$ *(the Jacobi identity)*

Moreover, there is a differential $d : \mathfrak{L}^\bullet \to \mathfrak{L}^{\bullet+1}$ such that $d^2 = 0$ and $d[x, y] = [dx, y] + (-1)^{|x|}[x, dy]$.

If $\mathfrak{L}$ is a dgla, then $\mathfrak{L}^0$ is an honest Lie algebra. There is a canonical representation of $\mathfrak{L}^0$ (the adjoint representation) $\mathrm{ad} : \mathfrak{L}^0 \to \mathrm{End}(\mathfrak{L}^\bullet)$ given by $\mathrm{ad}(x)y = [x, y]$.

Definition 3.3.7. *Let $\mathfrak{L}$ be a differential graded Lie algebra. The space of* Maurer-Cartan *elements in $\mathfrak{L}$ is*

$$\mathrm{MC}(\mathfrak{L}) = \left\{ x \in \mathfrak{L}^1 : dx + \frac{1}{2}[x, x] = 0 \right\}$$

If $\mathfrak{L}^1$ is finite-dimensional, then $\mathrm{MC}(\mathfrak{L})$ is actually a variety (in fact an intersection of quadrics in $\mathfrak{L}^1$). For $x \in \mathrm{MC}(\mathfrak{L})$, set

$$T_x \,\mathrm{MC}(\mathfrak{L}) = \left\{ v \in \mathfrak{L}' : dv + [x, v] = 0 \right\}.$$

This is precisely the Zariski tangent space to $\mathrm{MC}(\mathfrak{L})$. The action of $\mathfrak{L}^0$ on $\mathfrak{L}^1$ fixes $\mathrm{MC}(\mathfrak{L})$. If $\mathfrak{L}^0$ is a nilpotent Lie algebra, we can define the reduced Maurer-Cartan space to be

$$\overline{\mathrm{MC}}(\mathfrak{L}) = \mathrm{MC}(\mathfrak{L}) / \exp(\mathfrak{L}^0).$$

We will see that there exists a dgla $\mathfrak{L}_A$ such that $\mathrm{Def}(A, R) \simeq \overline{\mathrm{MC}}(\mathfrak{L}_A \otimes \mathfrak{m}_R)$.

We continue to assume that k is a field of characteristic zero. Let $\mathfrak{L} = \mathfrak{L}^\bullet$ be a dgla over k. The Maurer-Cartan space

$$\mathrm{MC}(\mathfrak{L}) = \left\{ x \in \mathfrak{L}^1 : dx + \frac{1}{2}[x, x] = 0 \right\}$$

can be regarded as a subscheme of $\mathfrak{L}^1$, where $\mathfrak{L}^1$ is viewed as affine space over k. We would like to compute the Zariski tangent space of $\mathrm{MC}(\mathfrak{L})$. Recall that if X is an affine variety (or scheme) over an algebraically closed field k, and $x \in X$ is a closed point, then there is (by the Nulstellensatz) a k-algebra homomorphism $\varphi : \mathscr{O}(X) \to k$ with $\mathrm{Ker}(\varphi) = \mathfrak{m}_x$.

Definition 3.3.8. *The* Zariski tangent space *of X at x is*

$$T_x X = \mathrm{Der}(\mathscr{O}(X), x) = \{\delta : \mathscr{O}(X) \to k : \delta(fg) = f\delta g + g\delta f\}$$

There is a canonical identification of $T_x X$ with $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$. For more details, see any good book on algebraic geometry.

Example 3.3.9. If $X = V = \operatorname{Spec}(\operatorname{Sym}(V^*))$ is a finite dimensional vector space over k, then $\mathscr{O}(X) = k[V^*]$, and there is a canonical isomorphism $V \to T_x V$ given by

$$v \mapsto \partial_{v,x} : f \mapsto \frac{d}{dt} f(x+vt)\bigg|_{t=0}$$

Let $f : X \to Y$ be a morphism of affine schemes. (Since X and Y are affine, it is equivalent to give a homomorphism $f^* : \mathscr{O}(Y) \to \mathscr{O}(X)$.)

Definition 3.3.10. *For $y \in Y$ define the* scheme-theoretic fiber *of f at y by*

$$f^{-1}(y) = \operatorname{Spec}\left(\mathscr{O}(X)/f^*(\mathfrak{m}_y)\mathscr{O}(X)\right).$$

The differential *of f at $x \in X$ is $df_x : T_x X \to T_{f(x)} Y$ given by $\delta \mapsto \delta \circ f^*$.*

One can check that there is a natural isomorphism $T_x f^{-1}(y) = \operatorname{Ker}(df_x)$.

Example 3.3.11. Let $X = V$, $Y = W$ where V, W are finite dimensional vector spaces over k viewed as affine schemes. Let $f : V \to W$ be a morphism. For $x \in V$, $y = f(x) \in W$, one can verify that there is an exact sequence

$$0 \longrightarrow T_x f^{-1}(y) \longrightarrow T_x V \xrightarrow{df_x} T_y W$$

Under the identifications $T_x V = V$, $T_y W = W$, the kernel $T_x f^{-1}(y)$ is

$$\left\{ v \in V : \frac{d}{dt}\bigg|_{t=0} f(x+vt) = 0 \right\}$$

Now we can redefine the Maurer-Cartan space $\operatorname{MC}(\mathfrak{L})$, viewing it as a scheme-theoretic fiber. From now on assume that $\mathfrak{L}^\bullet$ is locally finite-dimensional, i.e. $\dim_k \mathfrak{L}^p < \infty$ for all p. Define $f : \mathfrak{L}^1 \to \mathfrak{L}^2$ by $x \mapsto dx + \frac{1}{2}[x,x]$. This is a morphism, and we can redefine the Maurer-Cartan scheme to be the scheme-theoretic fiber $f^{-1}(0)$. For $x \in \operatorname{MC}(\mathfrak{L})$, we have

$$T_x \operatorname{MC}(\mathfrak{L}) = \left\{ v \in \mathfrak{L}^1 : \frac{d}{dt} f(x+vt)\bigg|_{t=0} = 0 \right\}$$

We can compute

$$\begin{aligned} (df_x)t &= \frac{d}{dt}\left(d(x+vt) + \frac{1}{2}[x+vt, x+vt] \right)\bigg|_{t=0} \\ &= dv + [x,v]. \end{aligned}$$

This allows us to make the following definition even if $\mathfrak{L}^1$ is not locally finite-dimensional.

Definition 3.3.12. *The (Zariski) tangent space to* $\operatorname{MC}(\mathfrak{L})$ *at x is*

$$T_x \operatorname{MC}(\mathfrak{L}) = \{ v \in \mathfrak{L}^1 : d_x v = 0 \}.$$

where $d_x = d + \operatorname{ad}_x$.

Lemma 3.3.13. *For all $x \in \mathrm{MC}(\mathfrak{L})$,*

1. $d_x^2 = 0$
2. *The assignment $\mathfrak{L}^0 \ni \xi \mapsto (x \mapsto d_x\xi) \in \mathrm{Der}(\mathrm{MC}(\mathfrak{L}))$ defines a Lie algebra homomorphism. In particular, $d_x\xi \in T_x \mathrm{MC}(\mathfrak{L})$.*

Proof. Part 2 follows from 1 trivially. To see that part 1 is true, we assume $dx + \frac{1}{2}[x, x] = 0$, and compute

$$\begin{aligned} d_x^2\xi &= (d + \mathrm{ad}_x)(d + \mathrm{ad}_x)\xi \\ &= (d + \mathrm{ad}_x)(d\xi + [x, \xi]) \\ &= d^x\xi + d[x, \xi] + [x, d\xi] + [x, [x, \xi]] \\ &= [dx, \xi] - [x, d\xi] + [x, d\xi] + [x, [x, \xi]] \\ &= -\frac{1}{2}[[x, x], \xi] + [x, [x, \xi]] \\ &= 0 \end{aligned}$$

The last equality follows from Jacobi identity:

$$\begin{aligned} [x, [x, \xi]] &= [[x, x], \xi] - [x, [x, \xi]] \\ &= \frac{1}{2}[[x, x], \xi] \end{aligned}$$

□

From here on out, assume that $\mathfrak{L}^0$ is a nilpotent Lie algebra, i.e. for all $\xi \in \mathfrak{L}^0$, the endomorphism $\mathrm{ad}_\xi \in \mathrm{End}(\mathfrak{L}^0)$ is nilpotent. Moreover, we assume that the adjoint action of $\mathfrak{L}^0$ on $\mathfrak{L}^1$ is nilpotent. Consider the group $\mathrm{Aff}(\mathfrak{L}^1)$ of affine linear transformations of $\mathfrak{L}^1$. This is just the semidirect product $\mathrm{GL}(\mathfrak{L}^1) \ltimes \mathfrak{L}^1$. Let $\mathfrak{aff}(\mathfrak{L}^1)$ be the Lie algebra of $\mathrm{Aff}(\mathfrak{L}^1)$. We have $\mathfrak{aff}(\mathfrak{L}^1) = \mathfrak{gl}(\mathfrak{L}^1) \ltimes \mathfrak{L}^1$, and there is an anti-homomorphism $\mathfrak{L}^0 \to \mathfrak{aff}(\mathfrak{L}^1)$ given by $\xi \mapsto d_x\xi = d\xi + [x, \xi]$. Exponentiation gives an anti-homomorphism $\exp(\mathfrak{L}^0) \to \mathrm{Aff}(\mathfrak{L}^1)$, which yields a right action of $\exp(\mathfrak{L}^0)$ on $\mathfrak{L}^1$. This action restricts to an action of $\exp(\mathfrak{L}^0)$ on $\mathrm{MC}(\mathfrak{L})$.

Definition 3.3.14. $\overline{\mathrm{MC}}(\mathfrak{L}) = \mathrm{MC}(\mathfrak{L})/\exp(\mathfrak{L}^0)$

Remark 3.3.15. For $x \in \mathrm{MC}(\mathfrak{L})$, we can consider the orbit $O_x = x \cdot \exp(\mathfrak{L}^0) \subset \mathrm{MC}(\mathfrak{L})$. It turns out that $T_x \mathrm{MC}(\mathfrak{L})/T_xO_x = \mathrm{Ker}(d_x)/\mathrm{Im}(d_x) = \mathrm{H}^1(\mathfrak{L}, d_x)$.

Let $\mathcal{R} = \mathsf{Art}$ be the category of local commutative k-algebras with finite-dimensional (hence nilpotent) maximal ideal and residue field k. For any dgla $\mathfrak{L}$ and for any $R \in \mathcal{R}$ with maximal ideal $\mathfrak{m}$, the Lie algebra $\mathfrak{L} \otimes \mathfrak{m}$ is nilpotent.

Definition 3.3.16. $\mathrm{MC}(\mathfrak{L}, R) = \mathrm{MC}(\mathfrak{L} \otimes \mathfrak{m}_R)$ *and* $\overline{\mathrm{MC}}(\mathfrak{L}, R) = \overline{\mathrm{MC}}(\mathfrak{L} \otimes \mathfrak{m}_R)$.

One main theorem of this section is the following.

Theorem 3.3.17. *1. Let $f : \mathfrak{L}_1 \to \mathfrak{L}_2$ be a quasi-isomorphism of dg Lie algebras. That is, f is a morphism of graded Lie algebras, $f \circ d_1 = d_2 \circ f$, and $f_* : \mathrm{H}^\bullet(\mathfrak{L}_1) \to \mathrm{H}^\bullet(\mathfrak{L}_2)$ is an isomorphism. For any $R \in \mathcal{R}$, f induces a natural bijection.*

$$\overline{\mathrm{MC}}(\mathfrak{L}_1, R) \xrightarrow{\sim} \overline{\mathrm{MC}}(\mathfrak{L}_2, R).$$

2. For any $x \in \mathrm{MC}(\mathfrak{L}_1, R)$ f also induces a quasi-isomorphism

$$(\mathfrak{L}_1 \otimes \mathfrak{m}, d_x) \xrightarrow{\sim} (\mathfrak{L}_2 \otimes \mathfrak{m}, d).$$

Any homomorphism $f : \mathfrak{L}_1 \to \mathfrak{L}_2$ gives a natural map $f_* : \mathrm{MC}(\mathfrak{L}_1) \to \mathrm{MC}(\mathfrak{L}_2)$. The nontrivial part of the theorem is that f_* induces a bijection on reduced Maurer-Cartan spaces.

3.4 The Gerstenhaber bracket

Let A be an associative k-algebra, where k is a field of characteristic zero. Recall that the Hochschild complex of A is $C^\bullet(A, A) = \bigoplus_{p\in\mathbb{Z}} C^p(A, A)$, where $C^p(A, A) = 0$ for $p < 0$, $C^p(A, A) = \mathrm{Hom}_k(A^{\otimes p}, A)$ for $p \geqslant 0$, and $d : C^p \to C^{p+1}$ is

$$(-1)^p(df)(a_0, \ldots, a_p) = a_0 f(a_1, \ldots, a_p) + \sum_{i=0}^{p-1}(-1)^i f(a_0, \ldots, a_i a_{i+1}, \ldots, a_p) + (-1)^{p-1} f(a_0, \ldots, a_{p-1}) \cdot a_p$$

Definition 3.4.1. *Define an insertion operation $\bullet : C^p \times C^q \to C^{p+q-1}$ by*

$$(f \bullet g)(a_1, \ldots, a_{p+q-1}) = \sum_{i=0}^{p}(-1)^{i(q-1)} f(a_1, \ldots, a_i, g(a_{i+1}, \ldots, a_{i+1}), a_{i+q+1}, \ldots, a_{p+q-1}).$$

Notice that the operation $\bullet$ is not associative in general. We can use $\bullet$ to define the following bracket.

Definition 3.4.2. *The G-bracket (Gerstenhaber bracket) $[-,-]_G : C^p \times C^q \to C^{p+q-1}$ is defined by*

$$[f, g]_G = f \bullet g - (-1)^{|f|\cdot|g|} g \bullet f.$$

Let $\mathfrak{L}^\bullet_{\mathrm{AS}}(A) = C^\bullet(A, A)[1]$. That is, $\mathfrak{L}^p_{\mathrm{AS}} = C^{p+1}(A, A)$. The Gerstenhaber bracket induces a bracket $[-,-]_G : \mathfrak{L}^p_{\mathrm{AS}} \times \mathfrak{L}^q_{\mathrm{AS}} \to \mathfrak{L}^{p+q}_{\mathrm{AS}}$ of degree zero.

Lemma 3.4.3 (Gerstenhaber)**.** *$(\mathfrak{L}^\bullet_{\mathrm{AS}}(A), [-,-]_G, d)$ is a dg Lie algebra.*

Proof. This can be checked directly, though checking the (super) Jacobi directly in this way is quite tedious. However, it follows from the following.

Since $\bullet$ is not associative we define the "associator"

$$A(f, g, h) = (f \bullet g) \bullet h - f \bullet (g \bullet h)$$

One can check that A is (super)symmetric in g and h, i.e. $A(f, g, h) = (-1)^{(|g|-1)(|h|-1)} A(f, h, g)$. This symmetry formally implies the Jacobi identity. ∎

Remark 3.4.4. The Lie algebra $(\mathfrak{L}^\bullet_{\mathrm{AS}}(A), [-,-]_G)$ only depends on A as a vector space. The multiplication on A enters in this picture as follows. We have $m \in C^2(A,A) = \mathfrak{L}^1_{\mathrm{AS}}(A)$, and it is easy to check that for all $f \in C^p(A,A)$, we have $[m,f] = -df$, where d is the Hochschild differential. We can rewrite this as $d = -\operatorname{ad}_m$.

We define the cup-product on $C^\bullet(A,A)$ by

$$(f \smile g)(a_1, \ldots, a_{p+q}) = f(a_1, \ldots, a_p)g(a_{p+1}, \ldots, a_{p+q})$$

where $f \in C^p, g \in C^q$. It is easy to check that $d(f \smile g) = df \smile g + (-1)^{|f|} f \smile dg$.

By the lemma 3.4.3, $(\mathrm{HH}^\bullet(A,A)[1], [-,-]_G)$ is a graded Lie algebra. In fact, $(\mathrm{HH}^\bullet(A,A), [-,-]_H, \smile)$ is a (graded commutative) *Gerstenhaber algebra.*

Definition 3.4.5. *A* Gerstenhaber algebra G *is a graded commutative algebra with product* $\cdot$ *and bracket* $\{-,-\} : G^p \times G^q \to G^{p+q-1}$ *such that*

1. $(G^\bullet[1], \{-,-\})$ *is a graded Lie algebra*
2. $\{a \cdot b, c\} = a \cdot \{b,c\} + (-1)^{(|a|-1)|b|}\{a,b\} \cdot c$

Example 3.4.6 (Gestenhaber algebra from differential geometry)**.** Let M be a smooth manifold over $\mathbb{R}$, and let TM be the tangent bundle of M. Set $\Theta_p(M) = \Gamma(M, \bigwedge^p TM)$. In coordinates, an element of $\Theta_p(M)$ looks like $\sum \xi_1 \wedge \cdots \wedge \xi_p$. We can set $\Theta_\bullet(M) = \bigoplus_{p \geqslant 0} \Theta_p(M)$. This is a graded commutative algebra with respect to the exterior product $\wedge$. The algebra $\Theta_p(M)$ has a *Schouten bracket* $\{-,-\}_S : \Theta_p \times \Theta_q \to \Theta_{p+q-1}$, defined by

$$\{\xi_1 \wedge \cdots \wedge \xi_p, \eta_1, \cdots \wedge \eta_q\} = \sum_{i=1}^{p} \sum_{j=1}^{q} (-1)^{i+j} [\xi_i, \eta_j] \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \eta_1 \wedge \cdots \wedge \widehat{\eta_j} \wedge \cdots \wedge \eta_q$$

This algebra $(\Theta_\bullet(M), \wedge, \{-,-\})$ is a Gerstenhaber algebra.

Theorem 3.4.7 (Gerstenhaber)**.** $(\mathrm{HH}^\bullet(A,A), [-,-]_G, \smile)$ *is a commutative Gerstenhaber algebra.*

3.5 Stasheff construction

The following is a construction by Jim Stasheff, 1993. Recall that an associative algebra is a vector space A with a map $m : A \otimes A \to A$ such that the following diagram commutes:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes 1} & A \otimes A \\ \downarrow{\scriptstyle 1 \otimes m} & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{m} & A \end{array}$$

Thus it is very natural to define a dual objects to algebras to be coalgebras, defined as follows.

Definition 3.5.1. *A (coassociative) coalgebra over k is a vector space C with a coproduct $\Delta : C \to C \otimes C$ such that the following diagram commutes:*

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \times C \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle 1\otimes\Delta} \\ C \otimes C & \xrightarrow{\Delta\otimes 1} & C \otimes C \otimes C \end{array}$$

Recall that $d : A \to A$ is a derivation if $d \circ m = m \circ (1 \otimes d + d \otimes 1)$. This makes it natural to make the following definition.

Definition 3.5.2. *A linear map $D : C \to C$ is called a* coderivation *if it satisfies the "coLeibniz" rule, i.e.*

$$\Delta \circ D = (1 \otimes D + D \otimes 1) \circ \Delta.$$

Starting with an algebra A, one can define a coalgebra $C = T^c(A[1])$, which as a k-vector space is $\bigoplus_{n\geqslant 1} A[1]^{\otimes n}$. There is a natural coproduct on C, given by

$$\Delta(x_1, \ldots, x_n) = \sum_{i=1}^{n-1} (x_1, \ldots, x_i) \otimes (x_{i+1}, \ldots, x_n)$$

where we write $x_1, \ldots, x_n$ for $x_1 \otimes \cdots \otimes x_n$. It is essentially trivial that (C, Δ) is a coassociative coalgebra. Note that for an algebra A the space $\mathrm{Der}(A)$ has a natural bracket $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$. Let $\mathrm{Coder}(C)$ be the space of all graded coderivations of C. This is a graded Lie algebra with bracket $[D_1, D_2] = D_1 \circ D_2 - (-1)^{|D_1|\cdot|D_2|} D_2 \circ D_1$.

For a vector space V, if $A = TV$ with embedding $i : V \hookrightarrow TV$, then the natural map $i^* : \mathrm{Der}(TV) \to \mathrm{Hom}(V, TV)$ is an isomorphism. Similarly, let $p : T^c(A[1]) \twoheadrightarrow A[1]$ be the canonical projection. Then the natural map

$$p_* : \mathrm{Coder}\,(T^c(A[1])) \to \mathrm{Hom}\,(T^c A[1], A[1]) \simeq C^\bullet(A, A)[1]$$

is an isomorphism. One can check that the induced bracket on $C^\bullet(A, A)[1]$ is nothing but the Gerstenhaber bracket.

The coalgebra $(T^c A[1], \Delta)$ is called the *bar construction* of A. A number of complicated constructions on algebras actually come from simple constructions on the bar construction.

Theorem 3.5.3. *The dg Lie algebra $\mathfrak{L}_{\mathrm{AS}}(A) = C^\bullet(A, A)[1]$ controls the deformations of A. Precisely, there is a* functorial bijection *for all $R \in \mathcal{R}$:*

$$\mathrm{Def}(A, R) \simeq \overline{\mathrm{MC}}(\mathfrak{L}_{\mathrm{AS}}(A) \otimes \mathfrak{m}_R).$$

Proof. Given $R \in \mathcal{R}$, then the corresponding star-product $* : A_R \otimes_R A_R \to A_R$ is determined by R-linearity to its restriction $B : A \otimes R \to A \otimes \mathfrak{m}$, i.e. $a * b = ab + B(a, b)$. One has $B \in C^2(A, A) \otimes \mathfrak{m} = \mathfrak{L}^1_{\mathrm{AS}}(A) \otimes \mathfrak{m}$. A key point is that $*$ is associative if and only if B satisfies the Maurer-Cartan equation, i.e.

$$dB + \frac{1}{2}[B, B]_G = 0$$

Moreover, $* \sim *'$ if and only if B and B' are in the same orbit in $\overline{\mathrm{MC}}(\mathfrak{L}_{\mathrm{AS}}(A) \otimes \mathfrak{m})$. □

3.6 Kontsevich Formality Theorem

Recall that for an algebra A, $(\mathfrak{L}_{\mathrm{AS}}(A) = (C^{\bullet}(A,A)[1], [-,-]_G)$ is a dg Lie algebra that "controls" deformations of A in the sense that

$$\overline{\mathrm{MC}}(\mathfrak{L}_{AS}(A), R)) \simeq \mathrm{Def}(A, R)$$

for all $R \in \mathcal{R}$.

Definition 3.6.1. *Two dg Lie algebras $\mathfrak{L}$ and $\mathfrak{L}'$ are called* homotopy equivalent *if there is a sequence of dg Lie algebras $\mathfrak{L}_1, \ldots, \mathfrak{L}_n$ with quasi-isomorphisms*

$$\mathfrak{L} = \mathfrak{L}_0 \to \mathfrak{L}_1 \leftarrow \cdots \to \mathfrak{L}_n = \mathfrak{L}'$$

(the arrows can be in either *direction).*

Lemma 3.6.2 (Goldman-Milson)**.** *Let $\mathfrak{L}$ and $\mathfrak{L}'$ be dg Lie algebras. The following are equivalent:*

1. *$\mathfrak{L}$ and $\mathfrak{L}'$ are homotopy equivalent*
2. *There exists $\mathfrak{L}_0$ such that that there are quasi-isomorphisms*

$$\begin{array}{ccc} & \mathfrak{L}_0 & \\ \swarrow & & \searrow \\ \mathfrak{L} & & \mathfrak{L}' \end{array}$$

3. *There is an $\mathfrak{L}_\infty$ homomorphism $\mathfrak{L} \to \mathfrak{L}'$*

We think of $\mathfrak{L}_0$ from the Goldman-Milson lemma as being a "generalized morphism" from $\mathfrak{L}$ to $\mathfrak{L}'$.

Let $A = C^\infty(M)$ be the ring of functions on a smooth manifold M. Recall that in Lemma 3.2.4 we have constructed a natural map Θ_A from formal deformations of A to Poisson brackets on A. Kontsevich proved that Θ_A is surjective. There is a dg Lie algebra $\mathfrak{L}_{\mathrm{Pois}}(A)$ that controls "Poisson deformations" of A with the trivial bracket. Kontsevich's theorem can be interpreted as saying that $\mathfrak{L}_{\mathrm{AS}}(A)$ and $\mathfrak{L}_{\mathrm{Pois}}(A)$ homotopy equivalent (i.e. there is an $\mathfrak{L}_\infty$-morphism $\mathfrak{L}_{\mathrm{AS}}(A) \to \mathfrak{L}_{\mathrm{Pois}}(A)$. The existence of this homotopy is known as the formality theorem. To be more precise, we need the following definitions.

Definition 3.6.3 (Chevally-Eilenberg (co)homology)**.** *Let $\mathfrak{g}$ be a Lie algebra with bracket $[-,-] : \bigwedge^2 \mathfrak{g} \to \mathfrak{g}$. Let V be a $\mathfrak{g}$ module (i.e. there is a homomorphism of Lie algebras $\mathfrak{g} \to \mathrm{End}\, V$). The* Chevally-Eilenberg complex *$C^\bullet(\mathfrak{g}, V)$ has $C^p(\mathfrak{g}, V) = 0$ if $p < 0$, and*

$$C^p(\mathfrak{g}, V) = \mathrm{Hom}_k \left(\textstyle\bigwedge^p \mathfrak{g}, V\right)$$

with differential

$$(-1)^p df(X_0,\dots,X_{p+1}) = \sum_{i<j}(-1)^{i+j+1} f\left([x_i,x_j],X_1,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_{p+1}\right) + \sum_i (-1)^i X_i\cdot f(X_1,\dots,\widehat{X_i},\dots,X_{p+1})$$

This definition is motivated by the following theorem:

Theorem 3.6.4 (H. Cartan)**.** *Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Then $\mathrm{H}^\bullet(G,\mathbb{R})\simeq \mathrm{H}^\bullet(\mathfrak{g},\mathbb{R})$, where the action of $\mathfrak{g}$ on $\mathbb{R}$ is trivial.*

As before, let M be a smooth manifold and $A = C^\infty(M)$. Let $\{-,-\}$ be a Poisson bracket on A, and denote by $\mathfrak{g}$ the Lie algebra $(A,\{-,-\})$. Consider $\mathfrak{g}$ as a $\mathfrak{g}$-module via the adjoint action $\mathfrak{g}\to \operatorname{End}\mathfrak{g}$, where $\xi\mapsto \mathrm{ad}_\xi : x\mapsto [\xi,x]$. Let $\mathfrak{L} = C^\bullet(\mathfrak{g},\mathfrak{g})[1]$. There is an analogue of the Gerstenhaber bracket on $\mathfrak{L}$. We set

$$f\bullet g = \sum_{s\in S_q} \operatorname{sgn}(\sigma) f(g(X_{\sigma(1)},\dots,X_{\sigma(q)}),X_{q+1},\dots,X_{p+q+1})$$

Define $[f,g]_{\mathrm{C}} = f\bullet g = (-1)^{(|f|-1)(|g|-1)} g\bullet f$. Then $(\mathfrak{L}_{\mathrm{CE}},[-,-]_{\mathrm{CE}})$ is a dg Lie algebra.

Theorem 3.6.5 (Kontsevich's formality theorem)**.** *For a smooth manifold M and $A = C^\infty(M)$, $\mathfrak{g}_A = (A,\{-,-\}=0)$, there is a homotopy equivalence*

$$\mathfrak{L}_{AS}(A)\sim \mathfrak{L}_{CE}(\mathfrak{g}_A)$$

It's very interesting to study the space of $\mathfrak{L}_\infty$ equivalences between $\mathfrak{L}_{\mathrm{AS}}(A)$ and $\mathfrak{L}_{\mathrm{CE}}(\mathfrak{g}_A)$. This group admits a faithful action of the Grothendieck-Teichmüller group $\widehat{GT}$. The group $\widehat{GT}$ contains $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ as a subgroup, and Grothendieck conjecture that $\widehat{GT} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. There is a conjecture that the action of $\widehat{GT}$ on $\mathfrak{L}_\infty$-equivalences between $\mathfrak{L}_{\mathrm{AS}}(A)$ and $\mathfrak{L}_{\mathrm{CE}}(\mathfrak{g}_A)$ is simply transitive!

3.7 Deformation theory in algebraic number theory

Another source of motivation for deformation theory comes from algebraic number theory. Let k be a field of characteristic not 2 or 3. Recall that an *elliptic curve* over a field k is the subset of $\mathbb{P}^2_k$ given by a homogeneous equation

$$y^2 z = x^3 + axz^2 + bz^3$$

where $a,b\in k$ are such that $\Delta = -16(4a^3+27b^2)\neq 0$. Let E be an elliptic curve. One can show using the Riemann-Roch theorem that E naturally has the structure of an *abelian variety* (projective group variety) with unit $(0:1:0)$ in projective coordinates. For each integer $N\geqslant 5$, there is a smooth projective curve $X_0(N)$ over $\mathbb{Q}$ that parameterizes "elliptic schemes with level N structure." Complex-analytically, $X_0(N)$ is a compactification of $\mathfrak{h}/\Gamma_0(N)$, where $\mathfrak{h} = \{z \in \mathbb{C} : \Re(z) > 0\}$ is the upper half plane and

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) : d \equiv 0 \pmod{N} \right\}$$

One says that an elliptic curve $E/\mathbb{Q}$ is *modular* if there non-constant rational map $X_0(N) \to E$ for some N. Equivalently, E is modular if there is a surjection $J_0(N) \twoheadrightarrow E$, where $J_0(N)$ is the Jacobian of $X_0(N)$. The *Taniyama-Shimura conjecture* claims that all elliptic curves over $\mathbb{Q}$ are modular. One can show that Fermat's Last Theorem follows from the Taniyama-Shimura conjecture. Andrew Wiles proved the Taniyama-Shimura conjecture by creating a "mod-p Galois representation" and studying its deformations! More precisely, if E is an elliptic curve with discriminant Δ and $p \nmid \Delta$, the *p-adic Tate module* of E is the $\mathbb{Z}_p$-module

$$T_p E = \varprojlim E[p^n](\bar{\mathbb{Q}}) \simeq \mathrm{H}^1_{\text{ét}}(E_{\bar{\mathbb{Q}}}, \mathbb{Z}_p)^*$$

As a $\mathbb{Z}_p$-module, $T_p \simeq \mathbb{Z}_p^{\oplus 2}$, so the action of $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on T_p gives us a representation

$$\rho_{E,p} : G_{\mathbb{Q}} \to \mathrm{GL}(2, \mathbb{Z}_p)$$

On the other hand, each modular form f of level N gives rise to an ideal $I_f \subset \mathrm{End}(J_0(N))$, and the quotient $J_0(N)/I_f$ is an elliptic curve. We denote the representation $\rho_{J_0(N)/I_f,p}$ by $\rho_{f,p}$. It is a theorem that E is modular if and only if $\rho_{E,p}$ is isomorphic to $\rho_{f,p}$ for some modular form f.

For a special class of elliptic curves, it was already known that the "mod p representation"

$$\bar{\rho}_{E,p} : G_{\mathbb{Q}} \to \mathrm{GL}(2, \mathbb{Z}_p) \twoheadrightarrow \mathrm{GL}(2, \mathbb{F}_p)$$

was modular. What Andrew Wiles did is consider the category $\mathcal{R}$ whose objects are finite local $\mathbb{Z}_p$-algebras with residue field $\mathbb{F}_p$, and define the functor $D_{\bar{\rho}_{E,p}} : \mathcal{R} \to \mathsf{Set}$ by

$$D_{E,p}(R) = \{\rho : G_{\mathbb{Q}} \to \mathrm{GL}(2, R) : \rho \equiv \bar{\rho}_{E,p} \pmod{\mathfrak{m}_R}\} / \sim$$

Here ρ and ρ' are equivalent if they are conjugate by an element of $\mathrm{Ker}(\mathrm{GL}(2, R) \to \mathrm{GL}(2, \mathbb{F}_p))$. It is a theorem that $D_{E,p}$ is representable in the sense that there exists a profinite $\mathbb{Z}_p$-algebra $R_{E,p}$ such that

$$D_{E,p}(R) \simeq \mathrm{Hom}_{\mathbb{Z}_p\text{-}\mathsf{TopAlg}}(R_{E,p}, R)$$

functorially in R. Wiles then considered a deformation functor D_E classifying a special class of lifts of $\bar{\rho}_{E,p}$. This functor is also (pro-) representable, with representing algebra R_E. There is another deformation functor classifying modular lifts of $\bar{\rho}_{E,p}$. This functor is also representable by a ring $T_{\bar{\rho}_{E,p}}$. It was known that $\bar{\rho}_{E,p}$ was modular for a specific p; this gave a homomorphism $T_{\bar{\rho}_{E,p}} \to R_E$. Wiles proved that this homomorphism is actually an isomorphism (i.e. that "$R = T$") and thus that $\rho_{E,p}$ corresponds with a modular representation $G_{\mathbb{Q}} \to \mathrm{GL}(2, \mathbb{Z}_p)$, from which it follows that E is modular.

There are many other places in number theory where one attempts to classify deformations of object X_0 defined over $\mathbb{F}_p$. One does this by defining a functor assigning to each $R \in \mathcal{R}$ some class of lifts of X to R, and then hoping that the "deformation functor" is representable. Unlike the situation of this course, where the fact that our deformation functor is representable (by a dgla) is trivial, in the number-theoretic context it is often very difficult to show that a given deformation problem is representable.

Chapter 3

Category theory

1 Basic category theory

1.1 Definition of categories

Definition 1.1.1. *A category $\mathcal{C}$ consists of*

- *a class* $\mathrm{Ob}\,\mathcal{C}$ *of objects (written* $X, Y, \dots \in \mathrm{Ob}\,\mathcal{C}$*)*
- *a class* $\mathrm{Mor}\,\mathcal{C}$ *of morphisms (written* $(\varphi : X \to Y) \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$*)*
- *composition maps* $\mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$ *(written* $(f, g) \mapsto f \circ g$*)*

satisfying the following axions

- $\varphi \in \mathrm{Mor}\,\mathcal{C}$ *uniquely determines* X, Y *with* $\varphi \in \mathrm{Hom}(X, Y)$
- *for all* $X \in \mathrm{Ob}\,\mathcal{C}$ *there is a distinguished* $\mathrm{id}_X \in \mathrm{Mor}\,\mathcal{C}$ *such that* $f = f \circ \mathrm{id}_X$*,* $g = \mathrm{id}_X \circ g$ *whenever defined*
- *composition is associative*

Definition 1.1.2. *A category $\mathcal{D}$ is called a* subcategory *of $\mathcal{C}$ if* $\mathrm{Ob}\,\mathcal{D} \subset \mathrm{Ob}\,\mathcal{C}$*,* $\mathrm{Mor}\,\mathcal{D} \subset \mathrm{Mor}\,\mathcal{C}$*, and composition in $\mathcal{D}$ agrees with that of $\mathcal{C}$.*

Definition 1.1.3. *We say that a subcategory $\mathcal{D}$ of $\mathcal{C}$ is* full *if for all* $X, Y \in \mathrm{Ob}\,\mathcal{D}$*, we have* $\mathrm{Hom}_{\mathcal{D}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$*. We say that $\mathcal{D}$ is a* strictly full subcategory *of $\mathcal{C}$ if for all* $Y \in \mathrm{Ob}\,\mathcal{C}$*,* $Y \simeq X$ *for* $X \in \mathrm{Ob}\,\mathcal{D}$ *implies* $Y \in \mathrm{Ob}\,\mathcal{D}$*.*

Yuri Manin divides examples of categories into the following three groups.

Example 1.1.4. The first group of examples consists of categories, where objects are sets with some additional structure, and morphisms in such categories are just morphisms of sets which preserve this structure. Here are basic examples (they are very well-known, but we will mention them to fix the notation):

- Set is the category of sets;
- Top is the category of topological spaces with morphisms continuous maps between them;
- Gr is the category of groups and group homomorphisms;
- Ab is the full subcategory of Gr which consists of abelian groups;
- Vect is the category of vector spaces and linear maps;
- Ring and ComRing are categories of rings and commutative rings respectively;
- Alg_k and ComAlg_k are categories of algebras and commutative algebras respectively over a fixed field (sometimes a commutative ring) k;
- $\mathsf{Com}(\mathcal{A})$ for any additive category $\mathcal{A}$ denotes the category of complexes over $\mathcal{A}$ (see subsection 2.1 below for the notion of additive category).

Example 1.1.5. The second group consists of categories, where objects are still sets with some structure, but the morphisms are not maps of the sets.

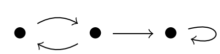
- Ho(Top) is the category, where objects are topological spaces, and morphisms are *homotopy classes* of continuous maps.
- Rel is the category with objects just sets, but morphisms between two objects X and Y are defined to be *binary relations* $R \subseteq X \times Y$. The composition $S \circ R$ of morphisms $R\colon X \to Y$ and $S\colon Y \to Z$ is defined by

$$S \circ R = \{(x, z) \in X \times Z \mid \exists y \in Y, \text{ s.t. } (x, y) \in R \text{ and } (y, z) \in S\} \subseteq X \times Z$$

  The identity morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathsf{Rel}}(X, X)$ is the equality relation $\{(x, x) \mid x \in X\}$.

Example 1.1.6. The third group of examples consists of classical structures that sometimes can be (usefully) considered as categories.

- Any (partially) ordered set I can be viewed as a category $\mathcal{C}(I)$ with $\mathrm{Ob}(\mathcal{C}(I)) = I$ and morphism sets $\mathrm{Hom}(x, y) = \{x \to y\}$ consisting of one arrow if $x \leqslant y$ and $\mathrm{Hom}(x, y) = \varnothing$ otherwise.
- For any topological space X we can make topology τ on X into a category $\mathsf{Open}(X)$ of open sets with morphisms identical inclusions (see section 3.1 in Chapter 1 where we defined presheaves on topological spaces).
- A *quiver* is a finite directed graph $Q = (Q_0, Q_1)$. For example,



is a quiver. Any quiver Q can be thought of as a category $\mathcal{Q}$, whose objects are vertices of Q, and morphisms between vertices v_i and v_j are all paths from v_i to v_j. Here "path" means a sequence of arrows $f_1, \dots, f_n$ such that f_1 starts at v_i and f_n ends in v_j. If $v_i = v_j = v$ then we also include the identity morphism id_v into $\mathrm{Hom}(v, v)$.

There is another category associated to a quiver. Namely, denote by $\mathcal{Q}_{\mathrm{com}}$ the category with $\mathrm{Ob}(\mathcal{Q}_{\mathrm{com}}) = \mathrm{Ob}(\mathcal{Q})$. Set of morphisms $\mathrm{Hom}_{\mathcal{Q}_{\mathrm{com}}}(v_i, v_j)$ contains unique element if $\mathrm{Hom}_{\mathcal{Q}}(v_i, v_j)$ is nonempty, and $\mathrm{Hom}_{\mathcal{Q}_{\mathrm{com}}}(v_i, v_j) = \varnothing$ otherwise. Intuitively this means that all paths between v_i and v_j define the same morphism in $\mathcal{Q}_{\mathrm{com}}$.

1.2 Functors and natural transformations

Definition 1.2.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A* (covariant) functor $F : \mathcal{C} \to \mathcal{D}$ *is a map $F : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$ and maps $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ for all $X, Y \in \mathrm{Ob}\,\mathcal{C}$, such that*

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$
- $F(f \circ g) = F(f) \circ F(g)$

whenever the composition is defined.

There is a dual notion, namely, the notion of a *contravariant functor*, which is just a functor on the *opposite category* $\mathcal{C}^\circ$. Here we define $\mathcal{C}^\circ$ to be the category with $\mathrm{Ob}(C^\circ) = \mathrm{Ob}(C)$ and

$$\mathrm{Hom}_{\mathcal{C}^\circ}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X),$$

with composition induced from $\mathcal{C}$.

Example 1.2.2. A an example of a contravariant functor consider the functor $(-)^* \colon \mathsf{Vect} \to \mathsf{Vect}$ which associates to any vector space V its linear dual space V^*.

Also, we have seen before in sections 2.2 and 3.1 of Chapter 1 that simplicial objects in a category and (pre)sheaves on topological spaces are also examples of contravariant functors.

Definition 1.2.3. *We call a functor $F : \mathcal{C} \to \mathcal{D}$*

- faithful *if the maps $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY)$ are injective for all $X, Y \in \mathrm{Ob}\,\mathcal{C}$*
- full *if the maps $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(FX, FY)$ are surjective*
- fully faithful *if F is both full and faithful*
- essentially surjective *if for all $Y \in \mathrm{Ob}\,\mathcal{D}$, there exists $X \in \mathrm{Ob}\,\mathcal{C}$ such that $Y \simeq F(X)$ in $\mathcal{D}$*

Definition 1.2.4. *If $\mathcal{C}$ and $\mathcal{D}$ are categories, then their product $\mathcal{C} \times \mathcal{D}$ is defined by* $\mathrm{Ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{Ob}\,\mathcal{C} \times \mathrm{Ob}\,\mathcal{D}$*, and*

$$\mathrm{Hom}_{\mathcal{C} \times \mathcal{D}}\left((X_1, Y_1), (X_2, Y_2)\right) = \mathrm{Hom}_{\mathcal{C}}(X_1, X_2) \times \mathrm{Hom}_{\mathcal{D}}(Y_1, Y_2).$$

Definition 1.2.5. *A* bifunctor *is just a functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, where $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ are categories. For example, the assignment $(X, Y) \mapsto \mathrm{Hom}(X, Y)$ is a bifunctor $\mathcal{C}^{\circ} \times \mathcal{C} \to \mathsf{Set}$.*

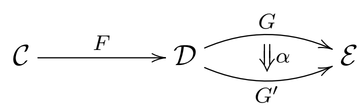
Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors.

Definition 1.2.6. *A* morphism of functors *$\alpha : F \to G$ is given by a collection $\{\alpha_X : F(X) \to G(X)\}_{X \in \mathrm{Ob}\, \mathcal{C}}$ such that whenever $f : X \to Y$ in $\mathcal{C}$, the following diagram commutes*

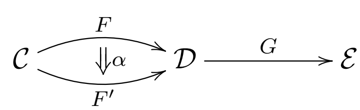
$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_X} & G(X) \\ \Big\downarrow{\scriptstyle F(f)} & & \Big\downarrow{\scriptstyle G(F)} \\ F(Y) & \xrightarrow{\alpha_Y} & G(Y) \end{array}$$

Example 1.2.7. Consider functor $GL_n(-)$: $\mathsf{ComRings} \to \mathsf{Gr}$ with $GL_n(R)$ being the group of $n \times n$ invertible matrices with coefficients in the ring R. Also, consider the functor $(-)^{\times}$: $\mathsf{ComRings} \to \mathsf{Gr}$ which associates to a ring R the subgroup $R^{\times}$ of its units. Then taking determinant of a matrix defines a natural transformation $\det : GL_n(-) \longrightarrow (-)^{\times}$.

Example 1.2.8 (Convolution of morphims of functors)**.** Suppose there are categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ with functors $F : \mathcal{C} \to \mathcal{D}$ and $G, G' : \mathcal{D} \to \mathcal{E}$, along with a natural transformation $\alpha : G \to G'$. We write



We write $\alpha \circ F : G \circ F \to G' \circ F$ for the morphism of functors given by $(\alpha \circ F)_X = \alpha_{F(X)} : G(F(X)) \to G'(F(X))$ for all $X \in \mathrm{Ob}\, \mathcal{C}$. Similarly, if we have a diagram



We write $G \circ \alpha : G \circ F \to G \circ F'$ for the morphism of functors determined by $(G \circ \alpha)_X = G(\alpha_X) : G(F(X)) \to G(F'(X))$ for all $X \in \mathrm{Ob}\, \mathcal{C}$.

We call a category $\mathcal{C}$ *small* if $\mathrm{Ob}\, \mathcal{C}$ is a set (as opposed to a proper class).

Definition 1.2.9. *If $\mathcal{C}$ is a small category and $\mathcal{D}$ any category, we define $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ by*

$$\mathrm{Ob}\, \mathsf{Fun}(\mathcal{C}, \mathcal{D}) = \textit{functors } \mathcal{D} \to \mathcal{D}$$
$$\mathrm{Mor}\, \mathsf{Fun}(\mathcal{C}, \mathcal{D}) = \textit{morphisms of functors}$$

If $\mathcal{C}, \mathcal{D}$ carry some extra structure (e.g. $\mathcal{D}$ and $\mathcal{D}$ are additive categories), then $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ is assumed to consist of functors preserving that structure

Definition 1.2.10. *A morphism of functors $\alpha : F \to G$ is an* isomorphism *(or natural equivalence, or natural isomorphism) if there is a morphism $\beta : G \to F$ such that $\beta \circ \alpha = \mathrm{id}_F$ and $\alpha \circ \beta = \mathrm{id}_G$.*

Lemma 1.2.11. *A morphism $\alpha : F \to G$ is an isomorphism of functors $\mathcal{C} \to \mathcal{D}$ if and only if $\alpha_X : F(X) \to G(X)$ is an isomorphism for all $X \in \operatorname{Ob}\mathcal{C}$.*

Proof. If α is an isomorphism, then it has an inverse $\beta : G \to F$ such that $\beta \circ \alpha = \mathrm{id}_F$ and $\alpha \circ \beta = \mathrm{id}_G$. But evaluating these equalities "pointwise," we see that $\beta_X \circ \alpha_X = \mathrm{id}_X$ and $\alpha_X \circ \beta_Y = \mathrm{id}_Y$ for all $Y \in \operatorname{Ob}\mathcal{D}$, i.e. each α_X is an isomorphism.

If each α_X is an isomorphism, define $\beta : G \to X$ by $\beta_X = \alpha_X^{-1}$. We need to check that β is a morphism of functors. Given $f : X \to Y$, we know that the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_X} & G(X) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\alpha_X} & G(Y) \end{array}$$

But this is easily seen to imply that

$$\begin{array}{ccc} G(X) & \xrightarrow{\beta_X} & F(X) \\ \downarrow{\scriptstyle G(f)} & & \downarrow{\scriptstyle F(f)} \\ G(Y) & \xrightarrow{\beta_Y} & F(Y) \end{array}$$

commutes, whence β is a natural transformation. □

Remark 1.2.12. It is convenient to think of the category Cat of all small categories as a *strict 2-category*. That is, objects of Cat are small categories, and for each $\mathcal{C}, \mathcal{D} \in \operatorname{Ob}\mathsf{Cat}$, the "hom-set" $\operatorname{Hom}_{\mathsf{Cat}}(\mathcal{C}, \mathcal{D})$ is actually the category $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. The 2-category Cat is strict because composition of 1-morphisms is strictly associative, i.e. $(F \circ G) \circ H = F \circ (G \circ H)$. In general, one requires $(F \circ G) \circ H = F \circ (G \circ H)$ being true only up to equivalence of functors.

1.3 Equivalences of categories

Definition 1.3.1. *Let $\mathcal{C}$, $\mathcal{D}$ be categories. We say that $\mathcal{C}$ is* isomorphic *to $\mathcal{D}$ if there is a pair of functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G = \mathrm{id}_{\mathcal{D}}$ and $G \circ F = \mathrm{id}_{\mathcal{C}}$.*

This definition is useless in practice because finding functors that are inverses "on the nose" is nearly impossible. Instead, we make the following definition.

Definition 1.3.2. *A functor $F : \mathcal{C} \to \mathcal{D}$ is an* equivalence of categories *if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G \simeq \mathrm{id}_{\mathcal{D}}$ and $G \circ F \simeq \mathrm{id}_{\mathcal{C}}$.*

We say that $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there is an equivalence $F : \mathcal{C} \to \mathcal{D}$. We write $\mathcal{C} \simeq \mathcal{D}$ if $\mathcal{C}$ and $\mathcal{D}$ are equivalent. If F is an equivalence, we call G a *quasi-inverse* for F. The quasi-inverse is far from unique, but it is unique up to natural equivalence.

Example 1.3.3. Let $\mathcal{D} = \mathsf{Vect}_k^n$ be the category of n-dimensional vector spaces over a field k. Let $\mathcal{C}$ be the full subcategory of $\mathcal{D}$ consisting of a single object $k^{\oplus n}$. The inclusion functor $i : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories, even though i is not a bijection at the level of objects. A quasi-inverse of $\mathcal{C}$ arises from choosing a basis for each $V \in \operatorname{Ob} \mathsf{Vect}_k^n$. To each $f : V \to W$, we assign the matrix representation of f in terms of our chosen bases of V and W.

This example is typical. Equivalent categories may have different objects, but the same isomorphism classes of objects. Also, the construction of a quasi-inverse typically requires the axiom of choice.

Recall that F is fully faithful if the maps $\operatorname{Hom}_{\mathcal{C}}(X, Y) \to \operatorname{Hom}_{\mathcal{D}}(F(X), F(Y))$ are bijections, and F is essentially surjective if for all $Y \in \operatorname{Ob} \mathcal{D}$, there exists $X \in \mathcal{C}$ for which $F(X) \simeq Y$.

Theorem 1.3.4 (Freud)**.** *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

Proof. We will prove a bit later more general result that will imply this theorem. □

If $\mathcal{C}$ is a category, a *skeleton* $\operatorname{sk} \mathcal{C}$ of $\mathcal{C}$ is a full subcategory of $\mathcal{C}$ with one object in each isomorphism class. The theorem shows that $\operatorname{sk} \mathcal{C} \hookrightarrow \mathcal{C}$ is an equivalence of categories.

Example 1.3.5 (Groupoids)**.** A *groupoid* is a (small) category in which all morphisms are isomorphisms. So a group is just a groupoid with one object. We say that a groupoid is *connected* if any two objects can be connected by arrows (possibly in both directions). It is easy to see that the skeleton of a connected groupoid is a group. The main example is the *fundamental groupoid* $\Pi(X)$ of a topological space X. Objects of $\Pi(X)$ are points in X, and $\operatorname{Hom}_{\Pi(X)}(x, y)$ is the set of homotopy classes of paths $x \to y$. If X is connected, the choice of a point $x \in X$ gives rise to an equivalence of categories $\pi_1(X, x) \hookrightarrow \Pi(X)$.

Example 1.3.6. There are several equivalences of categories relating algebra and geometry.

For example, if k is a field, the category of finitely generated commutative reduced k algebras is anti-equivalent to the category of affine varieties over k via the functor $A \mapsto \operatorname{Spec} A$. More generally, the category of all commutative k-algebras is anti-equivalent to the category of affine schemes over k, once again via Spec. In both cases, Spec has a quasi-inverse, namely $X \mapsto \mathscr{O}_X(X)$.

The category of all associative k-algebras does not have a good geometric analogue. Of course, one can define the category of "non-commutative affine schemes" to be the opposite of the category of associative k-algebras, but this is not reasonable, as is seen by the next section.

1.4 Representable functors and the Yoneda lemma

For a category $\mathcal{C}$, an object $X \in \operatorname{Ob} \mathcal{C}$ yields two functors. The first is contravariant, denoted $h_X : \mathcal{C}^\circ \to \mathsf{Set}$, defined by $Y \mapsto \operatorname{Hom}_{\mathcal{C}}(Y, X)$. For $g : Y \to Z$, the induced arrow $g^* = h_X(g) : h_X(Z) \to h_X(Y)$ is $s \mapsto s \circ g$. The covariant version of h_X is $h^X : \mathcal{C} \to \mathsf{Set}$. On objects, it is defined by $Y \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y)$, and for $g : Y \to X$, the map $g_* = h^X(g) : h^X(Y) \to h^Y(Z)$ is $s \mapsto g \circ s$.

Definition 1.4.1. *A functor $F : \mathcal{C}^\circ \to \mathsf{Set}$ is* representable *if there exists $X \in \mathrm{Ob}\,\mathcal{C}$ such that $F \simeq h_X$ in $\mathsf{Fun}(\mathcal{C}^\circ, \mathsf{Set})$. We say that $F : \mathcal{C} \to \mathsf{Set}$ is* corepresentable *if there exists $X \in \mathrm{Ob}\,\mathcal{C}$ such that $F \simeq h^X$.*

To simplicy notation, write $\widehat{\mathcal{C}}$ for the category $\mathsf{Fun}(\mathcal{C}^\circ, \mathsf{Set})$ of "presheaves" on $\mathcal{C}$. We want to extend $X \mapsto h_X$ to a (covariant) functor $h : \mathcal{C} \to \widehat{\mathcal{C}}$. So, h is defined on objects by $X \mapsto h_X$. For a morphism $f : X_1 \to X_2$, we define $h_f : h_{X_1} \to h_{X_2}$ by

$$h_f(Y) : h_{X_1}(Y) \to h_{X_2}(Y) \qquad , g \mapsto f_* g = f \circ g.$$

We need to check that h_f is a morphism of functors. Take a morphism $s : Z \to Y$ in $\mathcal{C}$. We need the following diagram to commute.

$$\begin{array}{ccc} h_{X_1}(Y) & \xrightarrow{h_f(Y)} & h_{X_2}(Y) \\ \downarrow{\scriptstyle s^*} & & \downarrow{\scriptstyle s^*} \\ h_{X_1}(Z) & \xrightarrow{h_f(Z)} & h_{X_2}(Z) \end{array}$$

The upper path sends $g : Y \to X_1$ to $(f \circ g) \circ s$, while the lower path sends g to $f \circ (g \circ s)$. The two are equal by associativity. It is easy to see that $h_{f \circ g} = h_f \circ h_g$, because this is equivalent to $(f \circ g)_* = f_* \circ g_*$.

Theorem 1.4.2 (Yoneda)**.** *Let $F \in \mathrm{Ob}(\widehat{\mathcal{C}})$. Then the map $\mathrm{Hom}_{\widehat{\mathcal{C}}}(h_X, F) \to F(X)$ given by $\varphi \mapsto \varphi(X)(\mathrm{id}_X)$ is a bijection.*

Proof. Let $y : \mathrm{Hom}_{\widehat{\mathcal{C}}}(h_X, F) \to F(X)$ be the map $\varphi \mapsto \varphi(X)(\mathrm{id}_X)$. We show that y is a bijection by constructing an explicit inverse. Given $x \in F(X)$, we want a morphism of functors $i(x) : h_X \to F$. This would consist of morphisms $i(x)(Y) : h_X(Y) \to F(Y)$ for each Y. Given $f \in h_X(Y) = \mathrm{Hom}(Y, X)$, we have a map $F(f) : F(X) \to F(Y)$. We define $i(x)(Y)f = F(f)(x)$. It is not difficult to show that $i(x)$ actually is a morphism of functors. We will show that i is an inverse to y.

First we show that i is a right inverse to y. For $x \in F(x)$, $i(x)$ is defined by $i(x)(Y)f = F(f)(x)$, so $y(i(x)) = i(x)(X)\mathrm{id}_X = F(\mathrm{id}_X)(x) = x$.

Now we show that i is a left inverse for h. Given $\varphi : h_X \to F$, let $x = y(\varphi) = \varphi(X)(\mathrm{id}_X)$. We need $i(x) = \varphi$, i.e. $\varphi(Z) = i(x)(Z)$ for all $Z \in \mathrm{Ob}\,\mathcal{C}$. This is just the claim that for all $f : Y \to X$, we have $\varphi(Z)(f) = F(f)(x)$. Apply the definition of "φ is a natural transformation" to $f : Y \to X$. We get a commutative diagram

$$\begin{array}{ccc} h_X(X) & \xrightarrow{\varphi(X)} & F(X) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle F(f)} \\ h_X(Y) & \xrightarrow{\varphi(Y)} & F(Y) \end{array}$$

The commutativity of this diagram is the fact that $F(f)(\varphi(X)s) = \varphi(X)(s \circ f)$ for any $s : X \to X$. If we choose $s = \mathrm{id}_X$, then we get $F(f)(\varphi(X)\mathrm{id}_X) = \varphi(X)(f)$, i.e. $i(x)f = F(f)(x) = \varphi(X)(f)$. □

Corollary 1.4.3. *Let $X, Y \in \mathrm{Ob}\,\mathcal{C}$. Then $h : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\widehat{\mathcal{C}}}(h_X, h_Y)$ is a bijection.*

Proof. Let $F = h_Y$. The Yoneda lemma says that the map $\mathrm{Hom}(h_X, h_Y) \to F(X) = \mathrm{Hom}(X, Y)$ given by $\varphi \mapsto \varphi(X)(\mathrm{id}_X)$ is a bijection. It is easy to check that this map is the inverse to $h : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(h_X, h_Y)$, so h is a bijection. □

In light of the corollary, we can use h to regard $\mathcal{C}$ as a full subcategory of $\widehat{\mathcal{C}}$ consisting of representable functors. If $F \in \mathrm{Ob}(\widehat{\mathcal{C}})$ is representable, then the representing object X is determined uniquely (up to canonical isomorphism). By this we mean the following. Suppose we have $F \in \mathrm{Ob}(\widehat{\mathcal{C}})$ and isomorphisms $\varphi : F \to h_X$, $\psi : F \to h_Y$. Thus $\psi \circ \varphi^{-1} : h_X \to h_Y$ is a natural isomorphism. The Yoneda lemma gives an isomorphism $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\widehat{\mathcal{C}}}(h_X, h_Y)$ Applying h^{-1} to $\psi \circ \varphi^{-1}$ gives a canonical isomorphism $X \to Y$. We can make this even more precise. If $F : \mathcal{C}^\circ \to \mathsf{Set}$ is representable ($\psi : h_X \xrightarrow{\sim} F$ for some X), one usually says that F is represented by a pair (X, σ), where $\sigma = \sigma_X = \psi_X(\mathrm{id}_X) \in F(X)$. The pair (X, σ) is unique up to unique isomorphism. That is, if (X', σ') also represents F, there is a unique isomorphism $f : X \to X'$ such that $F(f)(\sigma) = \sigma'$. Since $\mathrm{Aut}_{\widehat{\mathcal{C}}}(F) \simeq \mathrm{Aut}_{\mathcal{C}}(X)$, the functor F does not determine X up to unique isomorphism.

There is a dual Yoneda lemma, which states that the assignment $X \to h^X$ is a fully faithful contravariant functor $\mathcal{C}^\circ \to \mathsf{Fun}(\mathcal{C}, \mathsf{Set})$.

The representability of F can be redefined in terms of a universal property. The pair (X, σ) represents F if and only if for all objects $Y \in \mathrm{Ob}\,\mathcal{C}$, $\alpha \in F(Y)$, there exists a unique $f_\alpha : Y \to X$ such that $F(f_\alpha)(\sigma) = \alpha$. Dually, (X, σ) corepresents F if and only if for all $Y \in \mathrm{Ob}\,\mathcal{C}$ and $\alpha \in F(Y)$, there is a unique $f_\alpha : X \to Y$ such that $F(f_\alpha)(\sigma) = \alpha$.

Many objects in various categories can be constructed by first defining a functor (which is expected to be representable) and then by proving that the functor is representable. The representing object is the object one wants. This has an analogy in PDE theory. Given a system of PDEs, one wants to find a "nice" (smooth, for example) solutions. A natural way to do this is to first find a "generalized solution" (a distribution) and then prove that the generalized solution is regular enough.

Example 1.4.4 (Matrix representations)**.** Let k be a field, and fix a k-algebra A. A *matrix representation* of A over a commutative k-algebra B is a k-algebra homomorphism $\rho : A \to M_n(B)$. Given any $f : B \to B'$, we can define a new representation $f_*\rho : A \to M_n(B')$ by $f_*\rho = M_n(f) \circ \rho$. We say that $f_*\rho$ is induced from ρ by f. An obvious question is: "is there a universal representation?" That is, we seek a commutative ring A_n and a representation $\rho_n : A \to M_n(A_n)$ such that for any $\rho : A \to M_n(B)$, there is a unique homomorphism $f : A_n \to B$ such that $\rho = f_*\rho_n$.

Define the representation functor $\mathrm{Rep}_n(A) : \mathsf{ComAlg}_k \to \mathsf{Set}$ to be

$$\mathrm{Rep}_n(A)(B) = \mathrm{Hom}_{k\text{-}\mathsf{Alg}}(A, M_n(B)).$$

We are asking if $\mathrm{Rep}_n(A)$ is corepresentable. The answer is "yes"! Define $\sqrt[n]{A} = (A *_k M_n(k))^{M_n(k)}$, where $*$ denotes free product. In other words,

$$\sqrt[n]{A} = \{A \in A *_k M_n(k) : [a, m] = 0,\ \forall m \in M_n(k)\}.$$

Let $e_{ij} \in M_n(k)$ be the elementary matrix with 1 in the (i, j)-th coordinate. Then e_{ij} form a canonical k-basis for $M_n(k)$. For $a \in A *_k M_n(k)$, let

$$a_{ij} = \sum_{k=1}^{n} e_{ki} * a * e_{jk}.$$

We claim that $[a_{ij}, M_n(k)] = 0$, and that $\sqrt[n]{A}$ is spanned by the a_{ij}. Let $A_n = \sqrt[n]{A}/\langle[\sqrt[n]{A}, \sqrt[n]{A}]\rangle$ be the abelianization of the algebra $\sqrt[n]{A}$. It turns out that $\mathrm{Rep}_n(A)$ is represented by A_n. This construction is due to Bergman [Ber74].

Example 1.4.5 (Hilbert scheme)**.** Let k be an algebraically closed field of characteristic zero, and let X be a projective variety over k. Define the functor $\mathrm{Hilb}_X : \mathsf{Sch}_k \to \mathsf{Set}$ by

$$U \mapsto \{Z \subset U \times X \text{ closed subscheme such that } \pi_U : Z \to U \text{ is flat}\}$$

That is, $\mathrm{Hilb}_X(U)$ is the set of families of closed subschemes of $U \times X$ parameterized by U. One can prove that Hilb_X is representable. However, the representing object is a scheme, not a variety. One remedies this by stratifying Hilb_X via Hilbert polynomials.

For $u \in U$, let $Z_u = \pi^{-1}(u)$, and define the *Hilbert polynomial* of Z at u by $P_{Z,u}(m) = \chi(\mathscr{O}_{Z_u} \otimes \mathscr{O}_X(m))$, where χ denotes Euler characteristic. It is a theorem that $P_{Z,u}$ actually is a polynomial that is independent of u if U is connected. For some polynomial P, let

$$\mathrm{Hilb}_X^P(U) = \{Z \subset U \times X :\ Z \text{ is a flat subscheme with } P_Z = P\}.$$

It is a major theorem of Grothendieck that Hilb_X^P is representable by a projective variety.

Example 1.4.6 (PDEs)**.** Let $n \geqslant 1$, and let $U \subset \mathbb{C}^n$ be an open subset. Consider a differential operator

$$P = \sum_{|\alpha| \leqslant m} a_\alpha(z_1, \ldots, z_n)\partial_z^\alpha$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ ranges over multi-indices and where $\partial_z^\alpha = \partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n}$. Supposing the $a_\alpha \in \mathscr{O}^{\mathrm{an}}(U)$, we are interested in solutions to $Pu = 0$. Denote by $\mathscr{D} = \mathscr{D}(U)$ the ring of all linear differential operators with coefficients in $\mathscr{O}^{\mathrm{an}}(U)$. Let $\mathscr{D}$-Mod be the category of left $\mathscr{D}$-modules, and define a functor $\mathrm{Sol}_P : \mathscr{D}\text{-}\mathsf{Mod} \to \mathsf{Set}$ by

$$N \mapsto \{\text{solutions of } Pu = 0 \text{ in } N\}.$$

It is easy to see that Sol_P is represented by $M_P = \mathscr{D}/(\mathscr{D} \cdot P)$. Given $u \in N$ with $Pu = 0$, define $\mathscr{D} \to N$ by $D \mapsto Du$. This is a $\mathscr{D}$-modules homomorphism with kernel $\mathscr{D} \cdot P$, and it is easy to see that this correspondence is a bijection.

Given that Sol_P is determined by M_P, we can think of differential equations as $\mathscr{D}$-modules. If M is some $\mathscr{D}$-module, an "N-valued solution to the differential equation determined by M" is a $\mathscr{D}$-linear map $M \to N$. This suggests that we think of "higher solutions" as elements of $\mathrm{Ext}^i_{\mathscr{D}}(M, N)$. Even better, we can think of M as an object in the derived category of $\mathscr{D}$-Mod.

1.5 Adjoint functors

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. For $Y \in \mathrm{Ob}\,\mathcal{D}$, define the functor $\widetilde{F}_Y : \mathcal{C}^\circ \to \mathsf{Set}$ by

$$F(T) = \mathrm{Hom}_{\mathcal{D}}\left(F(T), Y\right).$$

In other words, $\widetilde{F}_Y = h_Y \circ F$.

Proposition 1.5.1. *Suppose $\widetilde{F}_Y$ is represented by some $X_Y \in \mathrm{Ob}\,\mathcal{C}$ for every Y. Then the assignment $Y \mapsto X_Y$ extends to a functor $G : \mathcal{D} \to \mathcal{C}$, for which there is a natural isomorphism of bifunctors $\mathcal{C}^\circ \times \mathcal{D} \to \mathsf{Set}$,*

$$\mathrm{Hom}_{\mathcal{D}}\left(F(-), -\right) \simeq \mathrm{Hom}_{\mathcal{C}}\left(-, G(-)\right).$$

In this situation, we say that F is *left adjoint* to G, and G is *right adjoint* to F. This situation we will denote by $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$.

Proof. For $Y \in \mathrm{Ob}\,\mathcal{D}$, choose a natural isomorphism $\psi : h_{X_Y} \to \widetilde{F}_Y$. This gives bijections $\psi_T : \mathrm{Hom}_{\mathcal{C}}(T, X_Y) \to \mathrm{Hom}_{\mathcal{D}}(F(T), Y)$, so we can define $\sigma_{X_Y} = \psi_{X_Y}(\mathrm{id}_{X_Y})$. Define $G : \mathcal{D} \to \mathcal{C}$ on objects by $G(Y) = X_Y$. For $f : Y \to \widetilde{Y}$ in $\mathcal{D}$, consider

$$\widetilde{\psi} : \mathrm{Hom}_{\mathcal{C}}(-, X_{\widetilde{Y}}) \to \mathrm{Hom}_{\mathcal{D}}(F(-), \widetilde{Y}).$$

We have $f \circ \sigma_{X_Y} : F(X_Y) \to \widetilde{Y}$, so we set

$$G(f) = \widetilde{\psi}^{-1}_{G(Y)}(f \circ \sigma_{G(Y)}) : G(Y) \to G(\widetilde{Y}).$$

It is tedious but straightforward to check that this construction actually makes G a functor. □

Example 1.5.2. If F and G are mutual quasi-inverses in an equivalence of categories, then (F, G) and (G, F) are adjoint pairs.

Example 1.5.3 (Abelianization)**.** Natural embedding $i\colon \mathsf{Ab} \hookrightarrow \mathsf{Gr}$ is right adjoint to the abelianization functor $\mathrm{ab}\colon \mathsf{Ab} \to \mathsf{Gr}$ that associates to any group G the group $G/[G, G]$. Analogous statements are true, for example, for embeddings $\mathsf{ComRing} \hookrightarrow \mathsf{Ring}$ and $\mathsf{ComAlg}_k \hookrightarrow \mathsf{Alg}_k$.

Example 1.5.4 (Free objects)**.** Consider forgetful functor for: $\mathsf{Gr} \to \mathsf{Set}$ that associates to each group itself, but viewed as a set (i.e. it "forgets" the group structure). Then $operatornamefor$ is right adjoint to the functor free: $\mathsf{Set} \to \mathsf{Gr}$ that associates to each set S free group free(S), generated by S. The same is true with forgetful functors from Ab, Ring, $\mathsf{ComRing}$, Alg_k et.c.

Example 1.5.5 (Matrix representations continued)**.** Recall we had a natural isomorphism

$$\mathrm{Hom}_{\mathsf{ComAlg}_k}(A_n, B) \simeq \mathrm{Hom}_{\mathsf{Alg}_k}(A, M_n(B)).$$

Now we realize that the functor $(-)_n : \mathsf{Alg}_k \to \mathsf{ComAlg}_k$, $A \mapsto A_n = \sqrt[n]{A}/[\sqrt[n]{A}, \sqrt[n]{A}]$, is left adjoint to the functor $M_n : \mathsf{ComAlg}_k \to \mathsf{Alg}_k$. It turns out that we have an adjoint pair

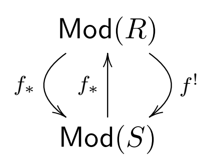
$$\sqrt[n]{-} : \mathsf{Alg}_k \rightleftarrows \mathsf{Alg}_k : M_n(-).$$

Example 1.5.6 (Tensor-Hom adjunction)**.** Let R and S be associative unital rings. Let $\mathcal{C} = \mathsf{Mod}(R)$, $\mathcal{D} = \mathsf{Mod}(S)$ be the categories of right modules over R and S respectively. Let B be an (R, S)-bimodule. Then we have a functor $(-) \otimes_R B : \mathsf{Mod}(R) \to \mathsf{Mod}(S)$ has right adjoint $\mathrm{Hom}_S(B, -)$. That is, there is a natural isomorphism

$$\alpha : \mathrm{Hom}_S(M \otimes_R B, N) \xrightarrow{\sim} \mathrm{Hom}_R(M, \mathrm{Hom}_S(B, N)).$$

This is easy to check. Given $f : M \otimes B \to N$, we can define the map $\alpha(f)(m) = f(m \otimes -)$. The map α has inverse $\alpha^{-1}(\lambda)(m \otimes b) = \lambda(m)b$.

Let $f : R \to S$ be any ring homomorphism. Then we have an "adjoint triple" of functors $(f^*, f_*, f^!)$. That is, f^* is left adjoint to f_*, which is left adjoint to $f^!$. We could write



The functor f_* is restriction of scalars via f. We have $f_*(N) = N \otimes_S S$, with right adjoint $f^! = \mathrm{Hom}_S(S, -)$. The left adjoint of f_* is $f^* = - \otimes_R S$.

As a concrete example, suppose $f : R \twoheadrightarrow R/I$ is the canonical surjection, where $I \subset R$ is a two-sided ideal. Then f^* assigns to an R-module the largest quotient killed by I, $f^!$ sends an R-submodule to the largest submodule which is also a submodule over S, and f_* sends an R-module to its quotient by I.

Let (F, G) be an adjoint pair, and let $\eta_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \to \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$ be a natural isomorphism witnessing this adjunction. If we take $Y = F(X)$, we get a map $\sigma : X \to GF(X)$ corresponding to $\mathrm{id}_{F(X)}$. The maps $\sigma_X = \eta_{x,F(X)}(\mathrm{id}_{F(X)})$ define a morphism of functors $\sigma : \mathrm{id}_{\mathcal{C}} \to GF$ called the *unit* of the adjunction. Dually, if we take $X = G(Y)$, we can define $\eta : FG \to \mathrm{id}_{\mathcal{D}}$ by $\eta_Y = \eta^{-1}_{G(Y),Y}(\mathrm{id}_{G(Y)})$. One calls η the *counit* of the adjunction. Note that we can define the convolutions $F\sigma : F \to FGF$ and $\eta G : FGF \to F$.

Lemma 1.5.7. *One has $(\eta F) \circ (F\sigma) = \mathrm{id}_F$ and $(G\eta) \circ (\sigma G) = \mathrm{id}_G$.*

Lemma 1.5.8. *Given functors $F : \mathcal{C} \to D$, $G : \mathcal{D} \to \mathcal{C}$, if there exist morphisms $\sigma : \mathrm{id}_{\mathcal{C}} \to GF$ and $\eta : FG \to \mathrm{id}_{\mathcal{D}}$ satisfying the above identities, then (F, G) is an adjoint pair.*

Example 1.5.9 (Traces in categories)**.** The following is an abstract version of the Bernstein trace. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor that has both left and right adjoints. So we have an adjoint triple (E, F, G), where $E, G : \mathcal{D} \to \mathcal{C}$. Suppose we have a natural transformation $\nu : G \to E$. Then for any $X, Y \in \operatorname{Ob}\mathcal{C}$, we have the *trace map* $\operatorname{tr} : \operatorname{Hom}_{\mathcal{D}}(F(X), F(Y)) \to \operatorname{Hom}_{\mathcal{C}}(X, Y)$ defined as follows. For $a : F(X) \to F(Y)$, we let $\operatorname{tr}(a)$ be the composite

$$X \xrightarrow{\eta_X} G \circ F(X) \xrightarrow{\nu_{F(X)}} E \circ F(X) \xrightarrow{E(a)} EF(Y) \xrightarrow{\sigma_Y} Y$$

Here $\eta : \mathrm{id} \to GF$ and $\sigma : EF \to \mathrm{id}$ come from the adjunction (E, F, G). Setting $X = Y$, we get a natural transformation $\operatorname{End}(F) \to \operatorname{End}(\mathrm{id}_{\mathcal{C}})$.

1.6 Limits and colimits

Let J be a fixed category, called the *index category*. We assume that J is finite, or at least small. For an arbitrary category $\mathcal{C}$, we write $\mathcal{C}^J$ for the functor category $\mathsf{Fun}(J, \mathcal{C})$. It is helpful to think of $\mathcal{C}^J$ as the category of "diagrams of shape J in $\mathcal{C}$."

For example, J could be a category with no non-identity morphisms (such categories are called *discrete*). Another very useful example is $J = \{\bullet \to \bullet \leftarrow \bullet\}$. Objects of $\mathcal{C}^J$ are called *pullback data* in $\mathcal{C}$. Both of these are special cases of when J is a poset, where we treat J as a category via

$$\operatorname{Hom}_J(i, j) = \begin{cases} \{*\} & \text{if } i \leqslant j \\ \varnothing & \text{otherwise} \end{cases}$$

There is an obvious functor $\Delta : \mathcal{C} \to \mathcal{C}^J$ that sends $X \in \operatorname{Ob}\mathcal{C}$ to the "constant diagram" $j \mapsto \Delta(X)(j) = X$, with $\Delta(X)(i \to j) = \mathrm{id}_X$. Given a morphism $\varphi : X \to Y$ in $\mathcal{C}$, define $\Delta(\varphi) : \Delta(X) \to \Delta(Y)$ by letting $\Delta(\varphi_j : \Delta(X)(j) = X \to \Delta(Y)(j) = Y$ be φ itself. We call Δ the *diagonal* (or *constant*) functor. It is natural to ask whether Δ has a left or right adjoint. Fix $F : J \to \mathcal{C}$ (i.e. $F \in \operatorname{Ob}(\mathcal{C}^J)$) and define $\widetilde{F} : \mathcal{C}^\circ \to \mathsf{Set}$ by $Y \mapsto \operatorname{Hom}_{\mathcal{C}^J}(\Delta(Y), F)$.

Definition 1.6.1. *If $\widetilde{F}$ is representable, we call the representing object X the* limit *of F, written $X = \varprojlim F$.*

Our definition requires that there be a natural isomorphism

$$\operatorname{Hom}_{\mathcal{C}^J}(\Delta(Y), F) \simeq \operatorname{Hom}_{\mathcal{C}}\left(Y, \varprojlim F\right).$$

Any such natural isomorphism comes from a morphism $s : \Delta(\varprojlim F) \to F$. We can make the requirement that $\varprojlim F$ represent $Y \mapsto \operatorname{Hom}(\Delta(Y), F)$ much more concrete.

For any $Y \in \operatorname{Ob}\mathcal{C}$, a natural transformation $t : \Delta(Y) \to F$ should be thought of as a "cone over F" with vertex Y. Given a morphism $Y' \to Y$, we can pull back a cone with vertex Y to get a come with vertex Y'. It is natural to ask if there is a terminal cone, i.e. a cone that induces all cones by pullback. This happens precisely when F has a limit. That is, $X = \varprojlim F$ with $s : \Delta(X) \to F$ is the "closest" (to $\mathcal{C}$) cone over F.

Example 1.6.2 (Pullbacks)**.** Let J be the category $\{\bullet \to \bullet \leftarrow \bullet\}$. Limits of diagrams of shape J are fibered products. The limit of a diagram $X \to Z \leftarrow Y$ is called the *fiber product* of X and Y over Z, and is denoted $X \times_Z Y$.

If it happens that for each $F \in \mathrm{Ob}(\mathcal{C}^J)$, $\widetilde{F}$ is representable by $\varprojlim F$, then $F \mapsto \varprojlim F$ can be extended to a functor $\varprojlim : \mathcal{C}^J \to \mathcal{C}$. The functor $\varprojlim$ is the right adjoint of Δ, i.e. there is an adjoint pair:

$$\Delta : \mathcal{C} \rightleftarrows \mathcal{C}^J : \varprojlim$$

We call a category $\mathcal{C}$ *complete* if all limits of shape J exist in $\mathcal{C}$ for all small categories J.

Dually, we can define *colimits*, which (if they exist) are left adjoint to Δ. That is, there is an adjoint pair

$$\varinjlim : \mathcal{C}^J \rightleftarrows \mathcal{C} : \Delta$$

The terminology is slightly confusing. A *limit* is often called a *projective (or inverse) limit*, especially if the index category is a poset and F is contravariant. Dually, colimits are called *inductive (or direct) limits*, especially if J is a poset and F is covariant. The notation $\operatorname{colim} F$ is sometimes used instead of $\varinjlim F$.

Example 1.6.3 (Initial and terminal objects)**.** If the index category $J = \emptyset$ is the empty category, then there exists unique functor $F\colon J \to \mathcal{C}$. The colimit and limit of F are initial and terminal objects in $\mathcal{C}$ respectively.

Example 1.6.4 (Products and coproducts)**.** A (small) category J is called *discrete* if for any $X, Y \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \emptyset$ if $X \neq Y$ and $\mathrm{Hom}_{\mathcal{C}}(X, X) = \{\mathrm{id}_X\}$. Schematically we have $J = \{\bullet \ \bullet \ \ldots \ \bullet\}$. Then a functor $F\colon J \to \mathcal{C}$ is just a collection of objects $\{X_j\}_{j \in J}$ in $\mathcal{C}$. Then $\varinjlim F$ is just the coproduct $\coprod\limits_{j \in J} X_j$ and $\varprojlim F$ is the product $\prod\limits_{j \in J} X_j$.

Example 1.6.5 (Pushouts)**.** If $J = \{\bullet \leftarrow \bullet \to \bullet\}$, then functors $F : J \to \mathcal{C}$ are just diagrams $X \leftarrow Y \to Z$, also known as "pushout data." The colimit of F corresponding to $X \leftarrow Y \to Z$ is the pushout of X and Z over Y, denoted $X \sqcup_Y Z$.

Let I be a small category. If we have a functor $F : I \to \mathcal{C}$, we think of F as a "diagram of shape I." For $i \in I$, write X_i for $F(i)$. We defined the limit of F, denoted $\varprojlim F$, to be a object X of $\mathcal{C}$ together with morphisms $\varphi_i : X \to X_i$, such that for all $f : i \to j$, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\varphi_i} & X_i \\ & \searrow^{f_j} & \downarrow{\scriptstyle F(f)} \\ & & X_j \end{array}$$

and such that (X, φ) is terminal with respect to this property. If $G : \mathcal{C} \to \mathcal{D}$ is any functor, we can apply G to (X, φ) to get computable morphisms $G(\varphi_i) : G(\varprojlim F) \to G(X_i)$, and thus a canonical morphism $\alpha_F : G(\varprojlim F) \to \varprojlim(G \circ F)$.

Definition 1.6.6. *The functor G* preserves limits *if α_F is an isomorphism whenever $\varprojlim F$ exists.*

Dually, given $F : I \to \mathcal{C}$, the colimit $X = \varinjlim F$ has morphisms $\psi : X_i \to X$. Applying $G : \mathcal{C} \to \mathcal{D}$, we get compatible morphisms $G(\psi) : G(X) \to G(X_i)$, hence a canonical morphism $\beta_F : \varinjlim(G \circ F) \to G(\varinjlim F)$.

Definition 1.6.7. *The functor G* preserves colimits *if β_F is an isomorphism whenever $\varinjlim F$ exists.*

Theorem 1.6.8. *For any object X in a category $\mathcal{C}$, the functor $h^X = \mathrm{Hom}_{\mathcal{C}}(X, -)$ preserves limits and the functor $h_X = \mathrm{Hom}_{\mathcal{C}}(-, X)$ maps colimits to limits.*

Proof. The fact that h^X preserves limits is easy to check. Just construct an inverse to α_F. Moreover, the fact that h_X maps colimits to limits follows trivially from the fact that h^X is limit-preserving. Treat $h_X : \mathcal{C}^\circ \to \mathsf{Set}$ as a covariant functor on the opposite category $\mathcal{C}^\circ$. In fact, $h_X = h^{X^\circ}$, so this follows from the fact that h^{X° preserves limits. □

Corollary 1.6.9. *If a functor $F : \mathcal{C} \to \mathsf{Set}$ is corepresentable, then F preserves limits. Dually, if $F : \mathcal{C}^\circ \to \mathsf{Set}$ is representable, then F sends colimits to limits.*

These are necessary conditions for (co)representability. Unfortunately, they are not sufficient conditions. In general, it is very difficult to prove that a given functor is (co)representable.

Corollary 1.6.10. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjoint pair of functors. Then F preserves colimits and G preserves limits.*

Proof. This is a standard trick, using Theorem 1.6.8 and the Yoneda Lemma. Consider $H : I \to \mathcal{C}$, and arbitrary $Y \in \mathrm{Ob}(\mathcal{D})$. Then

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}}(F(\varinjlim H), Y) &\simeq \mathrm{Hom}_{\mathcal{C}}(\varinjlim H, G(Y)) && \text{adjointness} \\
&\simeq \varprojlim \mathrm{Hom}_{\mathcal{C}}(H(-), G(Y)) \\
&\simeq \varprojlim\, (\mathrm{Hom}(-, G(Y)) \circ H) && \text{by Theorem 1.6.8} \\
&\simeq \varprojlim\, (\mathrm{Hom}_{\mathcal{D}}(F(-), Y) \circ H) \\
&\simeq \varprojlim\, (\mathrm{Hom}_{\mathcal{D}}(-, Y) \circ (F \circ H)) \\
&\simeq \mathrm{Hom}_{\mathcal{D}}(\varinjlim(F \circ H), Y) && \text{by Theorem 1.6.8.}
\end{aligned}$$

By the Yoneda lemma, we conclude that $F(\varinjlim H) \simeq \varinjlim(FH)$. □

2 Special topics in category theory

2.1 Brief introduction to additive categories

Example 2.1.1. Let Ab be the category of abelian groups. For two groups A, B, the set $\mathrm{Hom}_{\mathsf{Ab}}(A, B)$ is not only a set – it naturally has the structure of an abelian group in which composition is bilinear.

Definition 2.1.2. *A category $\mathcal{A}$ is* additive *if*

AB1 For all $X, Y \in \mathrm{Ob}\,\mathcal{A}$, the set $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ has the structure of an abelian group, and the composition maps

$$\mathrm{Hom}_{\mathcal{A}}(Y, Z) \times \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$$

are bilinear.

AB2 There is a (unique) object 0 for which $\mathrm{Hom}(0, X) = \mathrm{Hom}(X, 0) = 0$ for all $X \in \mathrm{Ob}\,\mathcal{C}$

AB3 Binary products and coproducts exist (and coincide) in $\mathcal{A}$

If a category $\mathcal{C}$ satisfies only the Axiom AB1, we say that $\mathcal{C}$ is a *preadditive category.* We also call preadditive categories $\mathbb{Z}$-categories, thinking of them as categories enriched over $\mathbb{Z}$.

Axiom 3 means that for any $X_1, X_2 \in \mathrm{Ob}\,\mathcal{A}$, there is an object Y with morphisms

$$X \underset{p_1}{\overset{i_1}{\rightleftarrows}} Y \underset{p_2}{\overset{i_2}{\leftrightarrows}} X_2$$

such that $p_a i_a = \mathrm{id}_{X_a}$, $p_a i_b = 0$ if $a \neq b$, and $i_1 p_1 + i_2 p_2 = \mathrm{id}_Y$. This is equivalent to the existence of squares

$$\begin{array}{ccc} Y & \xrightarrow{p_1} & X_1 \\ \downarrow{\scriptstyle p_2} & & \downarrow \\ X_2 & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccc} 0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow{\scriptstyle i_1} \\ X_2 & \xrightarrow{i_2} & Y \end{array}$$

the first of which is cartesian and the second of which is cocartesian.

Example 2.1.3. A prototypical example of a pre-additive category is a ring R. One considers R as a category with a one object $*$, and sets $\mathrm{Hom}(*, *) = (R, +)$. The composition on $\mathrm{Hom}(*, *)$ is induced by the multiplication on R.

Let $\mathcal{A}$ and $\mathcal{B}$ be pre-additive categories (also called $\mathbb{Z}$-categories).

Definition 2.1.4. *A functor $F : \mathcal{A} \to \mathcal{B}$ is* additive *if for all $A, B \in \mathrm{Ob}\,\mathcal{A}$, the map $F : \mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{B}}(F(A), F(B))$ is a homomorphism of abelian groups.*

From now on, we will tacitly assume that $\mathsf{Fun}(\mathcal{A}, \mathcal{B})$ is the category of additive functors from $\mathcal{A}$ to $\mathcal{B}$.

Example 2.1.5. If $\mathcal{R}$ is a $\mathbb{Z}$-category with one object $*$, then $R = (\mathrm{Hom}(*, *), \mathrm{id}_*)$ is an associative ring with unit. An additive functor $F : \mathcal{R} \to \mathsf{Ab}$ is just a left module over R. Indeed, set $M = F(*)$, and for $\varphi \in R = \mathrm{Hom}(*, *)$ and $m \in M$, set $\varphi \cdot m = F(\varphi)(m)$. The definition of a functor fores the action of R on M to be additive and associative. A right module over R is just a functor $\mathcal{R}^\circ \to \mathsf{Ab}$.

The moral is that additive functors on $\mathbb{Z}$-categories with values in Ab should be thought of as representations.

Example 2.1.6 (Quivers and their representations)**.** If $\mathcal{Q}$ is the category corresponding to a quiver Q (see Example 1.1.6 above), then a representation of Q is just a functor $\mathcal{Q} \to \mathsf{Ab}$. We can modify $\mathcal{Q}$ to get an additive category. Let $\widetilde{\mathcal{Q}}$ have the same objects as $\mathcal{Q}$, but let $\mathrm{Hom}_{\widetilde{\mathcal{Q}}}(v_i, v_j)$ be the free abelian group on the set of all paths from v_i to v_j. The category of representations of Q is just the category of additive functors $\widetilde{\mathcal{Q}} \to \mathsf{Ab}$. More explicitly, a representation F of quiver Q assigns to each vertex of Q an abelian group, and to each arrow – a $\mathbb{Z}$-linear map between abelian groups. So the category of representations of a quiver Q consists of all diagrams of abelian groups and their morphisms of fixed shape Q.

Similarly we can modify the category $\mathcal{Q}_{\mathrm{com}}$ from Example 1.1.6 to get an additive category $\widehat{\mathcal{Q}}_{\mathrm{com}}$. Then the category of representations of $\widehat{\mathcal{Q}}_{\mathrm{com}}$ is just the category of additive functors from $\widehat{\mathcal{Q}}_{\mathrm{com}}$ to Ab. Elements of this category are *commutative diagrams* of abelian groups of fixed shape Q.

For a nice introduction to the theory of representations of quivers, see for example [CB92].

Example 2.1.7. For any additive category $\mathcal{A}$ functors $F : \Delta^\circ \to \mathcal{A}$ will form an additive category. For example, functors $F : \Delta^\circ \to \mathsf{Ab}$ are called *simplicial modules.*

Example 2.1.8. Recall that abelian presheaves on a topological space X are just functors $F : \mathsf{Open}(X)^\circ \to \mathsf{Ab}$. In our context, we should think of a presheaf as being a "representation" of the underlying topological space.

2.2 Center of a category and Bernstein trace

Definition 2.2.1. *Let $\mathcal{A}$ be an additive category. The* center *of $\mathcal{A}$ is defined by*

$$\mathsf{Z}(\mathcal{A}) = \mathrm{End}_{\mathsf{Fun}(\mathcal{A})}(\mathrm{id}_{\mathcal{A}})$$

where we define $\mathsf{Fun}(\mathcal{A}) = \mathsf{Fun}(\mathcal{A}, \mathcal{A})$.

Proposition 2.2.2. *For any ring R, we have $\mathsf{Z}(R\text{-}\mathsf{Mod}) \simeq Z(R)$, where $Z(R) = \{x \in R \mid [x, r] = 0, \forall r \in R\}$ is the usual center of R.*

Proof. Suppose $\alpha \in \mathsf{Z}(R\text{-}\mathsf{Mod})$. It defines a morphism $\alpha_M \colon M \to M$ for every R-module M. In particular, we have a morphism $\alpha_R \colon R \to R$. Any morphism of R-modules $\beta \colon R \to M$ is defined by a single element of R, namely, by $\beta(1) = m \in M$. Indeed, then we have

$\beta(r) = \beta(r \cdot 1) = r \cdot \beta(1) = rm$. Moreover, for any element $m \in M$ there exist unique morphism $R \to M$ with $1 \mapsto m$. Hence, morphism α_R is completely defined by $\alpha_R(1) \in R$.

By the definition of natural transformation, for any $\beta\colon R \to R$ we have the following commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\beta} & R \\ {\scriptstyle \alpha_R}\downarrow & & \downarrow{\scriptstyle \alpha_R} \\ R & \xrightarrow{\beta} & R \end{array}$$

If $\beta(1) = s \in R$, then commutativity of the diagram implies $rs = sr$. Since this is true for all $s \in R$, $r = \alpha_R(1)$ is an element of the center $Z(R)$.

We want to show that α is completely defined by α_R. Take any R-module M and any $m \in M$. We want to prove that $\alpha_M(m) \in M$ is completely defined by $\alpha_R(1)$. Again, from the definition of natural transformation, we have commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\beta\colon 1 \mapsto m} & M \\ {\scriptstyle \alpha_R}\downarrow & & \downarrow{\scriptstyle \alpha_M} \\ R & \xrightarrow{\beta\colon 1 \mapsto m} & M \end{array}$$

From this diagram we have $\alpha_M(m) = \alpha_M(\beta(1)) = \beta(\alpha_R(1)) = rm$. This proves the claim.

The opposite direction is easy. Whenever we have an element $r \in Z(R)$ we can define a natural transformation $\alpha^r\colon \mathrm{id}_{R\text{-}\mathsf{Mod}} \to \mathrm{id}_{R\text{-}\mathsf{Mod}}$ by $\alpha^r_M\colon M \to M$, $\alpha^r_M(m) = rm$. Since $r \in Z(R)$, α_M will be indeed a morphism of R-modules since $\alpha(sm) = rsm = srm = s\alpha(m)$. Any diagram

$$\begin{array}{ccc} N & \xrightarrow{\beta} & M \\ {\scriptstyle \alpha_N}\downarrow & & \downarrow{\scriptstyle \alpha_M} \\ N & \xrightarrow{\beta} & M \end{array}$$

will be commutative since for all $n \in N$ we have $\beta\alpha_N(n) = \beta(rn) = r\beta(n) = \alpha_M\beta(n)$. □

Example 2.2.3 (Bernstein trace formula)**.** Let V be a finite-dimensional vector space over a field k, $a \in \mathrm{End}_k(V)$. We define the trace $\mathrm{tr}_V(a) \in k \simeq \mathrm{Hom}_k(k,k)$ to be the composite

$$k \longrightarrow \mathrm{End}_k(V) \xrightarrow{\sim} V \otimes V^* \xrightarrow{a \otimes 1} V \otimes V^* \xrightarrow{\langle\cdot,\cdot\rangle} k.$$

Here $\mathrm{End}\, V \to V \otimes V^*$ is the inverse of the canonical map $v \otimes f \mapsto [x \mapsto f(x)\cdot v]$. It is not hard to check that this agrees with the usual definition of the trace. Let M be any k-vector space (not necessarily finite-dimensional). Define a linear map $\mathrm{tr}_V : \mathrm{End}_k(M \otimes V) \to \mathrm{End}_k(M)$ by letting $\mathrm{tr}_V(a)$ be the composite

$$M \longrightarrow M \otimes \mathrm{End}\, V \xrightarrow{\sim} M \otimes V \otimes V^* \xrightarrow{a \otimes \mathrm{id}_{V^*}} M \otimes V \otimes V^* \xrightarrow{\mathrm{id}_M \otimes \langle\cdot,\cdot\rangle} M.$$

Lemma 2.2.4. *If M is finite-dimensional, the composite $\mathrm{tr}_M \circ \mathrm{tr}_V : \mathrm{End}(M \otimes V) \to k$ is the usual trace $\mathrm{Tr}_{M\otimes V}$ on the vector space $M \otimes V$.*

We can apply this "generalized trace" to representation theory. Let $\mathfrak{g}$ be a semisimple complex Lie algebra, e.g. $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Let $\mathcal{A} = \mathfrak{g}\text{-}\mathsf{Mod}$ be the category of (not necessarily finite-dimensional) representations of $\mathfrak{g}$. The category $\mathcal{A}$ is an additive tensor category. That is, given two $\mathfrak{g}$-modules M and V, then $M \otimes_k V$ is naturally a $\mathfrak{g}$-module via $x(m \otimes v) = (xm) \otimes v + m \otimes xv$. For fixed V, we define the functor $F_V : \mathcal{A} \to \mathcal{A}$ by $F_V(M) = M \otimes V$. For $f : M \to N$, the induced map $F_V(f) : M \otimes V \to N \otimes V$ is $f \otimes \mathrm{id}_V : m \otimes v \mapsto f(m) \otimes v$.

Lemma 2.2.5. *If V is a finite-dimensional $\mathfrak{g}$-module, then $\mathrm{tr}_V : \mathrm{End}(F_V(M)) \to \mathrm{End}(M)$ is functorial in M, in the sense that it induces a linear map*

$$\mathrm{tr}_V : \mathrm{End}_{\mathsf{Fun}(\mathcal{A})}(F_V) \to \mathrm{End}_{\mathsf{Fun}(\mathcal{A})}(\mathrm{id}_{\mathcal{A}}) = \mathsf{Z}(\mathcal{A}).$$

Proof. We define tr_V by the following rule. Given any $\mathfrak{g}$-module M, and any $a \in \mathrm{End}_{\mathsf{Fun}}(F_V)$, set

$$\mathrm{tr}_V(a) = \{\mathrm{tr}_V(a_M) : M \in \mathrm{Ob}\,\mathcal{A}\}.$$

We need to check that $a_M \in \mathrm{Mor}\,\mathcal{A}$ implies $\mathrm{tr}_V(a_M) \in \mathrm{Mor}\,\mathcal{A}$. For any $f : M \to N$, we need the following diagram to commute

$$\begin{array}{ccc} M & \xrightarrow{\mathrm{tr}_V(a_M)} & M \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ N & \xrightarrow{\mathrm{tr}_V(a_N)} & N \end{array}$$

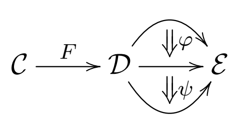
But this is easy to check. □

It is an open problem to compute $\mathrm{End}_{\mathsf{Fun}(\mathcal{A})}(F_V)$ in general.

Problem* Compute $\mathrm{End}_{\mathsf{Fun}(\mathcal{A})}(F_V)$ in the case where $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and $V = V_n$ the irreducible $(n+1)$-dimensional $\mathfrak{g}$-module.

We get around the computation of $\mathrm{End}_{\mathsf{Fun}(\mathcal{A})}(F_V)$ by constructing a map $\mathsf{Z}(\mathcal{A}) \to \mathrm{End}_{\mathsf{Fun}(\mathcal{A})}(F_V)$. Send a morphism of functors $\varphi : \mathrm{id}_{\mathcal{A}} \to \mathrm{id}_{\mathcal{A}}$ to the convolution $\varphi * F_V$, where $(\varphi * F_V)_M = \varphi_{F_V(M)}$.

Lemma 2.2.6. *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories with functors*



Then $(\psi \circ \varphi) * F = (\psi * F) \circ (\varphi * F)$.

Proof. We compute directly:

$$((\psi\circ\varphi)*F)_X = (\psi\circ\varphi)_{F(X)} = \psi_{F(X)}\circ\varphi_{F(X)} = (\psi*F)_X\circ(\varphi*F)_X = ((\psi*F)\circ(\varphi*F))_X.$$

□

Since $\mathfrak{g}\text{-}\mathsf{Mod}\simeq\mathcal{U}(\mathfrak{g})\text{-}\mathsf{Mod}$, we have $\mathrm{End}_{\mathsf{Fun}(\mathcal{A})}(\mathrm{id}_{\mathcal{A}})\simeq Z(\mathcal{U}(\mathfrak{g}))$ By the Poincaré-Birkhoff-Witt theorem, we can write $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h})\oplus(\mathfrak{n}^+\mathcal{U}(\mathfrak{g})+\mathcal{U}(\mathfrak{g})\mathfrak{n}^-)$, so we have a canonical projection $\psi:\mathcal{U}(\mathfrak{g})\twoheadrightarrow\mathcal{U}(\mathfrak{h})=\mathbb{C}[\mathfrak{h}^*]$. Here $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Denote by W the corresponding Weyl group. Then W acts on $\mathfrak{h}^*$ via the "dot action", i.e. $(w,\lambda)\mapsto w\cdot\lambda = w(\lambda+\rho)-\rho$, where $\rho=\frac{1}{2}\sum_{\alpha\in R^+}\alpha$ is one-half the sum of the positive roots.

Theorem 2.2.7 (Chevalley)**.** *The map $\psi|_{Z(\mathfrak{g})}: Z(\mathcal{U}(\mathfrak{g}))\to\mathbb{C}[\mathfrak{h}^*]^W$ is a ring isomorphism.*

For the next theorem, we need to set up some notation. We set $P(V)=\{\lambda\in\mathfrak{h}^*: V_\lambda\neq 0\}$, where $V_\lambda=\{v\in V: xv=\lambda(x)v,\ \forall x\in\mathfrak{h}\}$. The convolution of an element $f\in\mathbb{C}[\mathfrak{h}^*]$ with $P(V)\subset\mathfrak{h}^*$ is

$$(P(V)*f)(x) = \sum_{\lambda\in P(V)} f(x+\lambda).$$

Finally, the discriminant of W is the (skew-symmetric) polynomial

$$\Lambda = \Lambda(x) = \prod_{\alpha\in R^+}\langle\alpha, x\rangle.$$

Theorem 2.2.8. *The composite map*

$$\mathrm{Tr}_V:\mathbb{C}[\mathfrak{h}^*]\simeq Z(\mathcal{U}(\mathfrak{g}))\simeq\mathrm{End}_{\mathsf{Fun}(\mathcal{A})}(\mathrm{id}_{\mathcal{A}})\xrightarrow{F_V*}\mathrm{End}_{\mathsf{Fun}(\mathcal{A})}(F_V)\xrightarrow{\mathrm{tr}_V}\mathrm{End}_{\mathsf{Fun}(\mathcal{A})}(\mathrm{id}_{\mathcal{A}})\simeq Z(\mathcal{U}(\mathfrak{g}))\simeq\mathbb{C}[\mathfrak{h}^*]$$

is given by the formula

$$\mathrm{Tr}_V(f) = \frac{P(V)*(\Lambda f)}{\Lambda}. \tag{3.1}$$

Proof. Take a dominant integral weight $\lambda\in\mathfrak{h}^*$. This is, $\lambda=c_1\omega_1+\cdots+c_l\omega_l$ is a non-negative integral linear combination of fundamental weights $\omega_i\in\mathfrak{h}^*$. Fundamental weights are defined to be the basis of $\mathfrak{h}^*$ dual to the basis of simple roots $\Pi=\{\alpha_1,\ldots,\alpha_l\}$ in the sense that $\langle\omega_i,\alpha_j^\vee\rangle := \frac{2\langle\omega_i,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}=\delta_{ij}$. Suppose also that λ is regular, which means that $\langle\lambda+\rho,\alpha^\vee\rangle\neq 0$ for all roots $\alpha\in R$. In other words, λ is regular if its stabilizer in the Weyl group is trivial.

For such a weight λ define V_λ to be the irreducible (finite dimensional) $\mathfrak{g}$-module with highest weight $\rho-\lambda$. It is known that the the action of an element $z\in Z(\mathcal{U}(\mathfrak{g}))$ on V_λ is given by the multiplication by $f(\lambda)$, where $f\in\mathbb{C}[\mathfrak{h}^*]^W$ is the function on $\mathfrak{h}^*$ corresponding to z via Harish-Chandra isomorphism. Also, $\dim(V_\lambda)=c\cdot\Lambda(\lambda)$, where c is some constant (see [FH91][Cor.24.6]). This implies that $\mathrm{Tr}(z|_{V_\lambda}) = c\cdot(\Lambda f)(\lambda)$.

So to prove the formula (3.1) it is enough to show that the functions on both sides coincide for integral dominant regular λ. Choose such a λ. Let's compute the trace of the operator $\mathrm{Tr}_V(z)$ on V_λ. From one hand, it equals to $c\cdot(\Lambda\cdot\mathrm{Tr}_V(f))(\lambda)$. On the other hand, from Lemma 2.2.4 it equals $\mathrm{Tr}(z|_{V\otimes V_\lambda})$. It is well-known that $V \otimes V_\lambda$ is isomorphic to $\bigoplus_\mu V_{\lambda+\mu}$, where sum is taken over weights $\mu \in P(V)$ with multiplicities. Hence

$$\mathrm{Tr}(z|_{V\otimes V_\lambda}) = \sum_\mu \mathrm{Tr}(z|_{V_{\lambda+\mu}}) = c \cdot \sum_\mu \Lambda f(\lambda+\mu) = c \cdot (P(V) * (\Lambda f))(\lambda).$$

This proves the formula (3.1). □

2.3 Morita theory

For an associative unital ring A, $\mathsf{Mod}(A)$ denotes the category of all right (unital) A-modules, and A-Mod denotes the category of all left A-modules.

Definition 2.3.1. *Let A and B be (possibly noncommutative) rings. We call A and B* Morita-equivalent *(denoting $A \overset{M}{\sim} B$) if the categories $\mathsf{Mod}(A)$ and $\mathsf{Mod}(B)$ are equivalent.*

Remark 2.3.2. It turns out that $\mathsf{Mod}(A) \simeq \mathsf{Mod}(B)$ if and only if A-$\mathsf{Mod} \simeq B$-Mod, so there are no separate notions of "left Morita equivalence" and "right Morita equivalence."

Recall that a right A-module M is *projective* if it is the direct summand of a free module, i.e. there exists an A-module N such that $M \oplus N \simeq A^{\oplus I}$ as A-modules, where I is a (possibly infinite) index set.

Remark 2.3.3. It is obvious that any free module is projective. In general this is not true. For example, take $A = k[x,y,z]/(x^2+y^2+z^2=1)$ to be the coordinate ring of a 2-sphere. Consider $\varepsilon\colon A^3 \to A$ given by $\varepsilon(a,b,c) = ax+by+cz$. Since $\varepsilon(x,y,z)=1$, ε is onto, and so we have a splitting $A^3 = \mathrm{Ker}(\varepsilon) \oplus A$. Module $P = \mathrm{Ker}(\varepsilon)$ is by definition projective, and one can prove that P is not free, for example see [LS75][p.334].

Though, it turns out that for polynomial rings notions of projective modules and free modules coincide. The following theorem was originally known as Serre's problem.

Theorem 2.3.4 (Quillen-Suslin)**.** *If k is a field, then projective modules over $k[x_1,\dots,x_n]$ are free.*

More geometrically, the module P in the example above is not free meaning that that the tangent bundle on the sphere S^2 is not trivial. Moreover, the Quillen-Suslin theorem states that every algebraic vector bundle on $\mathbb{A}_k^n$ is trivial. Serre's conjecture was proven by Quillen and Suslin in 1976. Quillen's proof was much more intuitive. In the noncommutative setting, is it an open problem to classify projective modules over the ring of differential operators on $\mathbb{A}_k^n$.

Definition 2.3.5. *We say that M* generates *the category $\mathsf{Mod}(A)$ if $\mathrm{Hom}(M,-)$ is faithful, i.e. $f_1, f_2 : K \to L$ are equal if and only if $f_1 \circ g = f_2 \circ g$ for all $g : M \to K$. In this case we also call M a* generator *for the category $\mathsf{Mod}(A)$.*

An easy example of a generator is A itself, because $\mathrm{Hom}(A, -) \simeq \mathrm{id}_{\mathsf{Mod}(A)}$.

For a right A-module M, define $M^* = \mathrm{Hom}_A(M, A)$. The dual M^* is naturally a left A-module via $(a \cdot f)(x) = af(x)$. In fact, M^* is an $(A, \mathrm{End}_A M)$-bimodule, just as M is an $(\mathrm{End}_A M, A)$-bimodule.

Finally, let $MM^* = \mathrm{Im}(M \otimes_A M^* \to \mathrm{End}_A M)$ via the map $M \otimes M^* \to \mathrm{End}_A M$ that sends $m \otimes \varphi$ to the map $x \mapsto m\varphi(x)$. Similarly, let $M^*M = \mathrm{Im}(M^* \otimes_{\mathrm{End}_A M} M \to A)$, where $\varphi \otimes m \mapsto \varphi(m)$. Note that MM^* is a two-sided ideal in $\mathrm{End}_A M$, and M^*M is a two-sided ideal in A.

Theorem 2.3.6 (Dual basis)**.** *Let A be a ring, M an A-module.*

1. *M is projective if and only if there exists $m_i \in M$, $\varphi_i \in M^*$ such that for all $m \in M$, $\varphi_i(m) = 0$ for all but finitely many i, and one has*

$$m = \sum_i m_i \cdot \varphi_i(m).$$

2. *M is finitely generated and projective if and only if $MM^* = \mathrm{End}_A M$.*

If M is finitely generated and projective, one can choose a finite collection of $m_i \in M$, $\varphi_i \in M^*$ such that $m = \sum m_i\varphi_i(m)$ for all $m \in M$. One calls $\{m_i\}$ and $\{\varphi_i\}$ *dual bases*, even though $\{m_i\}$ may not be a basis of M.

As an exercise, prove the dual basis theorem, and show that M is a generator if and only if $M^*M = A$. There is a kind of duality here. The module M is finitely generated projective if and only if it is a direct summand of some $A^{\oplus n}$, while M is a generator if and only if A is a direct summand of some $M^{\oplus n}$.

Definition 2.3.7. *A right module M is a* progenerator *if M is a finitely generated projective generator in $\mathsf{Mod}(A)$.*

Theorem 2.3.8. *Let A and B be rings. The following are equivalent.*

1. $\mathsf{Mod}(A) \simeq \mathsf{Mod}(B)$
2. A-$\mathsf{Mod} \simeq B$-Mod
3. *there exists a progenerator M in $\mathsf{Mod}(A)$ such that $B \simeq \mathrm{End}_A M$*

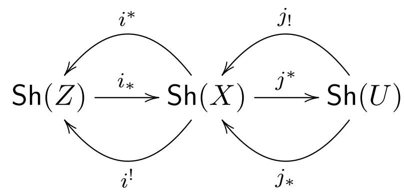
Example 2.3.9. Let A be a ring, $B = M_n(A)$ for some n. Since $M_n(A) \simeq \mathrm{End}_A(A^{\oplus n})$, the theorem shows that A and $M_n(A)$ are Morita equivalent.

Example 2.3.10. Let X be the affine line over a field k of characteristic zero. Let $A = \mathscr{D}(X) = k\langle x, \frac{d}{dx} : [\frac{d}{dx}, x] = 1\rangle$ be the ring of differential operators on X. Let $M = (x\frac{d}{dx} - 1)A + X^2A$; this is an ideal in A, and is in fact a progenerator for the category of A-modules. It turns out (Musson, 1991) that $\mathrm{End}_{\mathscr{D}(X)}(M) = \mathscr{D}(Y)$, where Y is the zero set of $y^2 - x^3$ on the affine plane.

As an exercise, show that if A and B are commutative rings, then A and B are Morita equivalent if and only if they are isomorphic.

2.4 Recollement (gluing) of abelian sheaves

Let X be a topological space, $Z \subset X$ a closed subspace, and let $U = X \setminus Z$ be its open complement. So we have a closed embedding $i : Z \hookrightarrow X$ and an open embedding $j : U \hookrightarrow X$. We would like to "decompose" the category of abelian sheaves $\mathsf{Sh}(X)$ on X using the categories $\mathsf{Sh}(Z)$ and $\mathsf{Sh}(U)$ of abelian sheaves on Z and U. The adjunctions we will obtain are a part of Grothendieck's "yoga" of the six functors, and fit into a diagram:



in which $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$ are adjoint triples.

It is easy to define the functors i^*, i_*, j^*, j_* because they make sense for any map between topological spaces. Indeed, if $f : X \to Y$ is continuous, recall that we defined $f_* : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$ by

$$(f_*\mathscr{F})(U) = \mathscr{F}(f^{-1}(U)).$$

Formally, one can define f^* by requiring it to be the left adjoint of f_*. To show that f^* exists, one constructs it in one of several ways. The most common is to let $f^*\mathscr{F}$ be the sheaf associated to the presheaf

$$U \mapsto \varinjlim_{\substack{V \supset f(U) \\ \text{open}}} \mathscr{F}(V).$$

Alternatively, we can define $f^*\mathscr{F}$ directly by

$$(f^*\mathscr{F})(U) = \{s : U \to \mathrm{Et}(\mathscr{F}) : s(x) \in \mathscr{F}_{f(x)} \text{ for all } x \in U\}.$$

Let's return to the setting where $Z \subset X$ is closed and $U = X \setminus Z$. The functor $j_!$ is "extension by 0," i.e.

$$(j_!\mathscr{F})(V) = \begin{cases} 0 & \text{if } V \not\subset U \\ \mathscr{F}(V) & \text{if } V \subset U \end{cases}$$

The functor $i^! : \mathsf{Sh}(X) \to \mathsf{Sh}(Z)$ is "restriction with compact support."

If R is a commutative ring I is an ideal in R. Let $S = R/I$, and write $i : \mathrm{Spec}(S) \hookrightarrow \mathrm{Spec}(R)$ for the induced embedding. Then $(i^*, i_*, i^!)$ are precisely the functors defined in the context of a closed embedding $i : Z \to X$ of topological spaces.

As an exercise, check the following properties of the six functors.

1. $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$ are adjoint triples

2. the unit $\mathrm{id}_{\mathsf{Sh}(U)} \to j^*j_!$ and counit $j^*j_* \to \mathrm{id}_{\mathsf{Sh}(U)}$ are isomorphisms

3. the unit $\mathrm{id}_{\mathsf{Sh}(Z)} \to i^! i_*$ and counit $i^* i_* \to \mathrm{id}_{\mathsf{Sh}(Z)}$ are isomorphisms

4. $i^! j_* = 0$ and $i^* j_! = 0$.

5. there are canonical (pointwise) exact sequences

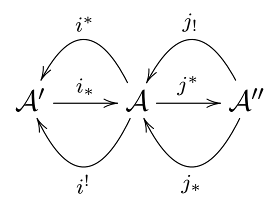
$$j_! j^* \longrightarrow \mathrm{id} \longrightarrow i_* i^* \longrightarrow 0$$

$$0 \longrightarrow i_* i^! \longrightarrow \mathrm{id} \longrightarrow j_* j^*$$

Later on, we will extend these sequences to exact triangles in the derived category.

Example 2.4.1 (projective plane)**.** Let $X = \mathbb{P}^2_{\mathbb{C}}$ be the projective plane over $\mathbb{C}$. Write $X = \operatorname{Proj} \mathbb{C}[x, y, z]$, and let $Z = \{z = 0\} = \operatorname{Proj} \mathbb{C}[X, Y]$ be the "line at infinity." The injection $i : Z \hookrightarrow X$ is induced by the graded homomorphism $\mathbb{C}[x, y, z] \to \mathbb{C}[x, y]$ that sends z to 0. If $\mathscr{F}$ is a quasicoherent sheaf on X, then $\mathscr{F} = \widetilde{M}$ for some graded $\mathbb{C}[x, y, z]$-module M. It turns out that $i^*\mathscr{F} = \widetilde{i^* M} = \widetilde{M/z}$. The sequence $M \to i_* i^* M \to 0$ is obviously exact, so sheafifying we get the exact sequence $\mathscr{F} \to i_* i^* \mathscr{F} \to 0$.

Definition 2.4.2. *An additive category $\mathcal{A}$ is said to be a* recollement *of $\mathcal{A}'$ and $\mathcal{A}''$ if there exist six functors*

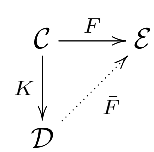


satisfying properties 1–5 above.

It is an easy consquence of the definitions that $i_* : \mathcal{A}' \to \mathcal{A}$ is an embedding with image $\{A : j^* A = 0\}$. A good reference for all of this is [BBD82]. An application to number theory (the Arvin-Verdier duality theorem for Galois cohomology) can be found in [Maz73].

2.5 Kan extensions

Consider two functors $F : \mathcal{C} \to \mathcal{E}$, $K : \mathcal{C} \to \mathcal{D}$. We are interested in finding $\bar{F}$ making the following diagram commute:



A typical example is as follows. Let $S \subset \operatorname{Mor} \mathcal{C}$ and $T \subset \operatorname{Mor} \mathcal{E}$ be classes of morphisms. We can construct categories $\mathcal{C}[S^{-1}]$ and $\mathcal{E}[T^{-1}]$, called the localizations of $\mathcal{C}$ and $\mathcal{E}$ at S and T.

We have a localization functors $\ell_S : \mathcal{C} \to \mathcal{C}[S^{-1}]$ and $\ell_T : \mathcal{E} \to \mathcal{E}[T^{-1}]$. Given $F : \mathcal{C} \to \mathcal{E}$, can we extend F to a functor $\bar{F} : \mathcal{C}[S^{-1}] \to \mathcal{E}[T^{-1}]$?

Our extension problem is obviously not possible in general. For instance, there might exist morphisms φ and ψ in $\mathcal{C}$ such that $F(\varphi) \neq F(\psi)$, but $K(\varphi) = K(\psi)$. Similarly, there might exist objects X and Y such that $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \varnothing$ and $\mathrm{Hom}_{\mathcal{E}}(FX, FY) = \varnothing$, but $\mathrm{Hom}_{\mathcal{D}}(KX, KY) \neq \varnothing$. We will content ourselves with asking for a "best possible approximation" of an extension of F to $\mathcal{D}$. One can do this by looking at universal natural transformations from F (left Kan extensions), or to F (right Kan extensions).

Definition 2.5.1. *A* left Kan extension *of F along K is a functor $\mathrm{Lan}_K(F) : \mathcal{D} \to \mathcal{E}$ together with $\eta : F \to \mathrm{Lan}_K(F) \circ K$ which is universal among all pairs $(G : \mathcal{D} \to \mathcal{E}, \varepsilon : F \to G \circ K)$, in the sense that for any such pair there exists a unique $\alpha : \mathrm{Lan}_K F \to G$ such that there is a commutative diagram*

$$\begin{array}{ccc} F & \xrightarrow{\ \varepsilon\ } & G \circ K \\ & \searrow^{\eta} & \uparrow{\scriptstyle \alpha K} \\ & & \mathrm{Lan}_K F \circ K \end{array}$$

A left Kan extension of $F : \mathcal{C} \to \mathcal{E}$ along $K : \mathcal{C} \to \mathcal{D}$ represents the functor $\mathcal{E}^{\mathcal{C}}(F, \circ K) : \mathsf{Fun}(\mathcal{D}, \mathcal{E}) \to \mathsf{Set}$ that sends $G : \mathcal{D} \to \mathcal{E}$ to the set $\mathrm{Hom}_{\mathsf{Fun}(\mathcal{C},\mathcal{D})}(F, G \circ K)$. In other words, there is a natural isomorphism

$$\mathrm{Hom}_{\mathsf{Fun}(\mathcal{D},\mathcal{E})}(\mathrm{Lan}_K F, G) = \mathrm{Hom}_{\mathsf{Fun}(\mathcal{C},\mathcal{E})}(F, G \circ K).$$

Let Cat be the category of all categories. A functor $K : \mathcal{C} \to \mathcal{D}$ is just a morphism in Cat. It induces, for any $\mathcal{E}$, a functor $K^* : \mathcal{E}^{\mathcal{D}} \to \mathcal{E}^{\mathcal{C}}$ given by $G \mapsto G \circ K$. The functor $\mathrm{Lan}_K : \mathcal{E}^{\mathcal{C}} \to \mathcal{E}^{\mathcal{D}}$ is the left adjoint to K^*.

Dually, we can define right Kan extensions.

Definition 2.5.2. *The* right Kan extension *of F along K, written $\mathrm{Ran}_K F$, is a functor $\mathrm{Ran}_K F : \mathcal{D} \to \mathcal{E}$ with a natural transformation $\sigma : \mathrm{Ran}_K(F) \circ K \to F$ which is universal among all pairs $(G : \mathcal{D} \to \mathcal{E}, \varepsilon : G \circ K \to F)$, in the sense that for any such pair there exists a unique $\beta : G \to \mathrm{Ran}_K F$ such that the following diagram commutes:*

$$\begin{array}{ccc} G \circ K & \xrightarrow{\ \beta K\ } & \mathrm{Ran}_K F \circ K \\ & \searrow_{\varepsilon} & \downarrow{\scriptstyle \sigma} \\ & & F \end{array}$$

In other words, there is a natural isomorphism

$$\mathrm{Hom}_{\mathsf{Fun}(\mathcal{D},\mathcal{E})}(G, \mathrm{Ran}_K F) \simeq \mathrm{Hom}_{\mathsf{Fun}(\mathcal{C},\mathcal{E})}(G \circ K, F)$$

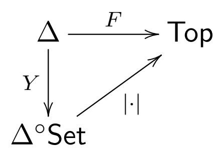
Assuming left and right Kan extensions exist, we have an adjoint triple $(\mathrm{Lan}_K, K^*, \mathrm{Ran}_K)$.

Lemma 2.5.3. *If K is fully faithful and* $\operatorname{Lan}_K F$ *exists, then* $\eta : F \to \operatorname{Lan}_K(F) \circ K$ *is a natural isomorphism.*

Example 2.5.4. In this example, Δ denotes the simplicial category, whose objects are the integers $1, 2, \ldots,$ and whose morphisms $f : n \to m$ are nondecreasing functions $[n] \to [m]$. Define F to be the realization functor $\Delta \to \mathsf{Top}$ that assigns to n the n-th simplex

$$\Delta_n = \left\{ (1_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum x_i = 1, x_i \geqslant 0 \right\}.$$

Let $K = Y : \Delta \to \Delta^\circ\mathsf{Set}$ be the Yoneda embedding from Δ into the category of simplicial sets. Write Δ^n for the image of $n \in \Delta$ under Y. We have a diagram:



It is a good exercise to check that $\operatorname{Lan}_Y F = |-|$. By Lemma 2.5.3, the natural transformation $F \to |\cdot| \circ Y$ is an isomorphism, i.e. $\Delta_n = |\Delta^n|$.

Chapter 4

Classical homological algebra

1 Abelian categories

1.1 Additive categories

Recall a category $\mathcal{A}$ is called a pre-additive (or $\mathbb{Z}$-) category if it satisfies the axiom AB1 below. We say that $\mathcal{A}$ is *additive* if $\mathcal{A}$ is preadditive, and also satisfies AB2 and AB3.

[**AB1**] each $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is given the structure of an abelian group in such a way that compositions

$$\mathrm{Hom}_{\mathcal{A}}(X, Y) \times \mathrm{Hom}_{\mathcal{A}}(Y, Z) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$$

are bilinear;

[**AB2**] $\mathcal{A}$ has initial object $\varnothing$, terminal object $*$ with $\varnothing = *$;

[**AB3**] $\mathcal{A}$ has finite products.

We call $\varnothing$ the *zero object*, denoted 0. (Note that we can define $\varnothing_{\mathcal{A}} = \varinjlim(\varnothing \to \mathcal{A})$, where $\varnothing$ is the empty category and $\varnothing \to \mathcal{A}$ is the unique functor.)

Let I be a set, and consider I as a category with no (non-identity) morphisms. Then a diagram of shape I in $\mathcal{A}$ is just a collection $\{A_i : i \in I\}$ of objects in $\mathcal{A}$. We set

$$\prod_{i \in I} X_i = \varprojlim\{I \to \mathcal{A}\} \qquad \text{(product)}$$

$$\coprod_{i \in I} A_i = \varinjlim\{I \to \mathcal{A}\} \qquad \text{(coproduct)}$$

The simplest case is when $I = \{0, 1\}$. One obtains products $X \times Y$ and coproducts $X \sqcup Y$.

Lemma 1.1.1. *Let $\mathcal{A}$ be an additive category. If $X \times Y$ exists, then so does $X \sqcup Y$ and $X \times Y \simeq X \sqcup Y$ (canonically).*

Proof. Recall that $X \times Y$ represents the functor $\mathcal{A}^\circ \to \mathsf{Set}$ defined by

$$Z \mapsto \mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y).$$

That is, there is a natural isomorphism $\psi : \mathrm{Hom}(Z, X \times Y) \to \mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y)$, that maps $\phi : Z \to X \times Y$ to the pair $(p_X \circ \phi, p_Y \circ \phi)$, where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ correspond to $\mathrm{id}_{X \times Y}$.

Define $i_X : X \to X \times Y$ and $i_Y : Y \to X \times Y$ by $i_X = \psi^{-1}(\mathrm{id}_X, 0)$ and $i_Y = \psi^{-1}(0, \mathrm{id}_Y)$. It is easy to check that

$$\begin{aligned} p_X \circ i_X &= \mathrm{id}_X \\ p_Y \circ i_Y &= \mathrm{id}_Y \\ p_X \circ i_Y &= 0 \\ p_Y \circ i_X &= 0 \end{aligned}$$

These identities formally imply $i_X \circ p_X + i_Y \circ p_Y = \mathrm{id}_{X \times Y}$. For, if we call the left-hand map ϕ, we get $p_X \circ \phi = p_X$ and $p_Y \circ \phi = p_Y$. But $\mathrm{id}_{X \times Y}$ also satisfies this, so uniqueness gives $\phi = \mathrm{id}_{X \times Y}$.

Given i_X and i_Y, we can define an isomorphism

$$\mathrm{Hom}(X, Z) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X \times Y, Z)$$

by $(\phi, \psi) \mapsto (\pi \circ p_X, \psi \circ p_Y)$, with inverse $\chi \mapsto (\chi \circ i_X, \chi \circ i_Y)$. Since $X \sqcup Y$ corepresents $\mathrm{Hom}(X, Z) \times \mathrm{Hom}(Y, Z)$, we have $X \times Y \simeq X \sqcup Y$. □

Exercise Let $\mathcal{A}$ be an additive category. Show that finite products and finite coproducts exist, and coincide. On the other hand, show that *infinite* products and coproducts need not be the same. For any $X, Y \in \mathcal{A}$, we can define the *diagonal* $\Delta_X : X \to X \times X$ by $\Delta_X = \mathrm{id}_X \times \mathrm{id}_X$, and the *folding map* $\nabla_Y : Y \sqcup Y \to Y$ by $\nabla_Y = \mathrm{id}_Y \sqcup \mathrm{id}_Y$. (In fact, these exist in any category with products and coproducts.) Show that the abelian group structure on $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is given by

$$f + g = \nabla_Y \circ (f \times g) \circ \Delta_X.$$

This exercise has a very important consequence: being additive is not "extra structure" on $\mathcal{A}$, but an intrinsic property of $\mathcal{A}$! An arbitrary category $\mathcal{A}$ is additive if and only if $\mathcal{A}$ has finite products and coproducts which coincide. (Note that the condition $\varnothing_{\mathcal{A}} \simeq *_{\mathcal{A}}$ is a consequence of requiring finite products and coproducts to coincide, since $\varnothing$ is empty coproduct and $*$ is the empty product.)

Exercise Find a categorical definition of $-f$ for any $f : X \to Y$. Also, show that if $\mathcal{A}$ is additive, then $\mathcal{A}^\circ$ is additive, and that $\mathcal{A} \times \mathcal{B}$ is additive whenever $\mathcal{A}$ and $\mathcal{B}$ are. Moreover, if $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are categories, show that there is an equivalence of categories

$$\mathsf{Fun}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \simeq \mathsf{Fun}(\mathcal{A}, \mathsf{Fun}(\mathcal{B}, \mathcal{C})).$$

If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are additive, show this equivalence restricts to an equivalence

$$\mathsf{Fun}_{\text{add}}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \simeq \mathsf{Fun}_{\text{add}}(\mathcal{A}, \mathsf{Fun}_{\text{add}}(\mathcal{B}, \mathcal{C})).$$

Recall that a functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories is *additive* if $F : \text{Hom}_{\mathcal{A}}(X, Y) \to \text{Hom}_{\mathcal{B}}(FX, FY)$ is a homomorphism of abelian groups for each $X, Y \in \text{Ob}\,\mathcal{A}$. If R, S are rings and B is an (R, S)-bimodule, then the functor $M \mapsto B \otimes_R M$ is an additive functor. On the other hand, if R is a commutative ring, the functor $M \mapsto M^{\otimes 2}$ on R-modules is *not* additive.

Exercise Show that if $F : \mathcal{A} \to \mathcal{B}$ is additive, then the canonical map $F(X \oplus Y) \to F(X) \oplus F(Y)$ is an isomorphism.

Exercise Show that if $\mathcal{A}$ is additive, then the "functors of points" $h_X : \mathcal{A}^\circ \to \mathsf{Set}$ are actually functors $h_X : \mathcal{A}^\circ \to \mathsf{Ab}$, and similarly for $h^X = \text{Hom}(X, -)$. Show that the Yoneda embedding $h : \mathcal{A} \to \mathsf{Fun}(\mathcal{A}^\circ, \mathsf{Ab})$ is additive.

Lemma 1.1.2. *If $F : \mathcal{A} \to \mathcal{B}$ is an additive functor and G is an adjoint to F, then G is additive. If G is right adjoint to F, then the isomorphisms*

$$\text{Hom}(FX, Y) \simeq \text{Hom}(X, GY)$$

are isomorphisms of abelian groups, and similarly if G is left adjoint to F.

Proof. Let $\psi : \text{Hom}(FX, Y) \to \text{Hom}(X, GY)$ be the isomorphism $\phi \mapsto \eta_Y \circ F(\phi)$, where $\eta : \text{id} \to GF$ is the counit of the adjunction. The map $\phi \mapsto F(\phi)$ is additive because F is. Moreover, $\phi \mapsto \eta_Y \circ \phi$ is additive because $\mathcal{B}$ is additive. It follows that ψ is a homomorphism. From the equivalence

$$\mathsf{Fun}_{\text{add}}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \simeq \mathsf{Fun}_{\text{add}}(\mathcal{A}, \mathsf{Fun}_{\text{add}}(\mathcal{B}, \mathcal{C})),$$

it follows that G must be additive. □

1.2 Non-additive bimodules

Let R be an associative unital ring, and let R-Mod be the category of left R-modules. Let R-Bimod be the category of bimodules over R. Both R-Mod and R-Bimod are abelian categories. Define $\mathsf{F}(R)$ to be the full subcategory of R-Mod consisting of objects isomorphic to $R^{\oplus n}$ for some $n \in \mathbb{N}$. Better yet, we could consider $\mathsf{P}(R)$, the category of finitely generated projective R-modules.

Definition 1.2.1. *A* non-additive bimodule *is just a functor $T : \mathsf{F}(R) \to R\text{-}\mathsf{Mod}$.*

Set $\mathscr{F}(R) = \mathsf{Fun}(\mathsf{F}(R), R\text{-}\mathsf{Mod})$. For example, the inclusion $\mathsf{F}(R) \hookrightarrow R$-$\mathsf{Mod}$ is a non-additive bimodule. There is a canonical functor $\Theta : R\text{-}\mathsf{Bimod} \to \mathscr{F}(R)$ defined by the rule $M \mapsto (M \otimes - : R^{\oplus n} \mapsto M^{\oplus n})$. Note that $\mathscr{F}(R)$ is an abelian category.

Theorem 1.2.2. *The functor Θ is fully faithful, with essential image the full subcategory of $\mathscr{F}(R)$ consisting of additive functors $\mathsf{F}(R) \to R\text{-}\mathsf{Mod}$.*

Proof. We give a construction showing that the image of Θ consists of additive functors. If $T \in \mathscr{F}(R)$ is additive, define $M = T(R)$. By definition, M is a left R-module. The right R-module structure on M is defined by a ring homomorphism $\lambda : R^\circ \to \operatorname{End}_R(M)$. Note that $R^\circ = \operatorname{End}_R(R)$, where R is treated as a left R-module. If we identify R with $\operatorname{End}_R(R)$, then the fact that T is a functor yields a homomorphism

$$\operatorname{End}_R(R) \to \operatorname{End}_R(T(R)) = \operatorname{End}_R(M)$$

hence M is an R-bimodule. □

Remark 1.2.3. It turns out that Hochschild (co)homology can be extended to the category of non-additive bimodules, yielding *topological Hochschild (co)homology.* Details can be found in [BL04].

Theorem 1.2.4. *The functor $\Theta : R\text{-}\mathsf{Bimod} \to \mathscr{F}(R)$ has both a left and right adjoint.*

Proof. Write Θ^* for the left adjoint, and $\Theta^!$ for the right adjoint of Θ. We will construct Θ^* and $\Theta^!$ directly. Given $X \in \operatorname{Ob}(R\text{-}\mathsf{Mod})$, define six natural morphisms $\delta^i : X \to X \oplus X$, $d_i : X \oplus X \to X$ for $i \in \{0, 1, 2\}$. We have

$$\begin{aligned} \delta^0(x) &= (0, x) \\ \delta^1(x) &= (x, x) \\ \delta^2(x) &= (0, x). \end{aligned}$$

Assume $X \in \operatorname{Ob}(\mathsf{F}(R))$. Then δ^i and d_i are elements of $\operatorname{Mor}(\mathsf{F}(R))$. Define, for any $T \in \operatorname{Ob}(\mathscr{F}(R))$,

$$\begin{aligned} \delta^X(T) &= T(\delta^0) - T(\delta^1) + T(\delta^2) : T(X) \to T(X \oplus X) \\ d_X(T) &= T(d_0) - T(d_1) + T(d_2) : T(X \oplus X) \to X \end{aligned}$$

It is easy to check that for any T,

$$\begin{aligned} X &\mapsto \operatorname{Ker}(\delta^X(T)) \\ X &\mapsto \operatorname{Coker}(d_X(T)) \end{aligned}$$

are additive functors. Tate $X = R$, viewed as a left R-module. We define

$$\begin{aligned} \Theta^!(T) &= \operatorname{Ker}(\delta^R(T) : T(R) \to T(R) \oplus T(R)) \\ \Theta^*(T) &= \operatorname{Coker}(d_R(T) : T(R) \oplus T(R) \to T(R)). \end{aligned}$$

We claim that this definition actually gives left and right adjoints to Θ. That is, for any $M \in R\text{-}\mathsf{Mod}$ and $T \in \operatorname{Ob}(\mathscr{F}(R))$, we have isomorphisms

$$\begin{aligned} \operatorname{Hom}_{R\text{-}\mathsf{Bimod}}(\Theta^*(T), M) &\simeq \operatorname{Hom}_{\mathscr{F}(R)}(T, M \otimes_R -) = \operatorname{Hom}_{\mathscr{F}(R)}(T, \Theta M) \\ \operatorname{Hom}_{R\text{-}\mathsf{Bimod}}(M, \Theta^!(T)) &\simeq \operatorname{Hom}_{\mathscr{F}(R)}(M \otimes_R -, T) = \operatorname{Hom}_{\mathscr{F}(R)}(\Theta(M), T). \end{aligned}$$

□

Exercise (Bernstein trace) Recall that if we have a functor $F:\mathcal{A}\to\mathcal{B}$ that sits in an adjoint triple (E,F,G), then any natural transformation $\gamma : G\to E$, we have a canonical trace $\operatorname{tr}(\gamma) : \operatorname{Hom}_{\mathcal{B}}(FX,FY)\to\operatorname{Hom}_{\mathcal{A}}(X,Y)$. Take $a\in R$ and $\widehat{a}\in\operatorname{End}_R(R)$ given by $x\mapsto x\cdot a$. Let $\nu_a:\Theta^!\to\Theta^*$ be a natural transformation, where $\nu_a(T)$ is defined by the composite

$$\Theta^!(T)\hookrightarrow T(R)\xrightarrow{T(\widehat{a})} T(R)\twoheadrightarrow\Theta^*(T)$$

The exercise is: compute $\operatorname{tr}(a):=\operatorname{tr}(\nu_a)$.

There is a notion of "polynomial approximations" of non-additive bimodules, based on the notion of a *polynomial maps* between abelian groups. The basic idea goes back to Eilenberg and MacLane. Given a (set-theoretic) map $f:A\to B$ between abelian groups, the *defect* of f is

$$(a_1\mid a_2)_f=f(a_1+a_2)-f(a_1)-f(a_2).$$

The map f is additive if and only if $(-\mid -)_f : A\times A\to B$ is the zero mapping. Inductively, we define the *n-th defect* of f as

$$(a_1\mid a_2\mid\cdots\mid a_n)_f=(a_1\mid\cdots\mid a_{n-1}+a_n)_f-(a_1\mid\cdots\mid a_{n-1})-(a_1\mid\cdots\mid\widehat{a}_{n-1}\mid a_n)_f.$$

We say that f is *polynomial of degree* $\leqslant n$ if the n-th defect of f is identically zero.

Definition 1.2.5. *A (non-additive) functor $T:\mathcal{A}\to\mathcal{B}$ between additive categories is* polynomial of degree $\leqslant n$ *if for all $X,Y\in\operatorname{Ob}\mathcal{A}$,*

$$T:\operatorname{Hom}_{\mathcal{A}}(X,Y)\to\operatorname{Hom}_{\mathcal{B}}(TX,TY)$$

is polynomial of degree $\leqslant n$.

Going back to our original example, we can define $\mathscr{F}_n(R)$ to be the full subcategory of $\mathscr{F}(R)$ consisting of polynomial functors of degree $\leqslant n$. There are canonical injections $\mathscr{F}_n(R)\hookrightarrow\mathscr{F}(R)$, and $\mathscr{F}_0(R)\simeq R\text{-}\mathsf{Bimod}$.

Example 1.2.6 (Theorem of the cube)**.** For a scheme X, the *Picard group* of X is the set of isomorphism classes of invertible sheaves, with group operation induced by the tensor product. One can prove that $\operatorname{Pic}(X)\simeq\operatorname{H}^1(X,\mathscr{O}_X^\times)$. Let k be a field, and let AbVar_k be the category of abelian varieties over k. The "theorem of the cube" states that the Picard functor $\operatorname{Pic}:\mathsf{AbVar}_k^\circ\to\mathsf{Ab}$ is quadratic. There is a way of defining "quadratic functors" for any pointed category, and in that generality, the theorem of the cube says that the Picard functor is quadratic on the whole category of pointed projective varieties.

1.3 Abelian categories

Recall that additive categories are categorized by some basic axioms (see 1.1).

Definition 1.3.1. *An additive category $\mathcal{A}$ is* abelian *if it satisfies an extra axiom AB4. We say $\mathcal{A}$ is a* Grothendieck category *if in addition it satisfies AB5 (see below).*

Before we can state the extra axioms, we need to define kernels and cokernels in arbitrary additive categories. Let $\mathcal{A}$ be an additive category, $\varphi : X \to Y$ a morphism in $\mathcal{A}$. Consider the functor $\underline{\mathrm{Ker}}(\varphi) : \mathcal{A}^\circ \to \mathsf{Ab}$ defined by

$$Z \mapsto \mathrm{Ker}_{\mathsf{Ab}}(\varphi_* : \mathrm{Hom}_{\mathcal{A}}(Z, X) \to \mathrm{Hom}_{\mathcal{A}}(Z, Y)).$$

If $\underline{\mathrm{Ker}}(\varphi)$ is representable, then its representing object is called the *kernel* of φ denoted by $\mathrm{Ker}(\varphi)$. If $\mathrm{Ker}(\varphi)$ exists, we have a (by definition) an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(Z, \mathrm{Ker}\,\varphi) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(Z, X) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(Z, Y).$$

A similar definition for $\mathrm{Coker}(\varphi)$ does not work (the obvious analog of the functor above is not representable). If we take $Z = K$, then $\psi(\mathrm{id}_K)$ is a morphism $k : K \to X$. Hence $\underline{\mathrm{Ker}}(\varphi)$ is represented by the pair $(K, k : K \to X)$. The kernel has a much easier definition. Let I be the category $\{\bullet \rightrightarrows \bullet\}$. A diagram of shape I is just a diagram

$$X_0 \underset{\sigma_1}{\overset{\sigma_0}{\rightrightarrows}} X_1.$$

A cone over this diagram is essentially a diagram

$$Y \xrightarrow{f} X_0 \underset{\sigma_1}{\overset{\sigma_0}{\rightrightarrows}} X_1.$$

such that $\sigma_0 f = \sigma_1 f$. The *equalizer* of the diagram is the limit of the corresponding functor $I \to \mathcal{A}$. It is a good exercise to check that our definition of the kernel is equivalent to letting $\mathrm{Ker}(\varphi)$ be the equalizer of the diagram

$$X \underset{0}{\overset{\varphi}{\rightrightarrows}} Y.$$

A naive definition of the cokernel would be to look at the functor $Z \mapsto \mathrm{Coker}(h_X(Z) \to h_X(Z))$. But this does not agree with the classical definition for abelian groups. In fact, this functor is not usually representable. Indeed, suppose this functor is represented by some C. Then we would have exact sequences

$$\mathrm{Hom}(Z, X) \longrightarrow \mathrm{Hom}(Z, Y) \longrightarrow \mathrm{Hom}(Z, C) \longrightarrow 0$$

for all $Z \in \mathrm{Ob}\,\mathcal{A}$. These sequences are not all exact even if $\mathcal{A} = \mathsf{Ab}$. For example, let $X = Y = \mathbb{Z}$, and let φ be multiplication by n. If $C = \mathbb{Z}/n$, then the sequence is

$$\mathrm{Hom}(Z, X) \longrightarrow \mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}) \longrightarrow 0$$

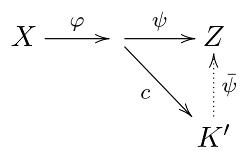
which is not exact. (Strictly speaking, this only shows that $C = \mathbb{Z}/n$ does not work.)

So let $\varphi : X \to Y$ be a morphism. The correct definition is the following.

Definition 1.3.2. Coker(φ) *is the representing object (if it exists) of the functor* $\underline{\text{Coker}}(\varphi) : \mathcal{A}^\circ \to \mathsf{Ab}$ *defined by*

$$Z \mapsto \operatorname{Ker}(\operatorname{Hom}(Y, Z) \xrightarrow{\varphi^*} \operatorname{Hom}(X, Z)).$$

If K' represents $\underline{\text{Coker}}(\varphi)$, we have a canonical morphism $c : Y \to K'$. The pair (K', c) has a universal property: if $\psi : Y \to Z$ is such that $\psi \circ \varphi = 0$, then there is a unique $\bar{\psi} : K' \to Z$ such that $\psi = \bar{\psi} \circ c$, as in the diagram



Alternatively, Coker(φ) is the coequalizer of

$$X \underset{0}{\overset{\varphi}{\rightrightarrows}} Y.$$

The axiom defining an abelian category is due to MacLane and Grothendieck. We say an additive category $\mathcal{A}$ is *abelian* if it satisfies

[AB4] Every $\varphi : X \to Y$ can be decomposed in the following way:

$$K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{k'} K'$$

where

1. $\varphi = j \circ i$
2. $(K, k) = \operatorname{Ker}(\varphi)$ and $(K', k') = \operatorname{Coker}(\varphi)$
3. $(I, i) = \operatorname{Coker}(k)$ and $(I, j) = \operatorname{Ker}(k')$.

Let's see what this axiom requires in the case $\mathcal{A} = \mathsf{Ab}$. Let $\varphi : A \to B$ be a homomorphism of abelian groups. Then we can decompose φ as

$$\operatorname{Ker}(\varphi) \overset{k}{\hookrightarrow} X \overset{i}{\twoheadrightarrow} X/\operatorname{Ker}(\varphi) = I \simeq \operatorname{Im}(\varphi) \overset{j}{\hookrightarrow} Y \overset{k'}{\twoheadrightarrow} \operatorname{Coker}(\varphi)$$

So essentially, axiom AB4 requires that the "first isomorphism theorem" holds in $\mathcal{A}$.

It was Grothendieck's insight that AB4 is equivalent to the combination of the following two axioms:

[AB4.1] Ker(φ) and Coker(φ) exist for all φ;

[AB4.2] If $\varphi : X \to Y$ is such that $\operatorname{Ker}(\varphi) = \operatorname{Coker}(\varphi) = 0$, then φ is an isomorphism.

Traditionally, if $\operatorname{Ker}(\varphi) = 0$ we say that φ is *mono* in $\mathcal{A}$, and if $\operatorname{Coker}(\varphi) = 0$ we say that φ is *epi*. Axiom AB4.2 just says that maps which are both mono and epi are isomorphisms.

Suppose we have AB4.1. Then for all $\varphi : X \to Y$, we can let $(K, k) = \operatorname{Ker}(\varphi)$ ad $(I, i) = \operatorname{Coker}(\varphi)$. Similarly, we can let $(K', k') = \operatorname{Coker}(\varphi)$ and $(I', i') = \operatorname{Ker}(k')$. In addition, the definition kernels and cokernels imply the existence of a canonical morphism $\ell : I \to I'$ such that $\varphi = j \circ \ell \circ i$. One can check this by using the universal property of kernels and cokernels. Moreover, $\operatorname{Ker}(\ell) = \operatorname{Coker}(\ell) = 0$. If AB4.2 holds, ℓ is an isomorphism, so the decomposition required by AB4 exists.

To see that AB4 implies AB4.2, note that $\operatorname{Coker}(0 \to X) = \mathrm{id}_X$ and $\operatorname{Ker}(Y \to 0) = \mathrm{id}_Y$. The axiom furnishes a canonical isomorphism $\operatorname{Coker}(0 \to X) \to \operatorname{Ker}(Y \to 0)$, hence the fact that φ is an isomorphism.

Example 1.3.3 (Abelian categories)**.** Categories of abelian groups, modules over a ring, quasicoherent sheaves on a scheme. If $\mathcal{A}$ is any abelian category and $\mathcal{C}$ is a small category, then $\mathsf{Fun}(\mathcal{C}, \mathcal{A})$ is an abelian category (we will prove this fact below).

Example 1.3.4. The category $\mathsf{Proj}(R)$ of finitely generated projective modules over a ring R is not generally abelian (it doesn't contain kernels and cokernels). Similarly, if X is a topological space, we can consider the category $\mathsf{Vect}(X)$ of vector bundles on X. This category is not usually abelian, for exactly the same reason.

The categories $\mathsf{Proj}(R)$ and $\mathsf{Vect}(X)$ are basic examples of exact categories (a notion due to Quillen). Every exact category is a full subcategory of an abelian category.

Example 1.3.5 (Filtered abelian groups)**.** Let AbF be the category of filtered abelian groups. Objects of AbF are abelian groups X equipped with an increasing filtration $F^\bullet X : \cdots \subset F^i X \subset F^{i+1} X \subset \cdots$. A morphism in AbF from $(X, F^\bullet X)$ to $(Y, F^\bullet Y)$ is a homomorphism $f : X \to Y$ such that $f(F^i X) \subset F^i Y$ for all i. Kernels and cokernels in AbF are defined as in AbF, with the following filtrations:

$$F^i \operatorname{Ker}(\varphi) = \operatorname{Ker}(\varphi) \cap F^i X$$
$$F^i \operatorname{Coker}(\varphi) = F^i Y / F^i Y \cap \operatorname{Im}(\varphi)$$

It is easy to check that part 1 of the axiom AB4 holds, but part 2 does not (i.e. mono + epi does not imply iso). As an example, choose some abelian group X that admits filtrations $F_1^\bullet \subsetneq F_2^\bullet$, i.e. $F_1^i X \subset F_2^i X$ for all i, but $F_1^i X \subsetneq F_2^i X$ for some i. The map $\mathrm{id}_X : X \to X$ is mono and epi, but is not an isomorphism. In general, in the factorization

$$K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{\ell} I' \xrightarrow{j} Y \xrightarrow{k'} K'$$

the map ℓ may not be an isomorphism on each component of the filtration.

Example 1.3.6. The category of topological abelian groups is also exact but not abelian. The reason is that epimorphisms are maps $f \colon X \to Y$ s.t. $f(X) \subset Y$ is dense. But in general density does not imply surjectivity. As an example, consider dense winding of the real 2-dimensional torus $f \colon \mathbb{R} \to \mathbb{T}^2$, $f(x) = (e^{ix}, e^{i\lambda x})$ for irrational λ.

Theorem 1.3.7. *If $\mathcal{C}$ is a small category, $\mathcal{A}$ any abelian category, then $\mathsf{Fun}(\mathcal{C}, \mathcal{A})$ is abelian.*

Proof. Let $F, G : \mathcal{C} \to \mathcal{A}$ be functors, and let $\varphi : F \to G$ be a natural transformation. We define $\mathrm{Ker}(\varphi)$ as a pair $(K : \mathcal{C} \to \mathcal{A}, k : K \to F)$ directly. For $X \in \mathrm{Ob}(\mathcal{C})$, set $K(X) = \mathrm{Ker}(\varphi(X) : F(X) \to G(X))$. On morphisms, given $f : X \to Y$, consider the following commutative diagram:

$$\begin{array}{ccccc} K(X) & \xrightarrow{k_X} & F(X) & \xrightarrow{\varphi_X} & G(X) \\ \downarrow{\scriptstyle K(f)} & & \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ K(Y) & \xrightarrow{k_Y} & F(Y) & \xrightarrow{\varphi_Y} & G(Y) \end{array}$$

The composite $\varphi_Y F(f) k_X : K(X) \to G(Y)$ is equal to $\varphi_Y G(f) \varphi_X k_X = 0$. By the universal property of kernels, there is a unique morphism $K(f) : K(X) \to K(Y)$ making the leftmost square commute. It follows that $k = \{k_X : K(X) \to F(X)\}$ is a natural transformation $K \to F$.

It is easy to check that (K, k) is the kernel of φ. Similarly, we define the cokernel "pointwise." In the canonical decomposition, we have pointwise isomorphisms

$$I(X) = \mathrm{Coker}(\mathrm{Ker}\,\varphi_X) \xrightarrow{\ell_X} I'(X) = \mathrm{Ker}(\mathrm{Coker}\,\varphi_X)$$

It follows that ℓ is a natural isomorphism. □

It follows that the category $\mathscr{F}(R)$ of nonadditive bimodules is abelian.

Remark 1.3.8. Similarly, one can prove the following useful observation.

Theorem 1.3.9. *If $\mathcal{A}$ is an abelian category, then the category $\mathsf{Com}(\mathcal{A})$ of complexes in $\mathcal{A}$ is also an abelian category.*

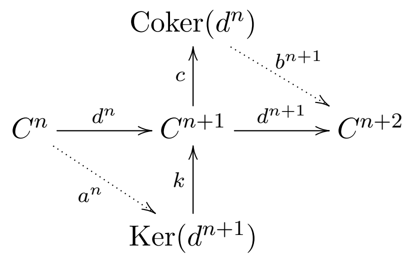
1.4 Complexes in abelian categories

In what follows, we will deal mostly with abstract abelian categories. How are we to think of these? The basic idea is that any general statement involving only finitely many objects and morphisms is true in any abelian category, if and only if it is true in a module category. This is justified by the following theorem:

Theorem 1.4.1 (Mitchell)**.** *If $\mathcal{A}$ is an abelian category, there is an associative unital ring R and a fully faithful exact functor $F : \mathcal{A} \to \mathsf{Mod}(R)$.*

Remark 1.4.2. In other words, Theorem 1.4.1 says that every abelian category can be thought of as being a full exact subcategory of some module category. Though, we prefer not to think about abelian categories this way. For example, the category $\mathsf{Qcoh}(X)$ of quasi-coherent sheaves on a projective scheme is abelian, but there is no obvious way to embed it into $\mathsf{Mod}(R)$ for some R. Moreover, this is not the way one usually thinks about sheaves. Nevertheless, Mitchell's theorem can be rather useful in proving facts about general abelian categories, because viewing objects of abelian categories as modules allows to pick elements.

Let $\mathcal{A}$ be an abelian category. We can define (co)chain complexes and (co)homology in $\mathcal{A}$, just as in module categories. It is not obvious that the notion of cohomology makes sense. Suppose we have a chain complex $(C^\bullet, d^\bullet)$, and look at a piece:



The arrows a^n, b^n are uniquely determined by the universal properties of $\mathrm{Ker}(d^{n+1})$ and $\mathrm{Coker}(d^n)$. There is a canonical morphism $\mathrm{Coker}(a^n) \to \mathrm{Ker}(b^{n+1})$, which is an isomorphism by axiom AB4. Thus we can define $\mathrm{H}^n(C) = \mathrm{Coker}(a^n) \simeq \mathrm{Ker}(b^{n+1})$.

Thus for any abelian category $\mathcal{A}$, we can define the category of complexes in $\mathcal{A}$, written $\mathsf{Com}(\mathcal{A})$, and it is an easy exercise to show that $\mathsf{Com}(\mathcal{A})$ is an abelian category.

Definition 1.4.3. *For all $n \in \mathbb{Z}$, the* n-th cohomology *is the functor* $\mathrm{H}^n : \mathsf{Com}(\mathcal{A}) \to \mathcal{A}$ *defined as above.*

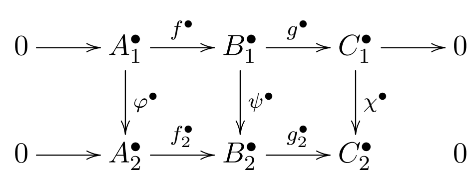
We can define homotopies between morphisms in $\mathsf{Com}(\mathcal{A})$ the same way we did that earlier for complexes of abelian groups. Homotopic morphisms induce the same morphism on cohomology.

Remark 1.4.4. There is much more general notion of a homotopy between two morphisms in a category. Namely, a notion of a homotopy in model categories. We will discuss model categories later.

Let $A^\bullet, B^\bullet, C^\bullet$ be objects in $\mathsf{Com}(\mathcal{A})$. It is easy to check that a sequence

$$0 \longrightarrow A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C \longrightarrow 0$$

is exact if and only if it is point-wise exact. Let $\mathsf{Exc}(\mathcal{A})$ be the category of short exact sequences in $\mathsf{Com}(\mathcal{A})$. Objects of $\mathsf{Exc}(\mathcal{A})$ are short exact sequences as above, and morphisms are commutative diagrams:



Define, for each $n \in \mathbb{Z}$, two functors from $\mathsf{Exc}(\mathcal{A})$ to $\mathcal{A}$ by

$$
\begin{aligned}
F^n(A^\bullet \to B^\bullet \to C^\bullet) &= \mathrm{H}^n(C^\bullet) & \qquad F^n(\varphi^\bullet, \psi^\bullet, \chi^\bullet) &= \mathrm{H}^n(\chi^\bullet) \\
G^n(A^\bullet \to B^\bullet \to C^\bullet) &= \mathrm{H}^{n+1}(C^\bullet) & \qquad G^n(\varphi^\bullet, \psi^\bullet, \chi^\bullet) &= \mathrm{H}^n(\varphi^\bullet)
\end{aligned}
$$

Definition 1.4.5. *The* connecting morphism *is a natural transformation $\delta^n : F^n \to G^n$ defined as follows. The morphism $\delta^n : (A^\bullet \to B^\bullet \to C^\bullet) : \mathrm{H}^n(C^\bullet) \to \mathrm{H}^{n+1}(C^\bullet)$ will be defined using the following diagram:*

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^n & \xrightarrow{f^n} & B^n & \xrightarrow{g^n} & C^n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_A^n} & & \downarrow{\scriptstyle d_B^n} & & \downarrow{\scriptstyle d_C^n} & & \\
0 & \longrightarrow & A^{n+1} & \xrightarrow{f^{n+1}} & B^{n+1} & \xrightarrow{g^{n+1}} & C^{n+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_A^{n+1}} & & \downarrow{\scriptstyle d_B^{n+1}} & & \downarrow{\scriptstyle d_C^{n+1}} & & \\
0 & \longrightarrow & A^{n+2} & \xrightarrow{f^{n+2}} & B^{n+2} & \xrightarrow{g^{n+2}} & C^{n+2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

We use the Mitchell embedding theorem and work as through everything were modules. Choose $c \in C^n$ such that $d_C^c(n) = 0$ (i.e. $[c] \in \mathrm{H}^n(C^\bullet)$). Since g^n is epi, there exists $b^n \in B^n$ such that $g^n(b) = c$. Note that $g^{n+1}(d_B^n b) = d_C^n g^n(b) = 0$, so $d^n b \in \operatorname{Ker} g^{n+1}$, which is the image of f^{n+1}. It follows that there exists $a \in A^{n+1}$ such that $f^{n+1}(a) = b$. We claim that $da = 0$. Indeed, $f^{n+2}(d^{n+1}a) = d_B^{n+1}(f^{n+1}b) = d^{n+1}(b) = 0$, whence $d^{n+1}a = 0$ since f^{n+2} is injective. Set $\delta^n(c) = a$.

Theorem 1.4.6. *For any short exact sequence of complexes*

$$0 \longrightarrow A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \longrightarrow 0 \tag{$*$}$$

the sequence

$$\cdots \longrightarrow \mathrm{H}^n(A^\bullet) \xrightarrow{\mathrm{H}^n(f)} \mathrm{H}^n(B^\bullet) \xrightarrow{\mathrm{H}^n(g)} \mathrm{H}^n(C^\bullet) \xrightarrow{\delta^n(f^\bullet, g^\bullet)} \mathrm{H}^{n+1}(A^\bullet) \longrightarrow \cdots \tag{$**$}$$

is exact.

The long exact sequence $(**)$ is functorial in $(*)$. So $(*) \mapsto (**)$ is a functor $\mathsf{Exc}(\mathcal{A}) \to \mathsf{Com}(\mathcal{A})$. Theorem 1.4.6 has a number of useful consequences.

Lemma 1.4.7 ("Snake lemma")**.** *Consider a diagram*

$$\begin{CD}
0 @>>> X^1 @>{f_1}>> Y^1 @>{g_1}>> Z^1 @>>> 0\\
@. @VV{\alpha}V @VV{\beta}V @VV{\gamma}V @.\\
0 @>>> X^2 @>{f_2}>> Y^2 @>{g_2}>> Z^2 @>>> 0
\end{CD}$$

Assume the rows are exact. Then there are two exact sequences with a connecting "snake" (natural in the sequences):

$$0 \longrightarrow \operatorname{Ker}\alpha \longrightarrow \operatorname{Ker}\beta \longrightarrow \operatorname{Ker}\gamma \xrightarrow{\ \delta\ } \operatorname{Coker}\alpha \longrightarrow \operatorname{Coker}\beta \longrightarrow \operatorname{Coker}\gamma \longrightarrow 0$$

Proof. Add zeros and think of the vertical sequences as complexes as in:

$$\begin{CD}
@. 0 @. 0 @. 0 @.\\
@. @VVV @VVV @VVV @.\\
0 @>>> X^1 @>{f_1}>> Y^1 @>{g_1}>> Z^1 @>>> 0\\
@. @VV{\alpha}V @VV{\beta}V @VV{\gamma}V @.\\
0 @>>> X^2 @>{f_2}>> Y^2 @>{g_2}>> Z^2 @>>> 0\\
@. @VVV @VVV @VVV @.\\
@. 0 @. 0 @. 0 @.
\end{CD}$$

We can apply Theorem 1.4.6 to the exact sequence $0 \to X^\bullet \to Y^\bullet \to Z^\bullet \to 0$, obtaining the result. □

Lemma 1.4.8 ("5-lemma")**.** *Suppose we have a diagram*

$$\begin{CD}
X_1 @>>> X_2 @>>> X_3 @>>> X_4 @>>> X_5\\
@VV{f_1}V @VV{f_2}V @VV{f_3}V @VV{f_4}V @VV{f_5}V\\
Y_1 @>>> Y_2 @>>> Y_3 @>>> Y_4 @>>> Y_5
\end{CD}$$

Assume the rows are exact, f_1 is epi, f_5 is mono, and that f_2 and f_4 are isomorphisms. Then f_5 is an isomorphism.

Lemma 1.4.9 (“3×3 lemma”)**.** *Consider a commutative diagram*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X_1 & \longrightarrow & Y_1 & \longrightarrow & Z_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X_2 & \longrightarrow & Y_2 & \longrightarrow & Z_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X_3 & \longrightarrow & Y_3 & \longrightarrow & Z_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

If the columns and middle row are exact, then if either the first or last row is exact, so is the other.

Proof. This is also an easy consequence of Theorem 1.4.6. □

1.5 Exact functors

Let $\mathcal{A}$, $\mathcal{A}'$ be abelian categories, $F : \mathcal{A} \to \mathcal{A}'$ an additive functor.

Definition 1.5.1. *We say F is* left exact *if for any short exact sequence in $\mathcal{A}$:*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0,$$

the following sequence is exact in $\mathcal{A}'$.

$$0 \longrightarrow FX \longrightarrow FY \longrightarrow FZ$$

We say F is right exact *if the analogous sequence (with 0 on the right) is exact. We say F is* exact *if it is both left and right exact.*

As motivation, consider the classical Riemann-Roch problem. Let X be a topological space, and let $\mathscr{F}$ be an “interesting sheaf” on X. One is usually interested in the global sections of $\mathscr{F}$. Often the sheaf $\mathscr{F}$ can be “decomposed” via a short exact sequence:

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}_2 \longrightarrow 0$$

If the functor $\mathscr{F} \mapsto \Gamma(X, \mathscr{F}) = \mathscr{F}(X)$ were exact, we would have a short exact sequence of abelian groups:

$$0 \longrightarrow \mathscr{F}_1(X) \longrightarrow \mathscr{F}(X) \longrightarrow \mathscr{F}_2(X) \longrightarrow 0$$

In the classical setting, $\mathscr{F}$ is a sheaf of complex vector spaces, and we are only interested in $\dim_{\mathbb{C}} \Gamma(X, \mathscr{F})$. The short exact sequence of global sections would give $\dim \Gamma(X, \mathscr{F}) = \dim \Gamma(X, \mathscr{F}_1) + \dim \Gamma(X, \mathscr{F}_2)$. Unfortunately, the functor $\Gamma(X, -)$ is almost never exact.

Example 1.5.2 (hom-functors)**.** If $\mathcal{A}$ is an abelian category, the Yoneda functors h_X and h^X factor through the category of abelian groups. That is, for $X \in \operatorname{Ob}\mathcal{A}$, we have functors

$$h^X = \operatorname{Hom}_{\mathcal{A}}(X, -) : \mathcal{A} \to \mathsf{Ab}$$
$$h_X = \operatorname{Hom}_{\mathcal{A}}(-, X) : \mathcal{A}^\circ \to \mathsf{Ab}$$

The fact that these functors are left exact is a direct consequence of our definition of the kernel.

Example 1.5.3 (Global sections)**.** If X is a topological space, then the functor $\Gamma : \mathsf{Sh}(X) \to \mathsf{Ab}$ is left exact. Though it is not at all obvious, this is a special case of the previous example. We will work out the details later.

Example 1.5.4 (Representation theory)**.** If G is a group, we define a functor $\mathrm{H}^0(G, -) : G\text{-}\mathsf{Mod} \to \mathsf{Ab}$ by assigning to a G-module M the group $M^G = \{m \in M : gm = m \text{ for all } g \in G\}$. Similarly, if $\mathfrak{g}$ is a Lie algebra over a field k, we set $\mathrm{H}^0(\mathfrak{g}, M) = M^{\mathfrak{g}} = \{m \in M : \mathfrak{g}x = 0\}$. More subtle is the *Zuckermann functor.* Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra (usually the Cartan subalgebra of a semisimple Lie algebra). We say a $\mathfrak{g}$-module M is $\mathfrak{h}$*-finite* if $\dim_k \mathcal{U}(\mathfrak{h})m < \infty$. We define the functor $(-)_{\mathfrak{h}} : \mathfrak{g}\text{-}\mathsf{Mod} \to \mathfrak{g}\text{-}\mathsf{Mod}$ by $M \mapsto M_{\mathfrak{h}} = \{m \in M : m \text{ is } \mathfrak{h}\text{-finite}\}$.

1.6 Adjointness and exactness

Theorem 1.6.1. *If $F : \mathcal{A} \to \mathcal{B}$ is an additive functor between abelian categories. Then*

- *If F has a right adjoint, F is right exact.*
- *If F has a left adjoint, F is left exact.*

Proof. Let $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ be an adjoint pair. Let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be an exact sequence in $\mathcal{A}$. Take $Y \in \operatorname{Ob}(\mathcal{B})$ and apply $\operatorname{Hom}(-, G(Y))$ to this exact sequence. By the definitions of kernels and cokernels, we get a commutative diagram with exact first row:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Hom}(X'', G(Y)) & \longrightarrow & \operatorname{Hom}(X, G(Y)) & \longrightarrow & \operatorname{Hom}(X', G(Y)) \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ 0 & \longrightarrow & \operatorname{Hom}(FX'', Y) & \longrightarrow & \operatorname{Hom}(FX, Y) & \xrightarrow{(Ff)^*} & \operatorname{Hom}(FX', Y) \end{array}$$

The exactness of the second row, FX'' represents the functor

$$Y \mapsto \operatorname{Ker}(\operatorname{Hom}(FX, Y) \xrightarrow{(Ff)^*} \operatorname{Hom}(FX', Y))$$

hence $(FX'', Fg) \simeq \operatorname{Coker}(Ff)$. By definition, this means the sequence

$$FX' \longrightarrow FX \longrightarrow FX'' \longrightarrow 0$$

is exact. The proof when F is a right adjoint is similar. □

In other words, left adjoints are right exact, and right adjoints are left exact.

Corollary 1.6.2. *If $F : \mathcal{A} \to \mathcal{B}$ is an additive functor between abelian functors that has both left and right adjoints, then F is exact.*

The converse of this theorem is *not* true – there are functors that are exact but have no adjoints.

Example 1.6.3. Let X be a topological space, $U \subset X$ an open subset. Let $\mathsf{Sh}(X)$ be the category of abelian sheaves on X, and consider the functor $\Gamma(U, -) : \mathsf{Sh}(X) \to \mathsf{Ab}$ given by $\Gamma(U, \mathscr{F}) = \mathscr{F}(U)$. We claim that $\Gamma(U, -)$ is exact. Indeed, let $\mathsf{PSh}(X)$ be the category of presheaves on X. Recall that the forgetful functor $i : \mathsf{Sh}(X) \hookrightarrow \mathsf{PSh}(X)$ has a left adjoint denoted $(-)^+ : \mathscr{F} \mapsto \mathscr{F}^+$, were

$$P^+(U) = \{s : U \to \mathrm{Et}(P) : \pi \circ s = \mathrm{id}_U\}$$

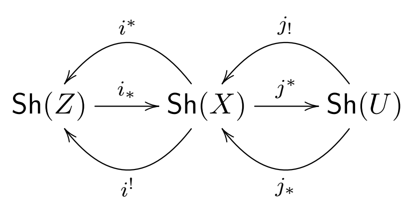
Here $\mathrm{Et}(P)$ is the total space of P and $\pi : \mathrm{Et}(P) \twoheadrightarrow X$ is the canonical projection.

Next, define $\mathbb{Z}_U \in \mathrm{Ob}(\mathsf{PSh}(X))$ by

$$\mathbb{Z}_U(V) = \begin{cases} 0 & \text{if } V \cap U = \varnothing \\ \mathbb{Z} & \text{if } V \cap U \neq \varnothing \end{cases}$$

The restriction maps are the obvious ones. For any presheaf P on X, we have $P(U) = \mathrm{Hom}_{\mathsf{PSh}(X)}(\mathbb{Z}_U, P)$. It follows that $\Gamma(U, -) == \mathrm{Hom}_{\mathsf{PSh}(X)}(\mathbb{Z}_U, -) \circ i$, so $\Gamma(U, -)$ is the composite of two left-exact functors, hence $\Gamma(U, -)$ is left-exact.

Example 1.6.4. Let X be a topological space, $Z \subset X$ a closed subspace, and let $U = X \setminus Z$ be the complement of Z. Write $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$ for the canonical inclusions. Recall that there is a diagram:



where $(i^*, i_* = i_!, i^!)$ and $(j_!, j^! = j^*, j_*)$ are adjoint triples. It follows that $i^!$ and j_* are left exact, i^* and $j_!$ are right exact, and i_*, j^* are exact.

Example 1.6.5. Let R, S be rings, $\mathcal{A} = \mathsf{Mod}(R)$, $\mathcal{B} = \mathsf{Mod}(S)$. Let B be a (R, S)-bimodule. Then we have an adjoint pair:

$$- \otimes_R B : \mathsf{Mod}(R) \rightleftarrows \mathsf{Mod}(S) : \mathrm{Hom}_S(B, -).$$

so $\otimes_R B$ s right exact. The converse is also true.

Theorem 1.6.6 (Watt). *Let R, S be rings, and let $f^* : \mathsf{Mod}(R) \to \mathsf{Mod}(S)$ be an additive functor that is right exact and commutes with direct sums. Then $f^*(R)$ has the structure of an (R,S)-bimodule, and there is a natural isomorphism $f^* \simeq - \otimes_R f^*(R)$.*

Suppose $\mathcal{C}, \mathcal{C}'$ are categories with coproducts and $F : \mathcal{C} \to \mathcal{C}'$ is a functor. For $\{X_\alpha\}_{\alpha\in I} \subset \operatorname{Ob}\mathcal{C}$, recall that $\coprod_{\alpha\in} X_\alpha = \varinjlim_I X$. This coproduct comes with maps $i\alpha : X_\alpha \to \coprod_\alpha X_\alpha$. Applying F, we get morphisms $F(i_\alpha) : F(X_\alpha) \to F(\coprod_{\alpha\in I} X_\alpha)$. By the universal properties of coproducts, we get a canonical morphism

$$\coprod_{\alpha\in I} F(X_\alpha) \xrightarrow{\coprod F(i_\alpha)} F\left(\coprod_{\alpha\in I} X_\alpha\right)$$

We say that F *commutes with coproducts* if $\coprod F(i_\alpha)$ is an isomorphism in $\mathcal{C}'$ for all collections $\{X_\alpha\}_{\alpha\in I}$. (Clearly this construction works for arbitrary colimits.)

Proof. Write $B = f^*(R)$. By definition, B is a right S-module. For each $x \in R$, define $\lambda_x : R \to R$, $a \mapsto x \cdot a$. The map λ gives us a ring homomorphism

$$\lambda : R \to \operatorname{Hom}_{\mathsf{Mod}(R)}(R,R).$$

Since f^* is additive, the following composite is also a ring homomorphism:

$$R \xrightarrow{\lambda} \operatorname{Hom}(R,R) \xrightarrow{f^*} \operatorname{Hom}_S(B,B).$$

Thus B is an (R,S)-bimodule. Explicitly, for $x \in R$ and $b \in B$, we put $x \cdot b = f^*(\lambda_x)(b)$. We need to construct a natural transformation $t : - \otimes_R B \to f^*$. For a right R-module M, we define $t_M : M \otimes_R B \to f^*(M)$ as follows. First, define for each $m \in M$ the map $\varphi_m : R \to M$ of right R-modules by $x \mapsto m \cdot x$. This gives $f^*(\varphi_m) \in \operatorname{Hom}_S(B, f^*M)$. We define

$$t_m(m \otimes b) = f^*(\varphi_m)(b).$$

If well-defined, it is easy to see that this is S-linear. We need to check that $t_M(m \otimes xb) = t_M(mx \otimes b)$ for all $x \in R$. A simple computation suffices:

$$\begin{aligned} f^*(\varphi_m)(x \cdot b) &= f^*(\varphi_m)(f^*(\lambda_x)(b)) \\ &= (f^*(\varphi_m) \circ f^*(\lambda_x))(b) \\ &= f^*(\varphi_m \circ \lambda_x)(b) \\ &= f^*(\varphi_{mx})(b) \end{aligned}$$

A similar routine computation shows that $t : - \otimes_R B \to f^*$ is natural.

Note that everything so far works for every additive functor (no exactness properties required). To show that t is a natural isomorphism, we need the stated hypotheses. It is sufficient to show that t_M is an isomorphism for each M in three steps:

1. Take $M = R$. Then $t_R : R \otimes_R B \to f^*(R)$ is an isomorphism by the definition of B.

2. Take any free R-module $F = \bigoplus_{\alpha \in I} R = R^{\oplus I}$, where I is a (possibly infinite) index set. Since f^* commutes with direct sums,

$$t_F : F \otimes_R B \simeq B^{\oplus I} \xrightarrow{f^*} f^*(B)^{\oplus I} \simeq f^*(B^{\oplus I}).$$

3. Finally, take any right R-module M. Take a presentation of M:

$$F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where F_0 and F_1 are free. Then in the following commutative diagram, the top and bottom rows are exact (by the right-exactness of f^* and $- \otimes_R B$), and the latter two vertical arrows are isomorphisms.

$$\begin{array}{ccccccc}
F_1 \otimes_R B & \longrightarrow & F_0 \otimes_R B & \longrightarrow & M \otimes_R B & \longrightarrow & 0 \\
\downarrow{\scriptstyle t_{F_1}} & & \downarrow{\scriptstyle t_{F_0}} & & \downarrow{\scriptstyle t_M} & & \\
f^*(F_1) & \longrightarrow & f^*(F_0) & \longrightarrow & f^*(M) & \longrightarrow & 0
\end{array}$$

By the 5-lemma, t_M is an isomorphism. □

2 Finiteness conditions

2.1 AB5 categories

We would like to define a class of abelian categories which are sufficiently large to have arbitrary direct sums, but still satisfy some finiteness properties.

Definition 2.1.1 (AB5)**.** *We say that an abelian category $\mathcal{A}$ satisfies AB5 if*

AB5 *$\mathcal{A}$ has* exact *(filtered) colimits.*

This axiom merits some explanation. Recall that a direct system is just a diagram indexed by a category that is actually a poset. Let $\{X_i, \varphi_{ij}^X\}_{i \in I}$, $\{Y_i, \varphi_{ij}^Y\}_{i \in I}$ and $\{Z_i, \varphi_{ij}^Z\}_{i \in I}$ be three direct systems, and suppose we have compatible exact sequences

$$0 \longrightarrow Y_i \longrightarrow Y_i \longrightarrow Z_i \longrightarrow 0$$

Then AB5 requires the the following sequences to be exact:

$$0 \longrightarrow \varinjlim X_i \longrightarrow \varinjlim Y_i \longrightarrow \varinjlim Z_i \longrightarrow 0$$

Theorem 2.1.2. *For any ring R, the category $\mathsf{Mod}(R)$ satisfies AB5.*

Example 2.1.3. The axioms for an abelian category are self-dual, so for any ring R, the category $\mathsf{Mod}(R)^\circ$ is abelian. However, the category $\mathsf{Mod}(R)^\circ$ is almost never AB5 because colimits in $\mathsf{Mod}(R)^\circ$ are just limits in $\mathsf{Mod}(R)$, and these are not necessarily exact.

Indeed, let R be a commutative local domain of dimension one with maximal ideal $\mathfrak{m} \subset R$. Consider the exact sequence of inverse systems:

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \mathfrak{m}^3 & \longrightarrow & \mathfrak{m}^2 & \longrightarrow & \mathfrak{m} \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & = & R & = & R & = & R \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & R/\mathfrak{m}^3 & \longrightarrow & R/\mathfrak{m}^2 & \longrightarrow & R/\mathfrak{m} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}
$$

i.e. for each n we have $0 \to \mathfrak{m}^n \to R \to R/\mathfrak{m}^n \to 0$. Taking limits, we get $0 \to 0 \to R \to \widehat{R} \to 0$, which is not exact unless R is already complete. Exactness on the left comes from $\varinjlim \mathfrak{m}^n = \bigcap \mathfrak{m}^n$ and

Theorem 2.1.4 (Krull's intersection theorem)**.** *If R is commutative local Noetherian ring with (unique) maximal ideal $\mathfrak{m}$, then* $\bigcap_{n=1}^{\infty} \mathfrak{m}^n = 0$.

Exercise [Mittag-Leffler condition] Let $\mathcal{A}$ be an abelian category. Suppose (X_i, φ_{ij}), (Y_i, φ_{ij}) and (Z_i, φ_{ij}) are inverse systems indexed by a directed poset I. Suppose there are compatible exact sequences

$$0 \longrightarrow X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i \longrightarrow 0$$

Prove that the sequence

$$0 \longrightarrow \varprojlim X_i \longrightarrow \varprojlim Y_i \longrightarrow \varprojlim Z_i \longrightarrow 0$$

provided (X_i, φ_{ij}) satisfies the *Mittag-Leffler condition*: for every $n \in I$, there exists $n_0 \geqslant n$ such that for all $i, j \geqslant n_0$, we have $\mathrm{Im}(\varphi_{in}) = \mathrm{Im}(\varphi_{jn})$.

If an abelian category $\mathcal{A}$ does not satisfy AB5, then some pathologies may occur. An object generated by simple subobjects may not be a *direct* sum of its simple subobjects (i.e. it may not be semisimple).

2.2 Grothendieck categories

Definition 2.2.1. *We say an object X in $\mathcal{A}$ is* Noetherian *if every increasing sequence*

$$X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots \subset X$$

is eventually stationary. The category $\mathcal{A}$ is Noetherian if every object in $\mathcal{A}$ is Noetherian.

Example 2.2.2. If $\mathcal{A} = \mathsf{Mod}(R)$ and $X \in \mathrm{Ob}(\mathcal{A})$, then X is Noetherian as an object of $\mathcal{A}$ if and only if X is Noetherian as an R-module. The category $\mathsf{Mod}(R)$ is never Noetherian, except in the trivial case $R = 0$. This is the case even if R is (right) Noetherian. Similarly, if X is a Noetherian scheme, the category $\mathsf{Qcoh}(X)$ of quasi-coherent sheaves is not Noetherian, but its full subcategory $\mathsf{coh}(X)$ of *coherent sheaves* is Noetherian.

Definition 2.2.3. *A category $\mathcal{C}$ has a* set of generators *if there exists a (small) set $\{X_i\}_{i\in I} \subset \mathrm{Ob}(\mathcal{C})$ satisfying the following property. For every $f, g : Y \to Z$ in $\mathcal{C}$ with $f \neq g$, there exists $i \in I$ and $h : X_i \to Y$ such that $f \circ h \neq g \circ h$.*

Lemma 2.2.4. *Assume $\mathcal{A}$ is an abelian category with arbitrary direct sums. Then the following are equivalent:*

1. *$\mathcal{A}$ has a set $\{X_i\}_{i\in I}$ of generators*
2. *$X = \bigoplus_i X_i$ is a generator for $\mathcal{A}$*
3. *every object of $\mathcal{A}$ is quotient of $X^{\oplus J}$ for some J*

Proof. This is a good exercise. □

Definition 2.2.5. *An abelian category $\mathcal{A}$ is called* locally Noetherian *if it has a set of Noetherian generators.*

Definition 2.2.6. *An abelian category $\mathcal{A}$ is called* Grothendieck category *if $\mathcal{A}$ satisfies AB5 and $\mathcal{A}$ is locally Noetherian.*

Remark 2.2.7. Sometimes in the literature the last condition of the definition of a Grothendieck category is weakened. Namely, sometimes only the existence of a set of generators is required, without assuming these generators are Noetherian.

Theorem 2.2.8. *Let R be a unital ring. Then $\mathsf{Mod}(R)$ is locally Noetherian if and only if R is right Noetherian.*

Proof. Let $S = \{X_i\}_{i\in I}$ be a set of Noetherian generators for $\mathsf{Mod}(R)$. Let $X = \bigoplus_i X_i$. By the lemma, X is a generator for $\mathsf{Mod}(R)$. Take any proper right ideal $J \subset R$, and consider the projection $p : R \twoheadrightarrow R/J$. The map $p \neq 0$ since $J \neq R$. Thus there exists $\varphi : X \to R$ such that $p \circ \varphi$ is not zero as a map $X \to R/J$. In particular, $\varphi(X) \not\subset J$. Thus $R = \sum \varphi(X)$, i.e. R is equal to the ideal generated by the $\{\mathrm{Im}(\varphi) : \varphi \in \mathrm{Hom}_R(X, R)\}$. It follows that R is a quotient of some large direct sum of Noetherian objects. In fact, R is a quotient of $X^{\bigoplus \mathrm{Hom}_R(X,R)}$. But then 1_R must be contained in the image of a *finite* direct sum of Noetherian modules, so R is actually a quotient of a finite sum of Noetherian R-modules. Thus R itself is Noetherian.

The other implication is trivial. □

2.3 Inductive closure of an abelian category

Let R be a right Noetherian ring. Let $\mathsf{mod}(R) \subset \mathsf{Mod}(R)$ be the full subcategory of right Noetherian modules. Similarly, if X is a Noetherian scheme, we have the category $\mathsf{coh}(X)$ as a subcategory of $\mathsf{Qcoh}(X)$. It is natural to ask if X and $\mathsf{coh}(X)$ determine each other in a categorical way. Similarly, one could ask if R and $\mathsf{mod}(R)$ determine each other. The answer to this involves the inductive closure of a category.

Definition 2.3.1. *A* coindex category *J is a small nonempty category such that*

1. *J is connected*
2. *For all $j' \leftarrow i \rightarrow j$, there is a k with a commutative diagram:*

$$\begin{array}{ccc} i & \longrightarrow & j \\ \downarrow & & \vdots \\ j' & \cdots\cdots\to & k \end{array}$$

3. *for pair of arrows $u, v : i \to j$ there exists $w : j \to k$ such that $wu = wv$.*

We would like "freely add colimits of coindex categories" in a category $\mathcal{C}$. Recall that $\mathcal{C}$ embeds in $\widehat{\mathcal{C}} = \mathsf{Fun}(\mathcal{C}^\circ, \mathsf{Set})$ via the Yoneda embedding. Given a diagram $F : J \to \mathcal{C}$, we define $\widehat{F} : J \to \widehat{\mathcal{C}}$ by $\widehat{F}(j)(Y) = \mathrm{Hom}(Y, F(j))$, i.e. $\widehat{F} = h_{F(-)}$.

Definition 2.3.2. *The* inductive closure *of $\mathcal{C}$ is the full subcategory $\widetilde{\mathcal{C}}$ of $\widehat{\mathcal{C}}$ consisting of all possible $L = \varinjlim_J \widehat{F}$ for $F : J \to \mathcal{C}$ with J a coindex category.*

Theorem 2.3.3 (Gabriel)**.** *Let $\mathcal{A}$ be a Noetherian abelian category. Then $\widetilde{\mathcal{A}}$ is a Grothendieck category. Moreover, $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ determine each other up to natural equivalence.*

Our main examples are $\mathsf{Qcoh}(X) = \widetilde{\mathsf{coh}(X)}$ and $\mathsf{Mod}(R) = \widetilde{\mathsf{mod}(R)}$. The main reason we are interested in $\widetilde{\mathcal{A}}$ is because it has "enough injectives". We will explain meaning of this later in 3.1.2.

We would like to have a good characterization of module categories among Grothendieck categories. First, we need to define projective objects in an arbitrary abelian category. Let $\mathcal{A}$ be an abelian category.

Definition 2.3.4. *An object $P \in \mathcal{A}$ is* projective *if the functor*

$$\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathsf{Ab}$$

is exact.

For any object X, the functor $\mathrm{Hom}(X, -)$ is left exact (see example 1.5.2). So P is projective if and only if $\mathrm{Hom}(P, -)$ is right exact. There is an equivalent, but very useful definition of projective objects. An object P is projective if and only if for all surjections $\pi : X \to X'$ and morphisms $\varphi : P \to X'$, there exists a lift $\psi : P \to X$ of φ as in the following diagram:

$$\begin{array}{ccccc} & & P & & \\ & \overset{\psi}{\swarrow} & \downarrow \varphi & & \\ X & \longrightarrow & X' & \longrightarrow & 0 \end{array}$$

To see this, suppose we have $\pi : X \twoheadrightarrow X'$. Then if $K = \mathrm{Ker}\,\pi$, the following sequence is exact.

$$0 \longrightarrow K \longrightarrow X \xrightarrow{\pi} X' \longrightarrow 0$$

Applying $\mathrm{Hom}(P, -)$, we get an exact sequence of abelian groups:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P, K) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P, X) \xrightarrow{\pi_*} \mathrm{Hom}_{\mathcal{A}}(P, X') \longrightarrow 0.$$

The surjectivity of π_* is precisely the lifting property we want P to have.

Lemma 2.3.5. *Let $\mathcal{A} = \mathsf{Mod}(R)$ for a ring R. Then P is projective if and only if P is the direct summand of a free module over R.*

Proof. First, we show that free modules are projective. If F is a free R-module, then by definition F possesses a basis $\{f_\alpha\}_{\alpha \in I}$. Given a surjection $\pi : X \twoheadrightarrow X'$, choose elements $x_\alpha \in X$ such that $\pi(x_\alpha) = \varphi(f_\alpha)$ for each α. Since F is free, we can define $\psi : F \to X$ by setting $\psi(f_\alpha) = x_\alpha$. This is well-defined precisely because F is free.

It easily follows that direct summands of free modules are projective. Let P be a direct summand of a free module F with complement Q. Given a surjection $\pi : X \twoheadrightarrow X'$ and $\varphi : P \to X'$, extend φ to $P \oplus Q = F$ by $\tilde{\varphi} = \varphi \oplus 0$:

$$\begin{array}{ccccc} & & P \oplus Q & & \\ & \swarrow & \downarrow & & \\ X & \longrightarrow & X' & \longrightarrow & 0 \end{array}$$

Since F is free, $\widetilde{\varphi}$ has an extension ψ to F, and its restriction $\psi|_P$ is the desired extension of φ to P.

Finally, we show that if P is projective, then there exists Q such that $F = P \oplus Q$ is free. Indeed, choose a free module F with a surjection $\pi : F \twoheadrightarrow P$. Consider the following diagram:

$$\begin{array}{ccccc} & & P & & \\ & \overset{s}{\swarrow} & \downarrow \mathrm{id} & & \\ F & \xrightarrow{\pi} & P & \longrightarrow & 0 \end{array}$$

By the lifting property, π has a lift s, which is a splitting of π, i.e. $F = P \oplus \mathrm{Ker}(\pi)$. $\square$

2.4 Finiteness conditions

Let $\mathcal{A}$ be an abelian category, X an object of $\mathcal{A}$.

Definition 2.4.1. *The object X is* compact (or small) *if the functor* $\mathrm{Hom}_{\mathcal{A}}(X, -)$ *commutes with direct sums.*

Let $\{X_\alpha\}_{\alpha \in I}$ be a (set-theoretically small) family of objects in $\mathcal{A}$. Recall that $\bigoplus_\alpha X_\alpha$ is by definition the colimit $\varinjlim_I X_\alpha$. By the definition of a colimit, there are canonical morphisms $i_\alpha : X_\alpha \to \bigoplus_\alpha X_\alpha$ for each α. For any object X in $\mathcal{A}$, we have maps

$$\mathrm{Hom}_{\mathcal{A}}(X, X_\alpha) \xrightarrow{i_{\alpha,*}} \mathrm{Hom}_{\mathcal{A}}\left(X, \bigoplus_\alpha X_\alpha\right).$$

These patch together to yield morphisms

$$\bigoplus_{\alpha \in I} \mathrm{Hom}_{\mathcal{A}}(X, X_\alpha) \xrightarrow{\phi_X} \mathrm{Hom}\left(X, \bigoplus_\alpha X_\alpha\right).$$

The object X is compact if ϕ_X is an isomorphism for every collection $\{X_\alpha\}$.

Definition 2.4.2. *The object X is* finitely presented *if* $\mathrm{Hom}_{\mathcal{A}}(X, -)$ *commutes with all small direct limits.*

Definition 2.4.3. *The object X is* finitely generated *if whenever $\{X_\alpha\}$ is a directed system of subobjects of X such that $\sum_\alpha X_\alpha = \varinjlim_\alpha X_\alpha = X$, there exists $\alpha_0 \in I$ such that $X_{\alpha_0} = X$.*

Definition 2.4.4. *The object X is* coherent *if X is finitely presented and every finitely generated subobject of X is finitely presented.*

Example 2.4.5. Let $R = k\langle x_1, \dots, x_n \rangle$ be the free algebra on n generators over a field k. Clearly R is finitely generated (by the unit) as an R-module. There are ideals $I \subset R$ that are not finitely generated, and the quotient R/I is finitely generated but *not* finitely presented.

For many purposes, coherent modules over non-Noetherian algebras are the correct substitute for finitely generated modules over a Noetherian ring.

Lemma 2.4.6. *If $\mathcal{A}$ is an abelian category satisfying AB5, then every finitely generated object is compact.*

Proof. This is a good exercise. □

The converse is false. There is a counter-example due to Rentschler: there exists a commutative integral domain R such that the field of fractions K of R is compact but not finitely generated. (**Need reference here!**)

Lemma 2.4.7. *Let $\mathcal{A}$ be an AB5 abelian category, and let $P \in \mathrm{Ob}(\mathcal{A})$ be projective. Then P is finitely generated if and only if P is compact, if and only if P is finitely presented.*

Proof. We will prove that if P is compact and projective, then P is finitely presented. First, observe the following. Let I be a directed set, and $\{(X_\alpha, f^\alpha_\beta)\}_{\alpha\in I}$ be a direct system of objects in $\mathcal{A}$. Then $\varinjlim_{\alpha\in I} X_\alpha$ is the colimit of the diagram $X : I \to \mathcal{A}$. Let $|I|$ be the underlying (discrete) set of I. Then $\varinjlim_{\alpha\in|I|} X_\alpha = \bigoplus_\alpha X_\alpha$ by definition, with canonical embeddings $i_\alpha \colon X_\alpha \hookrightarrow \bigoplus_\alpha X_\alpha$. We have canonical morphisms $j_\alpha : X_\alpha \to \varinjlim X_\alpha$; these patch together to yield a canonical morphism $j : \bigoplus_\alpha X_\alpha \to \varinjlim_\alpha X_\alpha$. The morphism j is always an epimorphism, and fits into an exact sequence

$$\bigoplus_{(\alpha,\beta)\in S} X_{(\alpha,\beta)} \longrightarrow \bigoplus_{\alpha\in I} X_\alpha \longrightarrow \varinjlim_{\alpha\in I} X_\alpha \longrightarrow 0$$

where $S = \{(\alpha,\beta) \in I\times I : \alpha \leqslant \beta\}$, $f^\alpha_\beta : X_\alpha \to X_\beta$ and $X_{(\alpha,\beta)} = \operatorname{Im}\left(i_\beta f^\alpha_\beta - i_\alpha\right)$.

Assume P is compact projective, and apply $\operatorname{Hom}_{\mathcal{A}}(P,-)$ to the sequence above. We get

$$\begin{array}{ccccccc}
\operatorname{Hom}\left(P, \bigoplus_{(\alpha,\beta)\in S} X_{(\alpha,\beta)}\right) & \longrightarrow & \operatorname{Hom}\left(P, \bigoplus_{\alpha\in I} X_\alpha\right) & \longrightarrow & \operatorname{Hom}\left(P, \varinjlim X_\alpha\right) & \longrightarrow & 0 \\
\uparrow\sim & & \uparrow\sim & & \uparrow\sim & & \\
\bigoplus_{(\alpha,\beta)\in S} \operatorname{Hom}\left(P, X_{(\alpha,\beta)}\right) & \longrightarrow & \bigoplus_{\alpha\in I} \operatorname{Hom}(P, X_\alpha) & \longrightarrow & \varinjlim \operatorname{Hom}(P, X_\alpha) & \longrightarrow & 0
\end{array}$$

The first two vertical arrows are isomorphisms because we assumed P is compact. Since P is projective, rows are exact. Then 5-lemma implies the third vertical arrow is an isomorphism. So $\operatorname{Hom}(P,-)$ commutes with arbitrary direct limits, hence P is finitely presented. □

Projective objects are analogs of vector bundles. This analogy can be made somewhat precise.

Theorem 2.4.8 (Swan)**.** *Let X be a para-compact topological space, and let $A = C(X)$ be the ring of continuous $\mathbb{C}$-valued functions on X. Then the category $\mathsf{Vect}(X)$ of vector bundles on X is equivalent to the category $\mathsf{Proj}(A)$ of projective A-modules via the functor $\Gamma(X,-)$.*

Theorem 2.4.9. *Let $X = \operatorname{Spec}(A)$ is a Noetherian affine scheme. Then the category $\mathsf{Vect}(X)$ of vector bundles on X is equivalent to the category $\mathsf{Proj}(A)$ of projective A-modules via the functor $\Gamma(X,-)$.*

Lemma 2.4.10. *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories. Then F is faithful if and only if $F(X) \neq 0$ whenever $X \neq 0$.*

Proof. Suppose f is faithful (injective on hom-sets). Then $X \neq 0$ implies $\mathrm{id}_X \neq 0$, which implies $F(\mathrm{id}_X) \neq 0$, hence $F(X) \neq 0$. (In other words, arbitrary faithful functors send nonzero objects to nonzero objects.)

Conversely, suppose that for all $X \neq 0$ we have $F(X) \neq 0$. Let f be a nonzero morphism in $\mathcal{A}$. Then $\operatorname{Im}(f) \neq 0$. Since F is exact, it commutes with taking kernels, cokernels and images. So $F(\operatorname{Im} f) = \operatorname{Im} F(f) \neq 0$, hence $F(f) \neq 0$. □

Corollary 2.4.11. *P is a projective generator (i.e.* $\mathrm{Hom}(P, -)$ *is faithful) if and only if* $\mathrm{Hom}(P, X) \neq 0$ *whenever* $X \neq 0$.

Theorem 2.4.12. *Let $\mathcal{A}$ be a Grothendieck category. Assume $\mathcal{A}$ has a compact projective generator P. Then the functor* $\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathrm{End}_{\mathcal{A}}(P)\text{-}\mathsf{Mod}$ *is an equivalence of categories. Conversely, module categories have compact projective generators.*

For example, we can use this to show that $\mathcal{A} = \mathsf{Qcoh}(\mathbb{P}^1)$ is not a module category. In fact, it has no nonzero projective objects. The same holds for $\mathsf{Qcoh}(X)$ whenever X is a projective variety of positive dimension.

3 Classical derived functors

3.1 Injectives and injective envelopes

Let $\mathcal{A}$ be an abelian category.

Definition 3.1.1. *An object E in $\mathcal{A}$ is* injective *if the funtor* $\mathrm{Hom}_{\mathcal{A}}(-, E)$ *is exact.*

In other words, E is injective if whenever we have an exact sequence

$$0 \longrightarrow X' \xrightarrow{f} X \xrightarrow{g} X'' \longrightarrow 0,$$

the following sequence is exact.

$$0 \longrightarrow \mathrm{Hom}(X'', E) \xrightarrow{g^*} \mathrm{Hom}(X, E) \xrightarrow{f^*} \mathrm{Hom}(X', E) \longrightarrow 0$$

We already know it is exact on the left, so the only nontrivial condition is that $\mathrm{Hom}(-, E)$ is right-exact.

Definition 3.1.2. *An abelian category $\mathcal{A}$ has* enough injectives *if every $X \in \mathrm{Ob}(\mathcal{A})$ is isomorphic to a subobject of an injective object in $\mathcal{A}$.*

Tautologically, E is injective in $\mathcal{A}$ if and only if E is projective in the dual category $\mathcal{A}^\circ$. So it would appear that injective and projective objects are "dual" in some sense. However, in practice we usually work with Grothendieck categories, and for $\mathcal{A}$ Grothendieck, $\mathcal{A}^\circ$ is Grothendieck if and only if $\mathcal{A}$ is the zero category. Proving this is a good exercise.

We will see that any Grothendieck category has enough injectives. On the other hand, a Grothendieck category does not necessarily have enough projectives (or any projectives at all). A typical example is $\mathcal{A} = \mathsf{Qcoh}(X)$ for X a projective scheme. This category has no projective objects.

If $\mathcal{A}$ is a Noetherian abelian category, then $\mathcal{A}$ as a rule does not have enough invectives. It is possible for such categories to have enough projectives – just take the category of finitely generated modules over a semisimple ring. In any case, even if $\mathcal{A}$ does not have enough injectives, its inductive closure $\widetilde{\mathcal{A}}$ always have enough injectives.

Theorem 3.1.3. *For any ring R, the categories $\mathsf{Mod}(R)$ and R-Mod have enough injectives.*

Lemma 3.1.4. *Let $\{E_\alpha\}_{\alpha\in I}$ be a set of injectives in $\mathcal{A}$. Assume that the product $E = \prod_\alpha E_\alpha$ exists. Then E is injective.*

Proof. Note that $\prod_\alpha E_\alpha = \varprojlim_I E_\alpha$. For all X in $\mathcal{A}$, we have

$$\mathrm{Hom}(X, E) \simeq \prod_{\alpha\in I} \mathrm{Hom}(X, E_\alpha).$$

The latter is a product of exact functors, hence exact. It follows that E itself is exact. □

Definition 3.1.5. *Let R be an associative unital ring. An element $x \in R$ is called* right regular *if the right R-module map $r \mapsto xr$ is injective on R, i.e. if $xr = 0$ implies $r = 0$.*

Definition 3.1.6. *Let M be a right R-module. We say M is* divisible *if $Mx = M$ for any right regular $x \in R$.*

In other words, M is divisible if for any $m \in M$ and right regular $x \in R$, there exists $m' \in M$ such that $m'x = m$.

Lemma 3.1.7. *Every injective module is divisible.*

Proof. If E is injective, then consider for any right regular $x \in R$ the map $x : R \to R$ given by $r \mapsto xr$. This injective, so by the fact that E is injective, the morphism $x^* : E \simeq \mathrm{Hom}(R, E) \to E \simeq \mathrm{Hom}(R, E)$ is surjective. It follows that $xE = E$, i.e. E is divisible. □

Theorem 3.1.8 (Baer)**.** *If every ideal in R is principal, then divisible R-modules are injective.*

Proof. Fill this in. □

Lemma 3.1.9. *If $\mathcal{A}$ is an AB5 category, then the canonical morphism $\coprod X_\alpha \to \prod X_\alpha$ is monic.*

Example 3.1.10. Let $R = \mathbb{Z}$. Then we claim that $\mathsf{Mod}(\mathbb{Z}) = \mathsf{Ab}$ has enough injectives. Indeed, $\mathbb{Q}$ is obviously divisible, and hence injective. By Lemma 3.1.4, any product of copies of $\mathbb{Q}$ is injective. Moreover, Baer's theorem shows that any quotient of injectives is injective. In particular, $\mathbb{Q}/\mathbb{Z}$ is injective. But any $\mathbb{Z}$-module can be identified with a subquotient of a product of copies of $\mathbb{Q}$, via

$$\begin{array}{ccc} \bigoplus_M \mathbb{Z} & \hookrightarrow & \prod_M \mathbb{Q} \\ \downarrow & & \\ M. & & \end{array}$$

Thus M is isomorphic to a submodule of an injective.

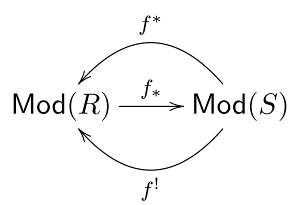
Proposition 3.1.11. *Let $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ be an adjoint pair of functors between abelian categories. If F is exact, then G preserves injectives. If G is exact, then F preserves projectives.*

Proof. We'll prove that if F is exact, then G preserves injectives. Let E be an injective object in $\mathcal{B}$; we want to show that $G(E)$ is injective in $\mathcal{A}$. But

$$\mathrm{Hom}_{\mathcal{A}}(-, G(E)) \simeq \mathrm{Hom}_{\mathcal{B}}(-, E) \circ F,$$

and the composition of exact functors is exact, so $G(E)$ is injective. □

We are now ready to prove Theorem 3.1.3. Recall that if $f : S \to R$ is a ring homomorphism, we have an adjoint triple



where $f^*(M) = M \otimes_S R$ and $f^!(M) = \mathrm{Hom}_S(R, M)$. In fact, $f^! f_* = \mathrm{id}$ and $f^* f_* = \mathrm{id}$. Since f_* has right and left adjoints, it is exact. By the proposition, f_* maps injectives to injectives. Now let $f : \mathbb{Z} \to R$ be the canonical unital homomorphism. Then $f^! : \mathsf{Mod}(\mathbb{Z}) \to \mathsf{Mod}(R)$ maps injectives to injectives. For any right R-module M, choose an injective abelian group E with an embedding $f_* M \hookrightarrow E$. This gives

$$M = f^! f_* M \hookrightarrow f^! E,$$

where $f^! E$ is an injective R-modules.

3.2 Canonical constructions on complexes

The following constructions are motivated by topology, but they actually make sense in much greater generality (i.e. for model categories, as we will see later in the course).

Suspension

Let $\mathcal{A}$ be an abelian category and $\mathsf{Com}(\mathcal{A})$ be the category of (cohomological) chain complexes $(X^\bullet, d^\bullet)$.

Definition 3.2.1. *Let $k \in \mathbb{Z}$. The k-th* suspension *functor $[k] : \mathsf{Com}(\mathcal{A}) \to \mathsf{Com}(\mathcal{A})$ is defined by $X^\bullet \mapsto X^\bullet[k]$, where $X^\bullet[k] = X^{i+k}$ and $d^i_{X[k]} = (-1)^k d^{i+k}_X$.*

Lemma 3.2.2. *The functor $[k]$ is an auto-equivalence of categories, with inverse $[-k]$. Moreover, $[k] \circ [m] = [k+m]$.*

Cone

We'll construct the functor cone explicitly. There is a more functorial construction, and we will describe it later when we will be talking about model categories. Given $f^\bullet : X^\bullet \to Y^\bullet$, set $\mathrm{cone}(f^\bullet) = Y^\bullet \oplus X^\bullet[1]$. The complex $\mathrm{cone}(f^\bullet)$ looks like

$$\cdots \longrightarrow Y^k \oplus X^{k+1} \xrightarrow{d^k_{\mathrm{cone}}} Y^{k+1} \oplus X^{k+2} \longrightarrow \cdots,$$

where

$$d^k_{\mathrm{cone}} = \begin{pmatrix} d^k_Y & f^{k+1} \\ 0 & -d^{k+1}_X \end{pmatrix}.$$

It is easy to check that $d^{k+1}_{\mathrm{cone}} \circ d^k_{\mathrm{cone}} = 0$ if and only if

$$\begin{aligned} d^{k+1}_Y d^k_Y &= 0 \\ d^{k+2}_X d^{k+1}_K &= 0 \\ d^{k+1}_Y \circ f^k &= f^k \circ d^k_X. \end{aligned}$$

Proposition 3.2.3. *A morphism $f^\bullet : X^\bullet \to Y^\bullet$ is a quasi-isomorphism if and only if* $\mathrm{cone}(f^\bullet)$ *is acyclic.*

Proof. Recall there is a short exact sequence

$$0 \longrightarrow Y^\bullet \xrightarrow{i^\bullet} \mathrm{cone}(f) \xrightarrow{p^\bullet} X^\bullet[1] \longrightarrow 0$$

Consider the associated long exact sequence:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \mathrm{H}^{i-1}(X[1]) & \longrightarrow & \mathrm{H}^i(Y^\bullet) & \longrightarrow \mathrm{H}^i(\mathrm{cone}\, f) \longrightarrow & \mathrm{H}^i(X[1]) & \longrightarrow & \mathrm{H}^{i+1}(Y^\bullet) & \longrightarrow \cdots \\ & \| & \nearrow_{f^i} & & & \| & \nearrow_{f^{i+1}} & & \\ & \mathrm{H}^i(X) & & & & \mathrm{H}^{i+1}(X) & & & \end{array}$$

The arrows f^i and f^{i+1} are isomorphisms, which is easily seen to imply $\mathrm{H}^i(\mathrm{cone}\, f) = 0$. □

Exercise Let $\mathsf{Com}^b(\mathcal{A})$ be the full subcategory of $\mathsf{Com}(\mathcal{A})$ consisting of *bounded complexes* (complexes $X^\bullet$ with $X^i = 0$ for all $|i| \gg 0$). The category $\mathcal{A}$ naturally embeds into $\mathsf{Com}^b(\mathcal{A})$. Show that $\mathsf{Com}^b(\mathcal{A})$ is "generated" by $\mathcal{A}$ in the sense that every $X^\bullet \in \mathsf{Com}^b(\mathcal{A})$ can be obtained by taking iterated suspensions and cones of objects in $\mathcal{A}$.

Cylinder

Definition 3.2.4. *Let $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism of complexes. The* cylinder *of $f^\bullet$ is*

$$\mathrm{Cyl}(f^\bullet) = \mathrm{cone}(\mathrm{cone}(f^\bullet)[-1] \xrightarrow{p^\bullet_X[-1]} X^\bullet).$$

Explicitly,

$$\mathrm{Cyl}(f^\bullet) = Y^\bullet \oplus X^\bullet \oplus X^\bullet[1]$$

with differentials

$$d^k_{\mathrm{cyl}} = \begin{pmatrix} d^k_Y & 0 & f^{k+1} \\ 0 & d^k_X & -\mathrm{id}_{X^{k+1}} \\ 0 & 0 & -d^{k+1}_X \end{pmatrix}$$

Note that we have a short exact sequences associated to $f^\bullet : X^\bullet \to Y^\bullet$:

$$0 \longrightarrow X^\bullet \xrightarrow{i_X} \mathrm{Cyl}(f) \xrightarrow{\pi_{\mathrm{cone}}} \mathrm{cone}(f) \longrightarrow 0$$

By Proposition 3.2.3, the morphism i_X is a quasi-isomorphism if and only if f is a quasi-isomorphism, which happens if and only if $\mathrm{cone}(f)$ is acyclic.

The other short exact sequence associated to $f^\bullet : X^\bullet \to Y^\bullet$ is

$$0 \longrightarrow Y^\bullet \xrightarrow{\alpha} \mathrm{Cyl}(f) \longrightarrow \mathrm{cone}(-\mathrm{id}_X) \longrightarrow 0.$$

Here α is the obvious inclusion. It turns out that α has a splitting.

Proposition 3.2.5. *For $f^\bullet : X^\bullet \to Y^\bullet$ consider the following diagram*

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & Y^\bullet & \xrightarrow{i_Y} & \mathrm{cone}(f) & \longrightarrow & X[1] \longrightarrow 0 \\
 & & & & \downarrow{\scriptstyle\alpha} & & \| & & \\
0 & \longrightarrow & X^\bullet & \longrightarrow & \mathrm{Cyl}(f) & \longrightarrow & \mathrm{cone}(f) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\varphi} & & \\
0 & \longrightarrow & X^\bullet & \xrightarrow{f} & Y^\bullet & \xrightarrow{g} & Z^\bullet & \longrightarrow & 0
\end{array}$$

where $(g, Z^\bullet)$ is defined to be cokernel of f, β is defined by $(y^k, x^k, x^{k+1}) \mapsto f^k(x^k) + y^k$ and φ is given by $\varphi\colon (x, y) \mapsto g(y)$. Then

1. $\beta \circ \alpha = \mathrm{id}_Y$
2. $\alpha \circ \beta \sim \mathrm{id}_{\mathrm{Cyl}(f)}$.
3. *φ is quasi-isomorphism.*

Mapping cylinders can be characterized for $\mathcal{A} = \mathsf{Mod}(\mathbb{Z})$. Define the complex $I = (0 \to \mathbb{Z}^{\oplus 2} \to \mathbb{Z} \to 0)$, where $(a, b) \mapsto a - b$. For any morphism $f^\bullet : X^\bullet \to Y^\bullet$, consider the inclusion $X^\bullet \hookrightarrow X^\bullet \otimes I^\bullet$. We claim that there is a cocartesian square:

$$\begin{array}{ccc}
X & \xrightarrow{i_0} & X \otimes I \\
\downarrow{\scriptstyle f} & & \downarrow \\
Y & \longrightarrow & \mathrm{Cyl}(f)
\end{array}$$

Proving this is a good exercise.

3.3 "Classical" definition of classical derived functors

We follow Grothendieck's construction. Recall that if $\mathcal{A}$ is an abelian category, then $\mathcal{A}$ has *enough injectives* if every object X in $\mathcal{A}$ embeds into an injective object.

Definition 3.3.1. *An* injective resolution *of an object X is an exact complex*

$$0 \longrightarrow X \xrightarrow{\varepsilon} I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

with each I^i injective.

Write $I^\bullet$ for the complex $0 \to I^0 \to I^1 \to \cdots$. If we think of X as a complex concentrated in degree zero, then it makes sense to write $\varepsilon : X \to I^\bullet$ for an injective resolution of X. Any injective resolution $\varepsilon : X \to I^\bullet$ is a quasi-isomorphism.

Definition 3.3.2. *Dually to the definition 3.1.2, we say $\mathcal{A}$ has* enough projectives *if every object X in $\mathcal{A}$ is the quotient of a projective object.*

Similarly, a *projective resolution* of X is an exact sequence of the form

$$\cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{\varepsilon} X \longrightarrow 0,$$

with each P^i projective. Again, we denote by $P^\bullet$ the complex $\cdots \to P^1 \to P^0 \to 0$ and think about projective resolutions as a quasi-isomorphism of complexes $\varepsilon : P^\bullet \to X$.

Lemma 3.3.3. *If $\mathcal{A}$ has enough injectives, then every object in $\mathcal{A}$ has an injective resolution.*

Dually, if $\mathcal{A}$ has enough projectives, every object in $\mathcal{A}$ has a projective resolution.

Proof. Just iterate injective embeddings as follows. Given $X \in \mathrm{Ob}(\mathcal{A})$, we have an embedding $\varepsilon : X \hookrightarrow I^0$. We can embed $\mathrm{Coker}(\varepsilon)$ into another injective I^1. Then just repeat the process. □

Proposition 3.3.4. *Suppose $X_1, X_2 \in \mathrm{Ob}(\mathcal{A})$ have injective resolutions $\varepsilon_1 : X_1 \to I_1^\bullet$ and $\varepsilon_2 : X_2 \to I_2^\bullet$. Then any morphism $f : X_1 \to X_2$ admits a lifting $\widetilde{f} : I_1^\bullet \to I_2^\bullet$ making the following diagram commute:*

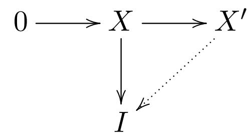
$$\begin{array}{ccc} X_1 & \xrightarrow{\varepsilon_1} & I_1^\bullet \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \widetilde{f}^\bullet} \\ X_2 & \xrightarrow{\varepsilon_2} & I_2^\bullet \end{array}$$

The lift $\widetilde{f}$ is unique up to homotopy.

Proof. Recall that P is projective exactly when h_P is exact. So we only need to check the existence of lifts in the following diagram.

$$\begin{array}{ccccc} & & P & & \\ & \swarrow & \downarrow & & \\ X' & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

Reversing arrows, we see that an object I is injective exactly when lifts exist in the following diagram.



An easy iteration of this diagram yields the existence of a lift $\widetilde{f}^\bullet : I_1^\bullet \to I_2^\bullet$ of $f : X_1 \to X_2$. We will show that $\widetilde{f}^\bullet$ is unique up to homotopy later after gaining some machinery. $\square$

A similar theorem holds for projective resolutions. Note that if F is any additive functor from $\mathcal{A}$ to $\mathcal{B}$, then we can extend F in the obvious way to a functor $F : \mathsf{Com}(\mathcal{A}) \to \mathsf{Com}(\mathcal{B})$. It is easy to see that F preserves homotopies. In other words, if $f^\bullet \sim \widetilde{f}^\bullet$, then $F(f^\bullet) \sim F^\bullet(\widetilde{f}^\bullet)$. Thus the following definition makes sense.

Definition 3.3.5. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact additive covariant functor. Assume $\mathcal{A}$ has enough injectives. Then we define the* (classical) right derived functors *of F, as functors $\mathsf{R}^iF : \mathcal{A} \to \mathcal{B}$ for $i \geqslant 0$, by*

$$\mathsf{R}^iF(X) = \mathrm{H}^i(F(I^\bullet)),$$

where $I^\bullet$ is any injective resolution of X. For $f : X \to X'$, we define

$$\mathsf{R}^i(F)(f) = \mathrm{H}^i(F(\widetilde{f}^\bullet)),$$

where $\widetilde{f}$ is a lift of f to injective resolutions.

Theorem 3.3.6. *Let $\mathcal{A}, \mathcal{B}$ and F be as above. Then*

1. *For all $i \geqslant 0$ the funtors R^iF are additive and independent of the choice of resolution.*

2. *There is a natural isomorphism $\mathsf{R}^0F \simeq F$.*

3. *For any short exact sequence $0 \to X' \to X \to X'' \to$ in $\mathcal{A}$, there are morphisms $\delta^i : \mathsf{R}^iF(X'') \to \mathsf{R}^{i+1}F(X')$, such that the following sequence is exact.*

$$\cdots \longrightarrow \mathsf{R}^iF(X') \longrightarrow \mathsf{R}^iF(X) \longrightarrow \mathsf{R}^iF(X'') \xrightarrow{\delta^i} \mathsf{R}^{i+1}F(X') \longrightarrow \cdots$$

4. *The δ^i are functorial in exact sequences. In other words, if we define $\mathsf{R}^iF(0 \to X' \to X \to X'' \to 0) = \mathsf{R}^iF(X'')$ and $\mathsf{R}^{i+1}F(0 \to X' \to X \to X'' \to 0) = \mathsf{R}^{i+1}F(X')$, then the δ^i are natural transformations $\mathsf{R}^iF \to \mathsf{R}^{i+1}F$, where these are viewed as functors $\mathsf{Exc}(\mathcal{A}) \to \mathcal{B}$.*

5. *If I is injective in $\mathcal{A}$, then $\mathsf{R}^iF(I) = 0$ for all $i > 0$.*

Definition 3.3.7. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact additive functor. An object $J \in \mathrm{Ob}(\mathcal{A})$ is called* F-acyclic *if $\mathsf{R}^iF(J) = 0$ for all $i > 0$.*

By Theorem 3.3.6, injective objects are "universally" acyclic, i.e. they are F-acyclic for *any* left-exact functor F.

Definition 3.3.8. *An* F-acyclic resolution *of* $X \in \mathrm{Ob}(\mathcal{A})$ *is an exact complex* $X \hookrightarrow J^\bullet$*, where* $J^\bullet$ *consists of* F*-acyclic objects.*

Proposition 3.3.9. *If* $X \to J^\bullet$ *is an* F*-acyclic resolution, then* $\mathsf{R}^i F(X) \simeq \mathrm{H}^i(F(J^\bullet))$ *for all* i*.*

Example 3.3.10. Flabby sheaves are acyclic relative to the global sections functor. Thus we can compute sheaf cohomology using flabby resolutions.

We can define (classical) left-derived functors similarly. Given a right-exact additive functor $F : \mathcal{A} \to \mathcal{B}$, define its *left-derived functors* $\mathsf{L}_i F : \mathcal{A} \to \mathcal{B}$ as follows. Given $X \in \mathrm{Ob}(\mathcal{A})$, choose a projective resolution $P^\bullet \to X$, and define

$$\mathsf{L}_i F(X) = \mathrm{H}^{-i}(F(P^\bullet)).$$

3.4 δ-functors

Classical derived functors satisfy a universal mapping property. Let $\mathcal{A}, \mathcal{B}$ be abelian categories.

Definition 3.4.1. *A (covariant)* δ-functor *is a collection* $T = (T^i)_{i \geqslant 0}$ *of additive functors* $\mathcal{A} \to \mathcal{B}$ *given together with natural transformations* $\delta^i : T^i \to T^{i+1}$*, where here* T^i *and* T^{i+1} *are functors* $\mathsf{Exc}(\mathcal{A}) \to \mathcal{B}$ *via* $T^i(0 \to X' \to X \to X'' \to 0) = T^i(X'')$ *and* $T^{i+1}(0 \to X' \to X \to X'' \to 0) = T^{i+1}(X')$*. We require that for any short exact sequence* $0 \to X' \to X \to X'' \to 0$*, the following sequence be exact:*

$$0 \longrightarrow T^0(X') \longrightarrow T^0(X) \longrightarrow T^0(X'') \xrightarrow{\delta^1} T^1(X') \longrightarrow \cdots$$

Definition 3.4.2. *A* δ*-functor* $T = (T^i)_{i \geqslant 0} : \mathcal{A} \to \mathcal{B}$ *is* universal *if for any* δ*-functor* $T' : \mathcal{A} \to \mathcal{B}$ *and any natural transformation* $f^0 : T^0 \to {T'}^0$*, there exists a unique extension* $f^\bullet : T^\bullet \to T'^\bullet$ *commuting with the* δ^i*.*

Definition 3.4.3. *A functor* $F : \mathcal{A} \to \mathcal{B}$ *is called* effaceable *if for every* X *in* $\mathcal{A}$*, there is a monic* $f : X \hookrightarrow E$ *such that* $F(f) = 0$*.*

Dually, a functor $F : \mathcal{A} \to \mathcal{B}$ *is called* coeffaceable *if for every* $X \in \mathrm{Ob}(\mathcal{A})$*, there is an epic* $f : P \to X$ *such that* $F(f) = 0$*.*

Theorem 3.4.4 (Grothendieck)**.** *Let* $(E, \delta) = (E_i, \delta_i)_{i \geq 0} \colon \mathcal{A} \to \mathcal{B}$ *be a coeffaceable left* δ*-functor. Then* E *is universal in the following sense. For any left* δ*-functor* $(T, \delta) \colon \mathcal{A} \to \mathcal{B}$ *and any* $f_0 \colon T_0 \to E_0$ *there exist unique transformations* $f_i \colon T_i \to E_i$ *commuting with* δ*'s.*

Proof. We will prove this theorem by induction. Given any δ-functor $(T,\delta)\colon \mathcal{A} \to \mathcal{B}$ and $f_0 \colon T_0 \to E_0$, assume we have constructed $f_i \colon T_i \to E_i$ for $i < k$.

We want to construct $f_k \colon T_k \to E_k$. For this we need to define $f_k(X)\colon T_k(X) \to E_k(X)$ for each $X \in \mathcal{A}$. For each X, choose $p\colon Y \twoheadrightarrow X$, s.t. $E_i(Y) = 0$ for all $i > 0$. Let $X' = \mathrm{Ker}(p)$, so that we have a SES in $\mathcal{A}$

$$0 \to X' \to Y \to X \to 0$$

Since (T,δ) and (E,δ) are both δ-functors, by applying T and E we get

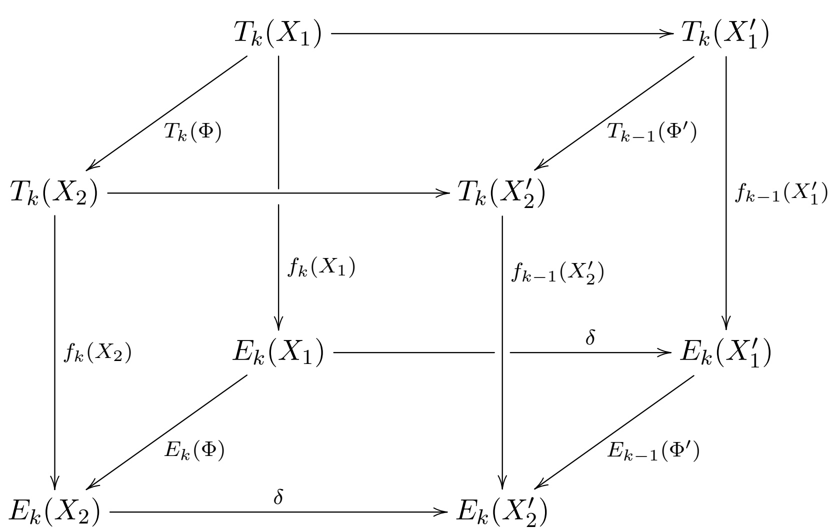
$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & T_k(X) & \xrightarrow{\delta_{k-1}} & T_{k-1}(X') & \xrightarrow{\delta_{k-1}} & T_{k-1}(Y) & \longrightarrow & T_{k-1}(X) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_k} & & \downarrow{\scriptstyle f_{k-1}(X')} & & \downarrow{\scriptstyle f_{k-1}(Y)} & & \downarrow{\scriptstyle f_{k-1}(X)} & & \\
\cdots & \longrightarrow & E_k(X) & \xrightarrow{\delta_{k-1}} & E_{k-1}(X') & \xrightarrow{\delta_{k-1}} & E_{k-1}(Y) & \longrightarrow & E_{k-1}(X) & \longrightarrow & \cdots
\end{array}$$

By assumption, $E_i(Y) = 0$ for $i > 0$, so $E_k(X) \to E_{k-1}(X')$ is actually an embedding. Since $\delta_{k-1} \circ f_{k-1}(X') \circ \delta_{k-1} = 0$, exactness of rows implies that $\mathrm{Im}\,(f_{k-1}(X') \circ \delta_{k-1}) \subset \mathrm{Im}(\delta^E_{k-1}) \subset E_{k-1}(X')$. This gives f_k as $\left(\delta^E_{k-1}\right) \circ f_{k-1}(X') \circ \delta_{k-1}$.

Now we need to check that actually maps $\{f_k\}$ do not depend on the choice Y and commute with δ's. Take $Y_1 \twoheadrightarrow X_1$ and $Y_2 \twoheadrightarrow X_2$, where $E_i(Y_1) = 0$ and $E_i(Y_2) = 0$ for $i > 0$. Consider a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_1' & \longrightarrow & Y_1 & \longrightarrow & X_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Phi'} & & \downarrow & & \downarrow{\scriptstyle \Phi} & & \\
0 & \longrightarrow & X_2' & \longrightarrow & Y_2 & \longrightarrow & X_2 & \longrightarrow & 0
\end{array}$$

This yields to a cubic diagram



In this diagram top and bottom faces commute since T and E are δ-functors. The front and back faces commute by construction of the maps f_k. The right face commutes by induction. So we only need to show that the left face commutes. Since the front face commutes, we get $\delta \circ f_k(X_2) \circ T_k(\Phi) = \delta \circ E_k(\Phi) \circ f_k(X_1)$. But $E(Y_2) = 0$, so $\delta \colon E_k(X_2) \to E_{k-1}(X_2')$ is injective. Hence $f_k(X_2) \circ T_k(\Phi) = E_k(\Phi) \circ f_k(X_1)$. $\square$

3.5 Main properties of resolutions

The goal of this section is to prove the existence of classical derived functors. Let $\mathcal{A}$ be an abelian category. We assume $\mathcal{A}$ has enough projectives, and as before write $\mathsf{Com}(\mathcal{A})$ for the category of chain complexes in $\mathcal{A}$. Recall that if $X^\bullet$ is a complex in $\mathcal{A}$, its *projective resolution (or approximation)* in $\mathsf{Com}(\mathcal{A})$ is a quasi-isomorphism $P^\bullet \to X^\bullet$, where $P^\bullet$ is (pointwise) projective.

Lemma 3.5.1. *If $X^\bullet \in \mathrm{Ob}(\mathsf{Com}(\mathcal{A}))$ is bounded from above ($X^i = 0$ for $i \gg 0$), then $X^\bullet$ admits a projective resolution.*

Proof. The proof is by induction. First, if $X^i = 0$ for $i \geqslant i_0$, put $P^i = 0$ and f^i for $i \geqslant i_0$. Now assume that we have already constructed P^i and f^i for $i \geqslant k+1$:

$$\begin{array}{ccccccccc} & & & & P^{k+1} & \xrightarrow{d_P^{k+1}} & P^{k+2} & \xrightarrow{d_P^{k+2}} & \cdots \\ & & & & \downarrow{\scriptstyle f^{k+1}} & & \downarrow{\scriptstyle f^{k+2}} & & \\ \cdots \longrightarrow X^{k-1} & \longrightarrow & X^k & \xrightarrow{d_X^k} & X^{k+1} & \xrightarrow{d_X^{k+1}} & X^{k+2} & \longrightarrow & \cdots \end{array}$$

Consider the cone of the morphism $f^\bullet : P^{\geqslant k+1} \to X^\bullet$. By assumption, $\mathrm{cone}(f)$ is acyclic in degree $\geqslant k+1$.

Recall that $\mathrm{cone}(f) = X^\bullet \oplus P^{\geqslant k+1}[1]$. The differential $d_{\mathrm{cone}} : X^k \oplus P^{k+1} \to X^{k+1} \oplus P^{k+2}$ is the matrix

$$\begin{pmatrix} d_X^k & f^{k+1} \\ 0 & -d_P^{k+1} \end{pmatrix}$$

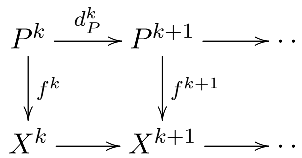
Let $Y = \mathrm{Ker}(d_{\mathrm{cone}}^k)$. Since $\mathcal{A}$ as enough projectives, we can choose $P^k \twoheadrightarrow Y$ with P^k projective. Consider the composition

$$P^k \twoheadrightarrow Y \xhookrightarrow{f^k \oplus -d_P^{k+1}} X^k \oplus P^{k+1}.$$

Since $Y = \mathrm{Ker}(d_{\mathrm{cone}}^k)$, we have

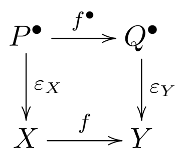
$$\begin{pmatrix} d_X^k & f^{k+1} \\ 0 & -d_P^{k+1} \end{pmatrix} \begin{pmatrix} f^k \\ -d_P^{k+1} \end{pmatrix} = 0$$

which occurs if and only if $d_X^k f^k - f^{k+1} d_P^{k+1} = 0$ and $d_P^{k+1} d_P^k = 0$. Hence, we can extend $f : P^\bullet \to X^\bullet$ by



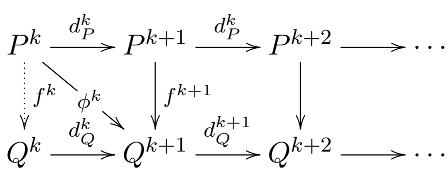
$\square$

Lemma 3.5.2. *Let $\varepsilon_X : P^\bullet \to X$ and $\varepsilon_Y : Q^\bullet \to Y$ be projective resolutions for $X, Y \in \mathrm{Ob}(\mathcal{A})$. Then any $f : X \to Y$ in $\mathcal{A}$ lifts to $f^\bullet : P^\bullet \to Q^\bullet$ making the following diagram commute:*

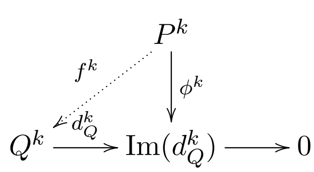


We can choose $f^\bullet$ so that $\mathrm{H}^0(f^\bullet) = f$. Moreover, any two such lifts $f^\bullet, g^\bullet : P^\bullet \to Q^\bullet$ are homotopic.

Proof. Let's write $P^1 = X$, $d_P^0 = \varepsilon_X$ and $Q^1 = Y$, $d_Q^0 = \varepsilon_Y$, so that we have *acyclic* complexes $\cdots \to P^{-1} \to P^0 \to P^1 \to 0$ and $\cdots \to Q^1 \to 0$. Moreover, put $f^1 = f$. By induction, assume that f^i exists for $i \geqslant k+1$:

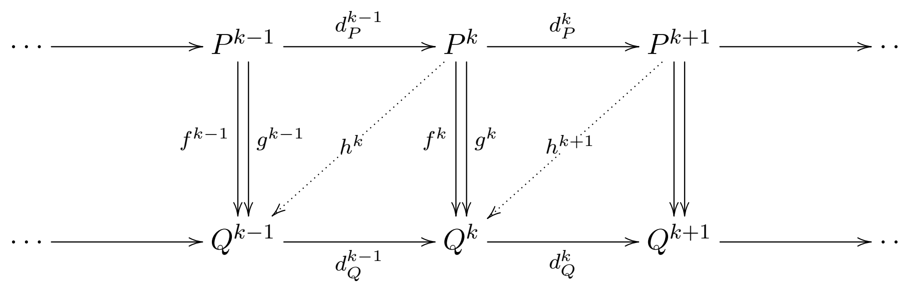


Put $\phi^k = f^{k+1} \circ d_P^k$. Then $d_Q^{k+1} \phi^k = d_Q^{k+1} f^{k+1} d_P^k = f^{k+2} d_P^{k+1} d_P^k = 0$, so $\mathrm{Im}(\phi^k) \subset \mathrm{Ker}(d_Q^{k+1}) = \mathrm{Im}(d_Q^k)$. Hence we have

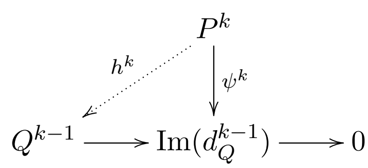


By the projectivity of P^k, there exists f^k such that $d_Q^k f^k = \phi^k = f^{k+1} d_P^k$. This finishes the induction.

Now we show that given $f^\bullet, g^\bullet : P^\bullet \to Q^\bullet$ lifting f with $\mathrm{H}^0(f^\bullet) = \mathrm{H}^0(g^\bullet)$, we want to construct a homotopy between $f^\bullet$ and $g^\bullet$:



such that for all k, $f^k - g^k = d_Q^{k-1}h^k + h^{k+1}d_P^k$. Put $h^0 = 0$. We argue by induction. Assume that we already have constructed $h^{k+1}, h^{k+2}, \ldots$ and we want to construct h^k. Consider $\psi^k = f^k - g^k - h^{k+1}d_P^k$, and note that $d_Q^k\psi^k = (f^{k+1} - g^{k+1} - d_Q^k h^{k+1})d_P^k = h^{k+2}d_P^{k+1}d_P^k = 0$. Consider



We get the existence of h^k by the projectivity of P^k. □

Note that in the proof of this lemma, we did *not* use the fact that the Q^i are projective (but we did use the fact that $\mathrm{H}^\bullet(Q) = 0$). Moreover, we didn't use the fact that $P^\bullet$ is acyclic. So we have the following corollary.

Corollary 3.5.3. *Let $P^\bullet$ be (pointwise) projective and bounded above, and let $Q^\bullet$ be acyclic. Then any $f^\bullet : P^\bullet \to Q^\bullet$ is homotopic to zero.*

3.6 Definition of classical derived functor via δ-functors

Recall the definition of a δ-functor. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories.

Definition 3.6.1. *A* (left) δ-functor *is a family of additive functors $T = (T_i)_{i\geqslant 0}$, $T_i : \mathcal{A} \to \mathcal{B}$ and morphisms $\delta_i : T_{i+1}(A'') \to T_i(A')$ of functors on the category of all short exact sequences $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$, such that for all such exact sequences, the following sequence is exact in $\mathcal{B}$.*

$$\cdots \longrightarrow T_1(A) \longrightarrow T_1(A'') \xrightarrow{\delta_0} T_0(A') \longrightarrow T_0(A) \longrightarrow T_0(A'') \longrightarrow 0$$

Definition 3.6.2. *Let $F : \mathcal{A} \to \mathcal{B}$ be a right-exact additive functor. The* classical left-derived functor *of F is a pair $(\mathsf{L}F, \alpha)$ consisting of a left δ-functor $\mathsf{L}F = (\mathsf{L}_i F, \delta_i)_{i\geqslant 0}$ and* *an isomorphism of functors* $\mathsf{L}_0 F \xrightarrow{\sim} F$*, which is universal among all left* δ*-functors in the following sense. For all left* δ*-functors* (T, δ) *and morphisms* $f_0 : \mathsf{L}_0 F \to T_0$*, there is a unique sequence of morphisms* $f_i : \mathsf{L}_i F \to T_i$ *extending* f_0*.*

Remark 3.6.3. It follows from the definition that $(\mathsf{L}F, \alpha)$ is determined up to unique isomorphism, if it exists. The functor $\mathsf{L}F$ is only determined up to automorphisms of F. Indeed, given $(\mathsf{L}F, \alpha)$ and $(\mathsf{L}'F, \alpha')$, by definition $\alpha : \mathsf{L}_0 F \xrightarrow{\sim} F$ and $\alpha' : \mathsf{L}'_0 F \xrightarrow{\sim} F$. Then $f_0 = (\alpha')^{-1}\alpha$ is an isomorphism $\mathsf{L}_0 F \to \mathsf{L}'_0 F$, which extends uniquely to an isomorphism $f : \mathsf{L}F \to \mathsf{L}'F$.

Previously we defined derived functors using injective (resp. projective) resolutions. Here, we prove that under various assumptions, this agrees with the above definition.

Theorem 3.6.4. *Let* $F : \mathcal{A} \to \mathcal{B}$ *be a right-exact additive functor. Assume* $\mathcal{A}$ *has enough projective objects. For any* $X \in \mathrm{Ob}(\mathcal{A})$ *choose a projective resolution* $P^\bullet \to X$ *in* $\mathsf{Com}(\mathcal{A})$*. Define, for* $i \geqslant 0$*,*

$$\mathsf{L}_i F(X) = \mathrm{H}^{-i}(F(P^\bullet))$$

Then $(\mathsf{L}_i F)_{i \geqslant 0}$ *is the classical left-derived functor of* F *in the sense of definition 3.6.2.*

Proof. Our main tools will be lemmas 3.5.1 and 3.5.2. In the proof we need to do the following steps.

1. prove that the $\mathsf{L}_i F$ are actually functors;
2. check independence of $\mathsf{L}_i F$ on the choice of resolutions (up to isomorphism);
3. prove that $\mathsf{L}_i F$ actually form a δ-functor. Namely, we need to show that there exist connecting morphisms $\delta_i : \mathsf{L}_{i+1} F(A'') \to \mathsf{L}_i F(A')$ satisfying the definition 3.6.1;
4. prove the existence of isomorphism $\mathsf{L}_0 F \xrightarrow{\sim} F$;
5. finally, we need to prove the universality of $\mathsf{L}F$.

First we prove the $\mathsf{L}_i F$ are functors. Let $f : X \to Y$ be a morphism in $\mathcal{A}$. Let $P^\bullet \to X$ and $Q^\bullet \to Y$ be our chosen resolutions. By Lemma 3.5.2, there is $f^\bullet : P^\bullet \to Q^\bullet$ lifting f. Define $\mathsf{L}_i F(f) = \mathrm{H}^{-i}(F(f^\bullet))$. By the same lemma, any two lifts $f^\bullet, g^\bullet : P^\bullet \to Q^\bullet$ are homotopic, i.e. we can write $f^\bullet - g^\bullet = dh + hd$. Since F is additive, we have $F(f^\bullet) - F(g^\bullet) = F(h)F(d) - F(d)F(h)$, so $F(f^\bullet)$ and $F(g^\bullet)$ are homotopic as morphisms $F(P^\bullet) \to F(Q^\bullet)$. It follows that $\mathrm{H}^{-i}(F(f^\bullet)) = \mathrm{H}^{-i}(F(g^\bullet))$, so $\mathsf{L}_i F(f)$ is well-defined.

Next we show that $\mathsf{L}_i F$ is independent of the choice of projective resolutions. Suppose we have two resolutions $P^\bullet, Q^\bullet \to X$. By Lemma 3.5.2, the map id_X lifts to morphisms $f^\bullet : P^\bullet \to Q^\bullet$ and $g^\bullet : Q^\bullet \to P^\bullet$ such that $f^\bullet \circ g^\bullet \sim \mathrm{id}_{F(Q^\bullet)}$ and $g^\bullet \circ f^\bullet \sim \mathrm{id}_{F(P^\bullet)}$. This tells us that $F(f^\bullet)$ and $F(g^\bullet)$ are inverses (up to homotopy). Thus $F(P^\bullet)$ and $F(Q^\bullet)$ are homotopy-equivalent, so $\mathrm{H}^{-i}(F(P^\bullet)) \simeq \mathrm{H}^{-i}(F(Q^\bullet))$. Thus $\mathsf{L}_i F$ is well-defined.

Now we construct the connecting homomorphisms δ_i. Consider a short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathcal{A}$. Replace X and Y by projective resolutions. By Lemma 3.5.2, f lifts to $f^\bullet : P^\bullet \to Q^\bullet$. Let $R^\bullet := \operatorname{cone}(f^\bullet)$. We claim that $R^\bullet$ is a projective resolution of Z. Recall there is an exact sequence

$$0 \longrightarrow Q^\bullet \longrightarrow R^\bullet \longrightarrow P^\bullet[1] \longrightarrow 0 \qquad (*)$$

of complexes in $\mathcal{A}$.

Remark 3.6.5. Instead of writing the sequence $(*)$ we could have written an exact *distinguished triangle*

$$P^\bullet \xrightarrow{f^\bullet} Q^\bullet \xrightarrow{i^\bullet} R^\bullet \xrightarrow{p^\bullet} P^\bullet[1] \xrightarrow{f^\bullet[1]} Q^\bullet[1] \longrightarrow \dots$$

This sequence is called "triangle" since we can write it as a triangle

$$\begin{array}{ccccc} & & Q^\bullet & & \\ & {}^{f^\bullet}\nearrow & & \searrow^{i^\bullet} & \\ P^\bullet & & \xleftarrow[{[1]}]{\qquad\qquad} & & R^\bullet \end{array}$$

Applying $\mathrm{H}^\bullet$ to $(*)$ gives a long exact sequence

$$\begin{array}{ccccccccccccccc} \cdots \to & \mathrm{H}^{-1}(Q^\bullet) & \to & \mathrm{H}^{-1}(R^\bullet) & \to & \mathrm{H}^{-1}(P[1]) & \to & \mathrm{H}^{0}(Q^\bullet) & \to & \mathrm{H}^{0}(R^\bullet) & \to & \mathrm{H}^{1}(P[1]) & \to \cdots \\ & \| & & \| & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \| & \\ & 0 & & 0 & & X & \xrightarrow{f} & Y & \longrightarrow & Z & & 0 & \end{array}$$

In this diagram we have $\mathrm{H}^{-1}(Q^\bullet) = 0$ since $Q^\bullet$ is acyclic in higher degrees, $\mathrm{H}^{-1}(P^\bullet[1]) \simeq \mathrm{H}^0(P^\bullet) \simeq P$. Since f is inclusion, the image of $\mathrm{H}^{-1}(R^\bullet)$ in $\mathrm{H}^{-1}(P^\bullet)[1]$ is zero, and since $\mathrm{H}^{-1}(Q^\bullet) = 0$ we conclude that $\mathrm{H}^{-1}(R^\bullet)$ is also zero. It is obvious from acyclicity of $P^\bullet$ and $Q^\bullet$ that $\mathrm{H}^{-i}(R^\bullet) = 0$ for $\forall i \geq 1$. Since $\mathrm{H}^0(P^\bullet[1]) = \mathrm{H}^1(P^\bullet) = 0$, it follows that $\mathrm{H}^0(R^\bullet) \simeq Y/X \simeq Z$. Hence $R^\bullet \to Z$ is a quasi-isomorphism, and so $R^\bullet$ is a resolution of Z.

But $R^i = Q^i \oplus P^{i+1}$ is termwise projective, so $R^\bullet$ is actually a *projective* resolution of Z.

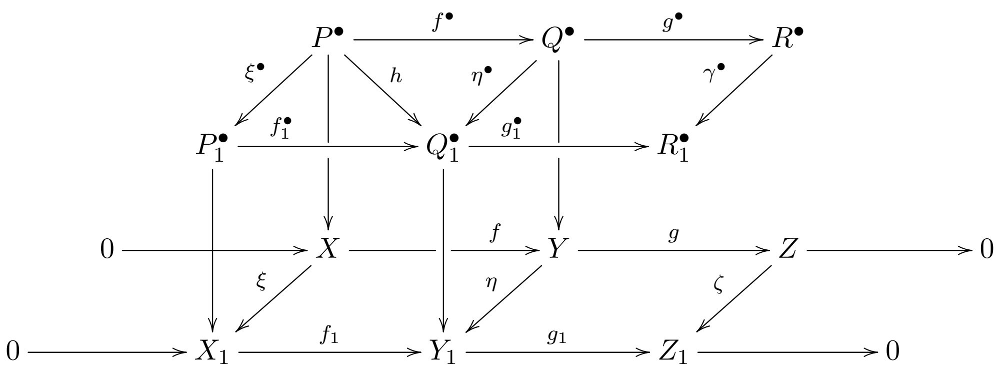
We can apply F to $(*)$. Since P is projective and F is right-exact, we get the short exact sequence

$$0 \longrightarrow F(Q^\bullet) \longrightarrow F(R^\bullet) \longrightarrow F(P^\bullet[1]) \longrightarrow 0$$

By the additivity of F, $F(R^\bullet) = \operatorname{cone}(F(f^\bullet))$. Moreover $F(P^\bullet[1]) = F(P)[1]$, so applying $\mathrm{H}^\bullet$, we get

$$\cdots \longrightarrow \mathsf{L}_0F(X) \longrightarrow \mathsf{L}_0F(Y) \longrightarrow \mathsf{L}_0F(Z) \xrightarrow{\delta_0} \mathsf{L}_{-1}F(X) \longrightarrow \cdots$$

We need to prove this exact sequence is functorial in short-exact sequences. Suppose we replaced the short exact sequence $0 \to X \to Y \to Z \to 0$ by $0 \to X_1 \to Y_1 \to Z_1 \to 0$. Pick projective resolutions as above. Then we get the following diagram



Then the upper square commutes up to homotopy, i.e. $\eta^\bullet \circ f^\bullet \sim f_1^\bullet \circ \xi^\bullet$. Choose a homotopy $h^k : P^k \to Q_1^{k-1}$ such that $\eta^k \circ f^k - f_1^k \circ \xi^k = d_{Q_1}^{k-1} \circ h^k + h^k \circ d_P^k$. Define $\gamma^k : R^k \to R_1^k$ by

$$\gamma^k : Q^k \oplus P^{k+1} \to Q_1^k \oplus P_1^{k+1}$$

given by the matrix

$$\gamma^k = \begin{pmatrix} \eta^k & h^{k+1} \\ 0 & \xi^{k+1} \end{pmatrix}$$

We claim that this gives us a morphism of exact triangles. Namely, we have the following diagram:

$$\begin{array}{ccccccc}
P^\bullet & \xrightarrow{f^\bullet} & Q^\bullet & \xrightarrow{i^\bullet} & R^\bullet & \xrightarrow{p^\bullet} & P^\bullet[1] \\
{\scriptstyle \xi}\downarrow & & {\scriptstyle \eta}\downarrow & & {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \xi[1]} \\
P_1^\bullet & \xrightarrow{f_1^\bullet} & Q_1^\bullet & \xrightarrow{i_1^\bullet} & R_1^\bullet & \xrightarrow{p_1^\bullet} & P_1^\bullet[1]
\end{array}$$

We need to check that the diagram above commutes up to homotopy. We have already seen that the first square commutes up to homotopy. Let's check the second square.

$$\begin{array}{ccc}
Q^k & \xrightarrow{i^k} & Q^k \oplus P^{k+1} \\
\downarrow & & \downarrow \\
Q_1^k & \xrightarrow{i^k} & Q_1^k \oplus P_1^{k+1}
\end{array}
\qquad
\begin{array}{ccc}
q^k & \longmapsto & (q^k, 0) \\
\downarrow & & \downarrow \\
\eta^k(q^k) & \longmapsto & (\eta^k(q^k), 0)
\end{array}
\tag{4.1}$$

So the second square commutes. Similarly for the third square.

If we now apply F to the diagram (4.1) and take cohomology, we will obtain the following commutative diagram between long exact sequences

$$\begin{array}{ccccccccccc}
\cdots & \to & \mathsf{L}_{i+1}F(Z) & \to & \mathsf{L}_iF(X) & \to & \mathsf{L}_iF(Y) & \to & \mathsf{L}_iF(Z) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \mathsf{L}_{i+1}F(Z) & \to & \mathsf{L}_iF(X) & \to & \mathsf{L}_iF(Y) & \to & \mathsf{L}_iF(Z) & \to & \cdots
\end{array}$$

This exactly means functoriality on short exact sequences. Thus, $(\mathsf{L}_i F, \delta)$ is a left δ-functor.

Next we need to prove existence of natural isomorphism $\mathsf{L}_0 F \simeq F$. Indeed, choose any $X \in \mathcal{A}$. Replace X by resolution $P^\bullet \twoheadrightarrow X$ and look at the first two terms

$$P^{-1} \xrightarrow{d} P^0 \longrightarrow X \longrightarrow 0$$

Applying right exact functor F to this exact sequence, we get an exact sequence

$$F(P^{-1}) \xrightarrow{F(d)} F(P^0) \longrightarrow F(X) \longrightarrow 0$$

Then $\mathsf{L}_0 F(X) := \mathrm{H}^0(F(P^\bullet)) = \operatorname{Coker} F(d) \simeq F(\operatorname{Coker}(d)) \simeq F(X)$. Notice that here we used right exactness of F again, which gave $\operatorname{Coker} F(d) \simeq F(\operatorname{Coker}(d))$.

Finally, we need to check the universal property.

Lemma 3.6.6. *If $\mathcal{A}$ has enough projectives then the functor $\mathsf{L}_i F, \delta$ is coeffaceable.*

Proof. Given $X \in \mathcal{A}$ we can replace it by a projective cover $P \twoheadrightarrow X$ for some projective $P \in \mathcal{A}$. Then we claim that $\mathsf{L}_i F(P) = 0$, $\forall i > 0$. But this is obvious, since P is projective, so we can choose its projective resolution to be the complex $P^\bullet = [0 \to P \to 0]$. □

Having proved lemma 3.6.6 we can apply theorem 3.4.4 to automatically get the universality of classical derived functors. □

3.7 Examples of derived functors

The functors Ext^i are classical derived functors of $\operatorname{Hom} : (X, Y) \mapsto \operatorname{Hom}(X, Y)$. We need to make this precise, because we can think of $\operatorname{Hom}(-, -)$ as a functor in either variable. We will assume that $\mathcal{A}$ is an abelian category with enough injectives and enough projectives. Then, for $X, Y \in \operatorname{Ob}(\mathcal{A})$, consider the functors

$$\operatorname{Hom}(-, Y) : \mathcal{A}^\circ \to \mathsf{Ab}$$
$$\operatorname{Hom}(X, -) : \mathcal{A} \to \mathsf{Ab}$$

Both these functors are left-exact, so we can define

$$\operatorname{Ext}^i_1(X, Y) = \mathsf{R}^i \operatorname{Hom}(-, Y)(X) = \mathrm{H}^{-i}(\operatorname{Hom}(P^\bullet, Y))$$
$$\operatorname{Ext}^i_2(X, Y) = \mathsf{R}^i \operatorname{Hom}(X, -)(Y) = \mathrm{H}^{i}(\operatorname{Hom}(X, I^\bullet)),$$

where $P^\bullet \to X$ is a projective resolution of X and $Y \to I^\bullet$ is an injective resolution of Y.

Theorem 3.7.1. *There is a natural isomorphism* $\operatorname{Ext}^i_1(X, Y) \simeq \operatorname{Ext}^i_2(X, Y)$.

We prove this by giving another (more explicit) construction of Ext, due to Yoneda. Let's start with $i = 1$. Fix X and Y in $\mathcal{A}$, and consider the set $E^1(X, Y)$ of all short exact sequences

$$\alpha = (0 \to Y \to Z \to X \to 0)$$

Define an equivalence relation on $E^1(X, Y)$ by setting $\alpha \sim \alpha'$ if there is a morphism $\varphi : Z \to Z'$ in $\mathcal{A}$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\varphi} & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

The snake lemma forces φ to be an isomorphism. Define $\mathrm{Ext}^1(X, Y) = E^1(X, Y)/\sim$. The additive structure on $\mathrm{Ext}^1(X, Y)$ is defined by $+ : \mathrm{Ext}^1(X, Y) \times \mathrm{Ext}^1(X, Y) \to \mathrm{Ext}^1(X, Y)$. Given $\alpha_1 = (0 \to Y \to Z_1 \to X \to 0)$ and $\alpha_2 = (0 \to Y \to Z_2 \to X \to 0)$, we define $\alpha_1 + \alpha_2$ as follows. There is an exact sequence $\alpha_1 \oplus \alpha_2$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \\ & & {\scriptstyle +}\uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & Y \oplus Y & \longrightarrow & \widetilde{Z} & \longrightarrow & X & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle\Delta} & & \\ 0 & \longrightarrow & Y \oplus Y & \longrightarrow & Z_1 \oplus Z_2 & \longrightarrow & X \oplus X & \longrightarrow & 0 \end{array}$$

Here $\widetilde{Z}$ is the pullback in the lower-right square, i.e. $\widetilde{Z} = X \times_{X \oplus X} (Z_1 \oplus Z_2)$. We define Z to be the pushout in the upper-left square, i.e.

$$Z = Y \sqcup_{Y \oplus Y} \widetilde{Z} = (X \times_{X \oplus X} (Z_1 \oplus Z_2)) \sqcup_{Y \oplus Y} Y.$$

The sequence $\alpha_1 + \alpha_2$ is $0 \to Y \to Z \to X \to 0$. We need to check that $+$ is compatible with $\sim$. The unit is the trivial split-exact sequence $0 \to Y \to Y \oplus X \to X \to 0$.

In general, for $i = k \geqslant 1$, we define $E^k(X, Y)$ to be the set of exact sequences of the form

$$\alpha = \left(0 \to Y \to \underbrace{Z_k \to Z_{k-1} \to \cdots \to Z_1}_{Z_\bullet} \to X \to 0 \right).$$

We say that α is *elementary equivalent* to $\alpha' \in E^k(X, Y)$ if there exists a morphism $\varphi_\bullet : Z_\bullet \to Z'_\bullet$ such that the following diagram commutes:

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Z_k & \longrightarrow & Z_{k-1} & \longrightarrow & \cdots & \longrightarrow & Z_1 & \longrightarrow & X & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\varphi_k} & & \downarrow{\scriptstyle\varphi_{k-1}} & & & & \downarrow{\scriptstyle\varphi_1} & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & Z'_k & \longrightarrow & Z'_{k-1} & \longrightarrow & \cdots & \longrightarrow & Z'_1 & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

We put $\alpha \sim \alpha'$ if there is a finite chain of elementary equivalences (of unspecified direction)

$$Z_\bullet \sim Z_\bullet^{(1)} \sim \cdots \sim Z_\bullet^{(n)} \sim Z'_\bullet.$$

More formally,given $X, Y \in \mathrm{Ob}(\mathcal{A})$ and $k \geqslant 1$, define the following category $\mathsf{Ext}^k(X, Y)$ with objects complexes of length k having homology

$$\mathrm{H}_i(Z_\bullet) = \begin{cases} Y & \text{if } i = k \\ X & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Morphisms in $\mathsf{Ext}^k(X, Y)$ are morphisms $\varphi : Z_\bullet \to Z'_\bullet$ of chain complexes satisfying $\mathrm{H}_1(\varphi_\bullet) = \mathrm{id}_X$, $\mathrm{H}_1(\varphi_\bullet) = \mathrm{id}_Y$. We could then define $\mathrm{Ext}^k(X, Y) = E^k(X, Y) = \pi_0(\mathsf{Ext}^k(X, Y))$.

(**the definition using Ext does not work when** $k = 1$.)

By convention, $\mathrm{Ext}^0(X, Y) = \mathrm{Hom}_{\mathcal{A}}(, Y)$.

Lemma 3.7.2. *The Yoneda construction of* Ext^k *defines a bifunctor, covariant in* Y *and contravariant in* X*, i.e.* $\mathrm{Ext}^k : \mathcal{A}^\circ \times \mathcal{A} \to \mathsf{Ab}$*.*

Proof. Let $\alpha = (0 \to Y \to Z_\bullet \xrightarrow{p_1} X \to 0$ and similarly for α', and let $f : X' \to X$ be a morphism. We need to construct $f^* : \mathrm{Ext}^k(X, Y) \to \mathrm{Ext}^k(X', Y)$. Let $Z'_1 = \mathrm{Ker}(Z_1 \oplus X' \to X)$, where $Z_1 \oplus X' \to X$ is the morphism $(z, x) \mapsto p_1(x) - f(x)$. Similarly, define $Z'_i = Z_i$ for $i = 2, \ldots, k$. We have a commutative diagram

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & Y & \longrightarrow & Z'_k & \longrightarrow & \cdots & \longrightarrow & Z'_2 & \longrightarrow & Z'_1 & \longrightarrow & X' & \longrightarrow & 0 \\
 & & \| & & \| & & & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & Y & \longrightarrow & Z_k & \longrightarrow & \cdots & \longrightarrow & Z_2 & \longrightarrow & Z_1 & \xrightarrow{p_1} & X & \longrightarrow & 0
\end{array}$$

Essentially, let $V_\bullet = (0 \to Y \to Z_k \to \cdots \to Z_2 \to 0$. We have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_\bullet & \longrightarrow & Z'_1 & = & Z' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & V_\bullet & \longrightarrow & Z_1 & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

We define $f^*\alpha$ to be the class of the exact sequence $0 \to Y \to Z_k \to \cdots \to Z_2 \to Z'_1 \to X \to 0$ in $\mathrm{Ext}^k(X', Y)$. As an exercise, verify that when compositions are defined, $(g \circ f)^* = f^* \circ g^*$. $\square$

Theorem 3.7.3. *If* $\mathcal{A}$ *has both enough injectives and enough projectives, then for all* $X, Y \in \mathrm{Ob}(\mathcal{A})$*, there are isomorphisms of abelian groups:*

$$\mathrm{Ext}_1^i(X, Y) \simeq \mathrm{Ext}^i(X, Y) \simeq \mathrm{Ext}_2^i(X, Y).$$

Proof. We prove the first isomorphism by showing that $\{\mathrm{Ext}^i(-,Y)\}$ is a universal δ-functor. First, we need to show that $\mathrm{Ext}^\bullet(-,Y)$ is effaceable, and second we need to show that it is a δ-functor. For the first, it suffices to show that $\mathrm{Ext}^i(P,Y)=0$ for $i \geqslant 1$ whenever P is projective. For $i=1$, $\mathrm{Ext}^1(P,Y)=0$ because any exact sequence $0 \to Y \to Z \to P \to 0$ has a splitting. For $i \geqslant 2$, it is easy to check that every exact sequence $0 \to Y \to Z_k \to \cdots \to Z_1 \to P \to 0$ is equivalent to a (trivial) sequence of the form $0 \to Y \to 0 \to \cdots \to P \to P \to 0$.

Now we show that $\mathrm{Ext}^\bullet(-,Y)$ is a δ-functor. Given $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$, we first construct $\widetilde{\delta}_k : E^{k-1}(X',Y) \to E^k(X'',Y)$. We want an exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^{k-1}(X'',Y) \longrightarrow \mathrm{Ext}^{k-1}(X,Y) \longrightarrow \mathrm{Ext}^{k-1}(X',Y) \xrightarrow{\delta^{k-1}} \mathrm{Ext}^k(X'',Y) \longrightarrow \cdots$$

Given $Z'_\bullet \in E^k(X',Y)$, define $Z''_1 = X$, and $Z''_{i+1} = Z'_i$ for $i = 1,\ldots,k-1$. We claim that $Z''_\bullet \in E^k(X'',Y)$, and that this assignment is compatible with the equivalence relation on $E^k(X',Y)$. $\square$

Exercise Let $\mathcal{A} = \mathsf{Ab}$. Prove that $\mathrm{Ext}^i_{\mathcal{A}}(X,Y) = 0$ for all $i \geqslant 2$, where X and Y are arbitrary abelian groups.

Section 3 in the Appendix A demonstrates an application of the classical derived functors to representation theory of quivers.

Chapter 5

Derived categories

1 Localization of categories

1.1 Motivation

We start with an example motivating the use of derived categories. Let k be an algebraically closed field of characteristic zero. Consider the projective line $\mathbb{P}^1_k$ over k, and let $\mathcal{A} = \mathsf{coh}(\mathbb{P}^1_k)$ be the category of coherent sheaves on $\mathbb{P}^1_k$. Let $\mathscr{E}$ be the object $\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1)$ in $\mathcal{A}$. One usually calls $\mathscr{E}$ *the tilting sheaf*.

It is easy to compute $\mathrm{End}_{\mathcal{A}}(\mathscr{E})$:

$$\begin{aligned}\mathrm{End}_{\mathcal{A}}(\mathscr{E}) &= \mathrm{Hom}_{\mathcal{A}}(\mathscr{O} \oplus \mathscr{O}(1), \mathscr{O} \oplus \mathscr{O}(1)) \\ &\simeq \begin{pmatrix} \mathrm{Hom}_{\mathcal{A}}(\mathscr{O}, \mathscr{O}) & \mathrm{Hom}_{\mathcal{A}}(\mathscr{O}, \mathscr{O}(1)) \\ \mathrm{Hom}_{\mathcal{A}}(\mathscr{O}(1), \mathscr{O}) & \mathrm{Hom}_{\mathcal{A}}(\mathscr{O}(1), \mathscr{O}(1)) \end{pmatrix} \\ &\simeq \begin{pmatrix} k & k^{\oplus 2} \\ 0 & k \end{pmatrix} \\ &\simeq k[\bullet \rightrightarrows \bullet].\end{aligned}$$

Write $B = \mathrm{End}_{\mathcal{A}}(\mathscr{E})$ for this ring.

Note that if $\mathscr{F} \in \mathsf{coh}(\mathbb{P}^1)$, then $\mathrm{Hom}_{\mathcal{A}}(\mathscr{E}, \mathscr{F})$ is naturally a right B-bimodule, with the action of $B = \mathrm{Hom}_{\mathcal{A}}(\mathscr{E}, \mathscr{E})$

$$\mathrm{Hom}_{\mathcal{A}}(\mathscr{E}, \mathscr{F}) \times \mathrm{Hom}_{\mathcal{A}}(\mathscr{E}, \mathscr{E}) \to \mathrm{Hom}_{\mathcal{A}}(\mathscr{E}, \mathscr{F})$$

coming from composition. Thus $\mathrm{Hom}(\mathscr{E}, -)$ induces a functor

$$F : \mathsf{coh}(\mathbb{P}^1) \to \mathsf{Mod}(kQ) \simeq \mathsf{Rep}_k(Q^\circ),$$

where Q is the quiver $\bullet \rightrightarrows \bullet$ and $\mathsf{Rep}_k(Q^\circ)$ is the category of its representations (see Section 3 in the Appendix A on quivers).

The functor F has a right-adjoint G, induced by tensoring with $\mathscr{E}$. One might hope that the functor F is an equivalence of categories, but this is not the case. For example, let $\mathscr{F} = \mathscr{O}_{\mathbb{P}^1}(-1)$. Then $F(\mathscr{F}) \simeq \Gamma(X, \mathscr{O}(-1)) \oplus \Gamma(X, \mathscr{O}(-2)) = 0$, but $\mathscr{F} \neq 0$.

Let's compute $\operatorname{Ext}^1(\mathscr{E}, \mathscr{F})$. This is, by definition $\operatorname{Ext}^1(\mathscr{E}, \mathscr{O}(-1))$, which is isomorphic by Grothendieck-Serre duality (**Reference here?**) to

$$\operatorname{Ext}^1(\mathscr{E}(-1), \mathscr{O}(-2)) \simeq \mathrm{H}^0(X, \mathscr{E}(-1))^\vee = \Gamma(\mathscr{O}(-1) \oplus \mathscr{O})^\vee \simeq \Gamma(\mathscr{O})^\vee \simeq k \neq 0.$$

So we see that even though $F(\mathscr{O}_{\mathbb{P}^1}(-1)) = 0$, the derived functor $\mathsf{R}F(\mathscr{O}_{\mathbb{P}^1}(-1)) = \operatorname{Ext}^1(\mathscr{E}, \mathscr{O}(-1))$ is non-zero, allowing us to retain some more information.

Theorem 1.1.1 (Beilinson, [Bei78])**.** *There is an equivalence of (triangulated) categories*

$$\mathsf{R}\operatorname{Hom}(\mathscr{E}, -) : \mathsf{D}^b(\mathsf{Coh}\,\mathbb{P}^1) \to \mathsf{D}^b(\mathsf{Rep}_k\, Q^\circ).$$

Here $\mathsf{D}^b(\mathsf{Rep}_k\, Q^\circ)$ denotes the *derived category* of the category of modules over the path algebra kQ. The derived category of an abelian category $\mathcal{A}$ can be defined as a category obtain from the category $\mathsf{Com}(\mathcal{A})$ of complexes in $\mathcal{A}$ by formally "inverting" all the quasi-isomorphisms. More precisely, we have the following definition.

Definition 1.1.2. *Let $\mathcal{A}$ be an abelian category. Let S be the class of all quasi-isomorphisms in $\mathsf{Com}(\mathcal{A})$. Then the* derived category $\mathsf{D}(\mathcal{A})$ *of $\mathcal{A}$ is the localization $\mathsf{D}(\mathcal{A}) = \mathsf{Com}(\mathcal{A})[S^{-1}]$ of $\mathcal{A}$ at S.*

To explain what this definition actually means, we now begin systematic study of localization of categories.

1.2 Definition of localization

For a category $\mathcal{D}$, let $\operatorname{Iso}(\mathcal{D}) \subset \mathsf{Mor}(\mathcal{D})$ be the class of all isomorphisms in $\mathcal{D}$.

Definition 1.2.1. *Let S be a (non-empty) class of morphisms in a category $\mathcal{C}$. The* localization of $\mathcal{C}$ at S *is a pair $(\mathcal{C}[S^{-1}], Q)$ consisting of a category $\mathcal{C}[S^{-1}]$ and a functor $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ such that*

- *(L1)* $Q(S) \subset \operatorname{Iso}(\mathcal{C}[S^{-1}])$
- *(L2) Q is universal among all $F : \mathcal{C} \to \mathcal{D}$ such that $F(S) \subset \operatorname{Iso}(\mathcal{D})$. In other words, for any such functor F there exists unique (up to isomorphism) functor $\widetilde{F} : \mathcal{C}[S^{-1}] \to \mathcal{D}$ such that $F \simeq \widetilde{F} \circ Q$.*

The category $\mathcal{C}[S^{-1}]$ is called the localization category *(or simply* localization*) of $\mathcal{C}$ at S and Q is called the* localization functor*. We will usually denote localization simple by $\mathcal{C}[S^{-1}]$ omitting the functor Q.*

Lemma 1.2.2. *If the localization $\mathcal{C}[S^{-1}]$ exists, then it is unique up to unique equivalence.*

Proof. It simply follows from the universal property (L2). $\square$

Definition 1.2.3. *Let S be a class of morphisms in a category $\mathcal{C}$. Denote by $\bar{S}$ the smallest class in* $\mathrm{Mor}(\mathcal{C})$ *such that*

(S1) $S \subset \bar{S}$

(S2) $\mathrm{Iso}(\mathcal{C}) \subset \bar{S}$

(S3) "two out of three property": if $f, g \in \mathrm{Mor}(\mathcal{C})$ are composable and two out of three morphisms f, g, gf are in $\bar{S}$, then the third one is also in $\bar{S}$

The class $\bar{S}$ is called saturation *of S. We say that $S \subset \mathrm{Mor}(\mathcal{C})$ is* saturated *if $\bar{S} = S$.*

Example 1.2.4. The following are examples of saturated classes of morphisms.

1. the class of all isomorphisms in a category $\mathcal{C}$;
2. the class of quasi-isomorphisms in $\mathsf{Com}(\mathcal{A})$ for any abelian category $\mathcal{A}$;
3. the class of all weak equivalences in a model category (we will discuss model categories in **Section?**).

Remark 1.2.5. We can regard any saturated class S of morphisms in $\mathcal{C}$ as a *wide* subcategory of $\mathcal{C}$ (i.e. $\mathrm{Ob}(S) = \mathrm{Ob}(\mathcal{C})$). Indeed, axiom (S3) implies S is closed under composition.

Lemma 1.2.6. *For a class $S \subset \mathrm{Mor}(\mathcal{C})$, localization $\mathcal{C}[S^{-1}]$ exists if and only if $\mathcal{C}[\bar{S}^{-1}]$ exists, and then they are isomorphic.*

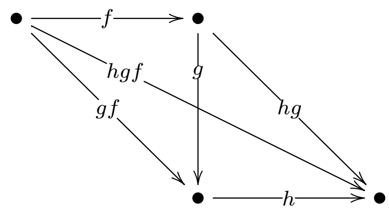
Proof. Suppose $\mathcal{C}[S^{-1}]$ exists. Let $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ be the localization functor, and let $F : \mathcal{C} \to \mathcal{D}$ be any functor such that $F(\bar{S}) \subset \mathrm{Iso}(\mathcal{D})$. Since $S \subset \bar{S}$, $F(S) \subset \mathrm{Iso}(\mathcal{D})$, so there exists a unique $\widetilde{F} : \mathcal{C}[S^{-1}] \to \mathcal{D}$ such that $F \simeq \widetilde{F} \circ Q$. Since Q is a functor, $Q(\bar{S}) \subset \overline{Q(S)} \subset \overline{\mathrm{Iso}(\mathcal{C}[S^{-1}])} = \mathrm{Iso}(\mathcal{C}[S^{-1}])$. Thus if $\mathcal{C}[S^{-1}]$ exists, so does $\mathcal{C}[\bar{S}^{-1}]$ and $\mathcal{C}[S^{-1}] \simeq \mathcal{C}[\bar{S}^{-1}]$.

Conversely, suppose $\mathcal{C}[\bar{S}^{-1}]$ and $\bar{Q} : \mathcal{C} \to \mathcal{C}[\bar{S}^{-1}]$ is the corresponding localization. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor such that $F(S) \subset \mathrm{Iso}(\mathcal{D})$. Then $F(\bar{S}) \subset \overline{F(S)} \subset \overline{\mathrm{Iso}(\mathcal{D})} = \mathrm{Iso}(\mathcal{D})$. This shows that there is a unique $\widetilde{F} : \mathcal{C}[\bar{S}^{-1}] \to \mathcal{D}$ such that $F \simeq \widetilde{F} \circ \bar{Q}$. Note that $\bar{Q}(S) \subset \bar{Q}(\bar{S}) \subset \mathrm{Iso}(\mathcal{C}[\bar{S}^{-1}])$, which completes the proof. $\square$

Lemma 1.2.7. *Let $S_1, S_2 \subset \mathrm{Mor}(\mathcal{C})$ be such that $\mathcal{C}[S_1^{-1}]$ and $\mathcal{C}[S_2^{-1}]$ exist. If $S_1 \subset S_2$, then $\mathcal{C}[S_2^{-1}] \simeq \mathcal{C}[S_1^{-1}][Q_{S_1}(S_2)^{-1}]$.*

Proof. This is a good exercise. $\square$

Remark 1.2.8. The "two-out-of-three" property can be strengthened to give the so-called "two-out-of-six" property (see [DHKS04]) which is defined as follows. Given any triple f, g, h of composable morphisms, then in the diagram



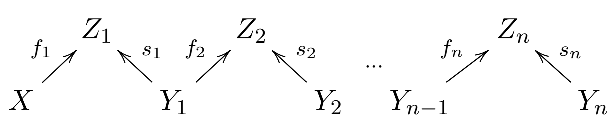
if $gf, hg \in S$, then f, g, h and hgf are in S. As an exercise, try showing that the collection of quasi-isomorphisms in $\mathsf{Com}(\mathcal{A})$ satisfies this property.

1.3 Calculus of fractions

Let $\mathcal{C}$ be a category and $S \subset \mathrm{Mor}(\mathcal{C})$. We will always assume S is saturated. Consider diagrams

with $s \in S$. We call such diagrams "left S-fractions" (resp. "right S-fractions.") We think of the diagram on the left as an avatar for $s^{-1} \circ f$, and the diagram on the right as an avatar for $f \circ s^{-1}$, and whenever it won't lead to confusion we will denote these diagrams simply by $s^{-1} \circ f$ and $s^{-1} \circ f$.

Consider a chain

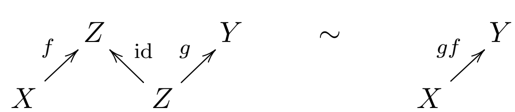


which we formally denote by $s_n^{-1} \circ f_n \circ \cdots \circ s_2^{-1} \circ f_2 \circ s_1^{-1} \circ f_1$. Such diagram is called *composit S-fraction.*

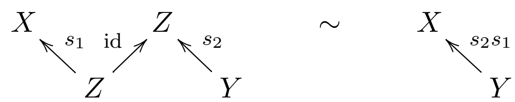
Definition 1.3.1. *Call two composite fractions* elementary equivalent *if one can be obtained from the other by one of the following rules*

(E1) adding (inserting inside the chain) $s^{-1} \circ s$ or $s \circ s^{-1}$ with $s \in S$

(E2) replacing $g \circ \mathrm{id}^{-1} \circ f$ by $g \circ f$



(E3) replacing $s_1^{-1} \circ \mathrm{id} \circ s_2^{-1}$ *by* $(s_2 \circ s_1)^{-1}$



We say that two composite fractions are equivalent *if there is a chain of elementary equivalences relating them.*

Definition 1.3.2. *Let* $\mathcal{C}$ *be a category,* $S \subset \mathrm{Mor}(\mathcal{C})$ *a saturated class of morphisms. We define* $\mathcal{C}[S^{-1}]$ *to be the category with the same objects as* $\mathcal{C}$*, and with*

$$\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X, Y) = \{\textit{equiv. classes of compos. } S\textit{-fractions from } X \textit{ to } Y\}.$$

Composition is induced by concatenation.

Remark 1.3.3. There is a subtle problem with this definition. Namely, $\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X, Y)$ might be a proper class and not a set.

Proposition 1.3.4. *If* $\mathcal{C}$ *is a small category and* $S \subset \mathrm{Mor}(\mathcal{C})$*, then the category* $\mathcal{C}[S^{-1}]$ *defined in 1.3.2 is indeed the localization of* $\mathcal{C}$ *at* S*.*

Proof. Define a functor $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ that is the identity on objects, and sends a morphism $f : X \to Y$ to the formal fraction $\mathrm{id}_Y^{-1} \circ f$. It is easy to see that Q is a functor; we have $\mathrm{id}^{-1} \circ (f \circ g) \sim (\mathrm{id}^{-1} \circ f) \circ (\mathrm{id}^{-1} \circ g)$.

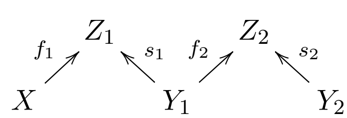
Suppose we have a functor $F : \mathcal{C} \to \mathcal{D}$ with $F(S) \subset \mathrm{Iso}(\mathcal{D})$. We define $\widetilde{F} : \mathcal{C}[S^{-1}] \to \mathcal{D}$ by $\widetilde{F}(QX) = F(X)$ and

$$\widetilde{F}(s_n^{-1} \circ f_n \circ s_{n-1}^{-1} \circ \cdots \circ s_1^{-1} \circ f_1) = F(s_n)^{-1} \circ \cdots \circ F(f_1).$$

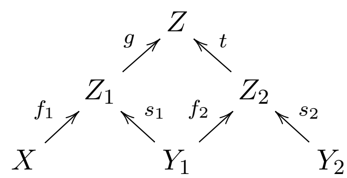
It is easy to check that $\widetilde{F}$ is unique up to natural isomorphism. □

Remark 1.3.5. Another problem with localization is that the set of "composite fractions" tends to be very complicated. Moreover, it is difficult to see what structure will be preserved under localization. For example, if $\mathcal{C}$ is additive, it is not clear whether $\mathcal{C}[S^{-1}]$ is again additive. For other examples of bad behavior of localization of categories see [Toë11].

Notice that fractions of the form $s_2^{-1} \circ f_2 \circ s_1^{-1} \circ f_1$ and $s_2^{-1} \circ t^{-1} \circ g \circ f$ are equivalent, provided $t \in S$ and $tf_1 = gs_1$. In other words, diagram



is equivalent to the fraction $s_2^{-1} \circ t^{-1} \circ g \circ f$ provided by the diagram



Indeed, we have the following calculation

$$\begin{aligned} s_2^{-1} \circ f_2 \circ s_1^{-1} \circ f_1 &\sim s_2^{-1} \circ \mathrm{id} \circ \mathrm{id}^{-1} \circ f_2 \circ s_1^{-1} \circ f_1 \\ &\sim s_2^{-1} \circ \mathrm{id} \circ t^{-1} \circ t \circ \mathrm{id}^{-1} \circ f_2 \circ s_1^{-1} \circ f_1 \\ &\sim (ts_2)^{-1} \circ (tf_2) \circ s_1^{-1} \circ f_1 \\ &= (ts_2)^{-1} \circ (gs_1) \circ s_1^{-1} \circ f_1 \\ &\sim (ts_2)^{-1} \circ (g \circ \mathrm{id}^{-1} \circ f_1) \\ &\sim (ts_2)^{-1} \circ (gf_1). \end{aligned}$$

Remark 1.3.6. If $\mathcal{C}$ has a null object (i.e. an object 0 which is both initial and terminal) then the corresponding object 0 in $\mathcal{C}[S^{-1}]$ is a null object in the localized category. To see this, we need to show that $\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(0, -)$ and $\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(-, 0)$ are both constant functors with value $\{*\}$. This is easy.

Example 1.3.7 (Universal (Cohn) localization of rings)**.** Let R be a unital ring. Let Σ be a set of morphisms $\sigma : P \to Q$, where P, Q range over some finitely-generated projective (left) R-modules. The *universal localization* of R at Σ is a ring $R[\Sigma^{-1}]$ together with a homomorphism $\rho : R \to R[\Sigma^{-1}]$ such that that the corresponding induction functor $\rho^* : R\text{-}\mathsf{Mod} \to R[\Sigma^{-1}]\text{-}\mathsf{Mod}$ sends all $\sigma \in \Sigma$ to isomorphisms.

Theorem 1.3.8 (Bergman)**.** *If R is hereditary (see definition 3.4.16 in the Appendix A), then all the universal localizations $R[\Sigma^{-1}]$ are also hereditary. Moreover, universal localization is* pseudo-flat *in the sense that* $\mathrm{Tor}_1^R(R[\Sigma^{-1}], R[\Sigma^{-1}]) = 0$.

1.4 Ore localization

Definition 1.4.1. *We say that $S \subset \mathrm{Mor}(\mathcal{C})$ satisfies the* left Ore conditions *if*

(LO1) S is saturated;

(LO2) for any right S-fraction $g \circ t^{-1} : X \to Y$, there exist a left S-fraction $s^{-1} \circ f$ such that $ft = sg$;

(LO3) if $fs = gs$ with $f, g \colon X \to Y$ and $s \colon Z \to X$, $s \in S$, then there exists $t \in S$ such that $tf = tg$

$$Z \xrightarrow{s} X \underset{g}{\overset{f}{\rightrightarrows}} Y \quad \Rightarrow \quad X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{\exists t} W$$

Similarly, one can define *right* Ore conditions.

Definition 1.4.2. *We say that $S \subset \mathrm{Mor}(\mathcal{C})$ satisfies the* left Ore conditions *if*

(RO1) S is saturated;

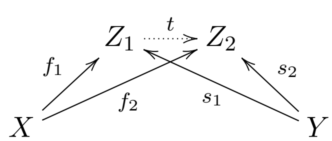
(RO2) for any left S-fraction $s^{-1} \circ f$ with $f \in \mathrm{Mor}(\mathcal{C})$, $s \in S$, there exists a right S-fraction $g \circ t^{-1}$ with $t \in S$, $g \in \mathrm{Mor}(\mathcal{C})$ such that $f \circ t = s \circ g$;

(RO3) if $tf = tg$ with $f, g\colon X \to Y$ and $t\colon Y \to Z$, $t \in S$, then there exists $s \in S$, $s\colon W \to X$ such that $fs = gs$.

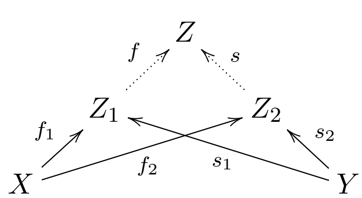
Exercise(hard) What happens to the localization if in the above definition one replaces the "two-out-of-three" property with the "2-of-6 property" (see Remark 1.2.8)?

Definition 1.4.3. *Fix $S \subset \mathrm{Mor}(\mathcal{C})$. For any $X, Y \in \mathrm{Ob}(\mathcal{C})$, we define an equivalence relation $\sim_\ell$ on the set of "left S-fractions" from X to Y as follows. Two fractions*

are $\sim_\ell$-equivalent if there exists $t \in S$, $t : Z_1 \to Z_2$ such that $f_2 = tf_1$ and $s_2 = ts_1$. In other words, we say that $s_1^{-1} \circ f \sim_\ell (ts_1)^{-1} \circ (tf_1)$. This definition can be represented by the following commutative diagram:



Remark 1.4.4. A more standard way to define an equivalence relation on left fractions (under the assumption that S is multiplicatively closed, but not necessarily saturated) is via the following commutative diagram:

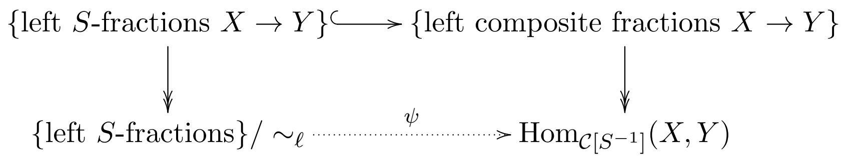


If S is saturated, then this definition is equivalent to the definition 1.4.3 above. All "interesting" examples of localizations one meets in real life are localizations over a saturated class of morphisms, so our definition 1.4.3 is usually enough. For example, quasi-isomorphisms in the category of complexes in an abelian category is a saturated class.

Lemma 1.4.5. *If S is left Ore, then*

$$\mathrm{Hom}_{\mathcal{C}[S^{-1}]} = \{\textit{left } S\textit{-fractions } X \to Y\}/\sim_\ell .$$

Proof. Observe that every left S-fraction is trivially a left composite fraction. We claim that the following commutative diagram can be filled in:



We begin by showing that the map ψ defined in the obvious way as a map

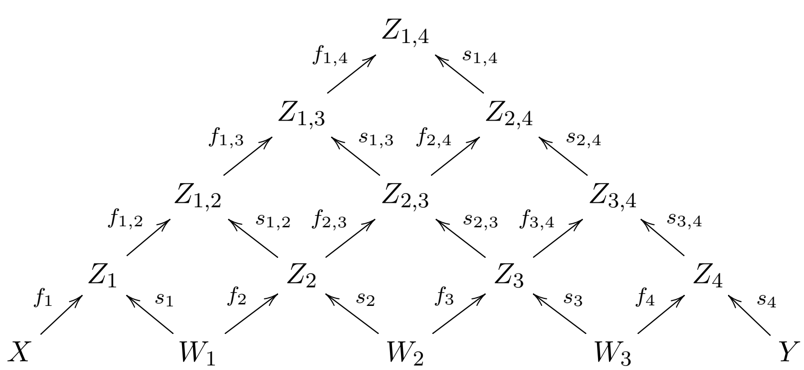
$$\{\text{left } S\text{-fractions } X \to Y\} \to \mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X, Y)$$

actually descends to the quotient $\{\text{left } S\text{-fractions}\}/\sim_\ell$.

If $s_1^{-1} \circ f_1 \sim_\ell s_2^{-1} \circ f_2$, then there exists $t \in S$ such that $s_2 = ts_1$ and $f_2 = tf_1$. We then compute:

$$\begin{aligned} s_1^{-1} \circ f_1 &\sim s_1^{-1} \circ \mathrm{id} \circ \mathrm{id}^{-1} \circ f_1 \\ &\sim s_1^{-1} \circ \mathrm{id} \circ t^{-1} \circ t \circ \mathrm{id}^{-1} \circ f_1 \\ &\sim (ts_1)^{-1} \circ (tf_1) \\ &= s_2^{-1} \circ f_2. \end{aligned}$$

Now we construct the map φ inverse to ψ. Take a composite fraction $a = s_n^{-1} \circ f_n \circ \cdots \circ s_1^{-1} \circ f_1$. Using (LO2), we replace each right fraction $f_i \circ s_{i-1}^{-1}$ with $s_{i-1,i}^{-1} \circ f_{i-1,i}$. Similarly, we can replace $f_{i,i+1} \circ s_{i-1,i}^{-1}$ with $s_{i-1,i+1}^{-1} \circ f_{i-1,i+1}$. For $n = 4$, the following commutative diagram illustrates the described process:



In general, let f be the composite $f_{1,n} \circ \cdots \circ f_{1,n} \circ f_1$ and let $s = s_{1,n} \circ \cdots \circ s_{n-1,n} \circ s_n$. We would like to define φ by $\varphi(a) = s^{-1} \circ f$.

We need to check that $\varphi(a) = s^{-1} \circ f$ is independent (up to $\sim_\ell$-equivalence) of the choices of the $s_{i,j}$ and $f_{i,j}$ coming from the Ore axioms. For example, replace $s_{1,2}^{-1} \circ f_{1,2}$ by $t_{1,2}^{-1} \circ g_{1,2}$. We have $f_{1,2} \circ s_1 = s_{1,2} f_2$ and $g_{1,2} s_1 = t_{1,2} f_2$. Consider the right fraction $s_{1,2} \circ t_{1,2}^{-1}$ and apply (LO2) to find a left fraction $u^{-1} \circ v = s_{1,2} \circ t_{1,2}^{-1}$. Then $us_{1,2} = vt_{1,2}$. Since S is saturated, $us_{1,2} = vt_{1,2}$ implies $v \in S$. Multiply $g_{1,2}s_1 = t_{1,2}f_2$ on the left by v. We get

$$\begin{aligned} vg_{1,2}s_1 &= vt_{1,2}f_2 \\ &= us_{1,2}f_2 \\ &= uf_{1,2}s_1. \end{aligned}$$

By (LO3), there exists $w \in S$ such that $wvg_{1,2} = wuf_{1,2}$. Hence

$$\begin{aligned} (s_{1,2}s_2)^{-1} \circ (f_{1,2}f_1) &\sim_\ell (wus_{1,2}s_2)^{-1} \circ (wvf_{1,2}f_1) \\ &\sim_\ell (wvt_{1,2}s_2)^{-1} \circ (wvg_{1,2}f_1) \\ &\sim_\ell (t_{1,2}s_2)^{-1} \circ (g_{1,2}f_1). \end{aligned}$$

It remains to check that replacing $s_i^{-1} f_i$ with $(fs_i)^{-1} \circ (tf_i)$ does not change the value of φ, and also that φ and ψ are actually inverses of each other. The proof is a (tedious) calculation similar to the one we've done above. We leave it as an exercise to the interested reader. □

Lemma 1.4.6. *Let $\mathcal{C}$ be an additive category, and $S \subset \mathrm{Mor}(\mathcal{C})$ be an Ore family of morphisms. Then $\mathcal{C}[S^{-1}]$ is additive, and the localization functor $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ is additive.*

Proof. (sketch) We define an additive structure on $\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X, Y)$ as follows. Given two fractions $s^{-1} \circ f, t^{-1} \circ g : X \to Y$, replace $s \circ t^{-1}$ by $u^{-1}v$ for $u \in S$, using (LO2). Let $w = vt = us \in S$. Define

$$s^{-1} \circ f + t^{-1} \circ g = w^{-1} \circ (uf + vg).$$

It is a straightforward check that this induces a well-defined commutative group structure on $\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X, Y)$, that composition in $\mathcal{C}[S^{-1}]$ is bilinear, and that Q is an additive functor, i.e. it induces homomorphisms of abelian groups $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}[S^{-1}]}(QX, QY)$.

We need to check that $\mathcal{C}[S^{-1}]$ satisfies the axiom (AB3). Since $\mathcal{C}$ is additive, for $Y_1, Y_2 \in \mathrm{Ob}(\mathcal{C})$, there exists an object $Y_1 \times Y_2$, together with canonical projections and injections $p_i : Y_1 \times Y_2 \to Y_i$ and $i_j : Y_j \to Y_1 \times Y_2$. These satisfy $p_k i_\ell = \delta_{k\ell}$ and $i_1p_1 + i_2p_2 = \mathrm{id}$. Now if we apply the *additive* functor Q, knowing that Q is identity on objects of $\mathcal{C}$, we obtain $QY_1 \times QY_2 = Q(Y_1 \times Y_2)$ and all the structure morphisms p_i, i_i for the category $\mathcal{C}[S^{-1}]$. This shows that $\mathcal{C}[S^{-1}]$ satisfies (AB3), and so is additive. □

One might wonder if localization preserves the property of "being abelian." We will see shortly that the answer is usually "no".

2 Localization of abelian categories

2.1 Serre quotients

Let $\mathcal{A}$ be an abelian category, and $S \subset \operatorname{Mor}\mathcal{A}$ a saturated class of morphisms in $\mathcal{A}$. We can view the localization $\mathcal{A}[S^{-1}]$ in two different ways:

- $\mathcal{A}[S^{-1}]$ is the localization of $\mathcal{A}$ obtained by adding "more" morphisms (i.e. inverses to $s \in S$)

- $\mathcal{A}[S^{-1}]$ is obtained from $\mathcal{A}$ by "quotienting out" the subcategory of objects in $\mathcal{A}$ that become the zero object in the localization

Morally, there is a formula "$\mathcal{A}[S^{-1}] = \mathcal{A}/\mathcal{N}$". We will try to make this "quotienting out" precise, and explain why the true. Our main references will be [Gab62], [Ste75] and [AZ94].

For the remainder of this section, assume that the abelian category $\mathcal{A}$ has arbitrary colimits and limits, and moreover has a set of generators (see 2.2.3).

Definition 2.1.1. *A (nonempty) full subcategory $\mathcal{T}$ of $\mathcal{A}$ is called a* Serre *(or dense) subcategory if $\mathcal{T}$ satisfies: if $0 \to M' \to M \to M'' \to 0$ is exact in $\mathcal{A}$, then $M \in \operatorname{Ob}\mathcal{T}$ if and only if both $M', M'' \in \operatorname{Ob}(\mathcal{T})$.*

Traditionally, objects in $\mathcal{T}$ are called *torsion objects*.

Lemma 2.1.2. *Serre subcategories are abelian subcategories.*

Proof. Let $\mathcal{T}$ be a Serre subcategory of $\mathcal{A}$. Since $\mathcal{T}$ is closed under subobjects and quotients. Moreover, the inclusion $\mathcal{T} \hookrightarrow \mathcal{A}$ preserves kernels and cokernels. Checking the rest of the definition is easy. For example, since $\mathcal{T}$ is closed under extensions, it is closed under finite direct sums and finite products, and these coincide with the sums and products computed in $\mathcal{A}$. □

Example 2.1.3. Let $\mathcal{A} = \mathsf{Vect}_k$ be the category of all vector spaces over a field k, and $\mathcal{T} = \mathsf{Vect}_k^{\mathrm{fd}}$ be the subcategory of finite-dimensional vector spaces.

Example 2.1.4. Let A be a commutative ring, $S \subset A$ a multiplicatively closed subset. Let $\mathcal{A} = \mathsf{Mod}(A)$. Call an A-module M *S-torsion* if for all $m \in M$, there exists $s \in S$ such that $s \cdot m = 0$. We can let $\mathcal{T}$ be the full subcategory of all S-torsion modules in $\mathcal{A}$.

Example 2.1.5. Suppose $A \subset B \subset \operatorname{Frac}(A)$, and assume B is flat over A. Let $\mathcal{A} = \mathsf{Mod}(A)$ and $\mathcal{T} = \{M \in \operatorname{Ob}\mathcal{A} : M \otimes_A B = 0\}$.

Example 2.1.6. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive exact functor between abelian categories. Put $\mathcal{T} = \{M \in \operatorname{Ob}(\mathcal{A}) : F(M) = 0\} = \operatorname{Ker}(F)$.

Example 2.1.7. Let A be a graded connected k-algebra, where k is a field. Here, "graded connected" means that $A = \bigoplus_{n \geqslant 0} A_n$, and $A_0 = k$. Let $\mathcal{A} = \mathsf{GrMod}(A)$ be the category of right graded A-modules. If $M \in \mathrm{Ob}(\mathcal{A})$, call $m \in M$ *torsion* if for all $n \gg 0$, $mA_n = 0$. The torsion elements in M form a graded submodule. Let $\tau(M)$ be this submodule. We say that M is torsion free if $\tau(M) = 0$. Let $\mathcal{T} = \mathrm{Tors}(A)$ be the full subcategory of $\mathcal{A}$ consisting of torsion A-modules. (Without further hypothesis, this is *not* a Serre subcategory.)

Lemma 2.1.8. *Let A be a Noetherian graded connected k-algebra. Then* $\mathrm{Tors}(A)$ *is a dense subcategory.*

Proof. Use the following easy exercise: any graded connected Noetherian k-algebra is locally finite, in the sense that each $\dim_k A_n < \infty$. It follows that $m \in \tau(M)$ if and only if $\dim_k(m \cdot A) < \infty$. Thus $\tau(M)$ is the sum of all finite-dimensional submodules of M. Now suppose $0 \to L \to M \to N \to 0$ is an exact sequence of graded A-modules. First, we suppose $M \in \mathrm{Ob}(\mathcal{T})$ and show that $L, N \in \mathrm{Ob}(\mathcal{T})$. But given the above discussion, this is obvious.

Conversely, suppose L, N are in $\mathcal{T}$. Take $m \in M$ and consider

$$0 \to mA \cap L \to mA \to mA/(mA \cap L) \to 0.$$

Since L is torsion, so is $mA \cap L$, and since $mA/(mA \cap L) \simeq (mA + L)/L \hookrightarrow N$, we know that $mA/(mA \cap L)$ is torsion. Since $mA\cap$ is finitely generated, we get that $mA \cap L$ is finite-dimensional. □

Definition 2.1.9. *Let $\mathcal{A}$ be an abelian category, $\mathcal{T}$ a Serre subcategory. Define the quotient $\mathcal{A}/\mathcal{T}$ (Serre quotient) by* $\mathrm{Ob}(\mathcal{A}/\mathcal{T}) = \mathrm{Ob}(\mathcal{A})$*, and by letting*

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{T}}(M, N) = \varinjlim_{(M', N')} \mathrm{Hom}(M', N/N'),$$

the limit being taken over all pairs (M', N') such that $M' \subset M$ and $N' \subset N$ satisfying $M/M', N' \in \mathrm{Ob}(\mathcal{T})$.

Lemma 2.1.10. *This definition of the Serre quotient makes sense.*

Proof. Let $I = \{(M', N') : M' \subset M, N' \subset N, M/M', N' \in \mathrm{Ob}(\mathcal{T})\}$. We put $(M', N') \leqslant (M'', N'')$ if $M'' \subset M'$ and $N' \subset N''$. The set $(I, \leqslant)$ is directed because whenever $(M_1', N_1'), (M_2', N_2') \in I$, the pair $(M_1' \cap M_2', N_1 + N_2)$ is also in I. The $\mathrm{Hom}_{\mathcal{A}}(M', N/N')$ form a directed system as follows. If $i : M'' \hookrightarrow M'$, $j : N' \hookrightarrow N''$ are injections, we get

$$\mathrm{Hom}(M', N/N') \xrightarrow{i^*} \mathrm{Hom}(M'', N/N') \xrightarrow{j_*} \mathrm{Hom}(M'', N/N'')$$

making $\{\mathrm{Hom}_{\mathcal{A}}(M', N/N')\}$ a direct system.

Note that $(M, 0) \in I$ because $0 = M/M \in \mathrm{Ob}(\mathcal{T})$. This gives us a map $\mathrm{Hom}_{\mathcal{A}}(M, N) \to \varinjlim_{\mathcal{A}}(M', N/N')$. So every morphism in $\mathcal{A}/\mathcal{T}$ is represented by a morphism in $\mathcal{A}$.

It is pretty straightforward to define the composition in $\mathcal{A}/\mathcal{T}$, we leave it as an exercise. □

The definition 2.1.9 is motivated by the following theorem.

Theorem 2.1.11 (Serre)**.** *Let A be a graded quotient of a polynomial ring $k[x_1, \dots, x_n]$. Let $X = \operatorname{Proj}(A)$ be the corresponding projective scheme over k. Let $\mathsf{Qcoh}(X)$ be the category of quasi-coherent sheaves on X. Then there is a natural equivalence of categories*

$$\mathsf{GrMod}(A)/\operatorname{Tors}(A) \xrightarrow{\sim} \mathsf{Qcoh}(X).$$

Write $\mathsf{Tails}(A)$ for the quotient category appearing in this theorem. If $M \in \mathsf{Tails}(A)$ is finitely-generated, then it turns out that one has $\operatorname{Hom}_{\mathsf{Tails}}(M, N) = \varinjlim \operatorname{Hom}(M/M_{\geqslant n}, N)$ for any N.

There is a *projection functor* $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{T}$, that assigns to $M \in \mathcal{A}$ the same object πM, and that sends f to the induced morphism πf.

Lemma 2.1.12. *An object M is zero in $\mathcal{A}/\mathcal{T}$ if and only if it $M \in \operatorname{Ob}(\mathcal{T})$.*

Proof. If $\pi M = 0$ in $\mathcal{A}/\mathcal{T}$, then $\operatorname{Hom}_{\mathcal{A}/\mathcal{T}}(\pi M, \pi M) = 0$, whence $\mathrm{id}_{\pi M} = 0$. Thus there exist $M', N' \subset M$ such that the induced map $M' \hookrightarrow M \xrightarrow{\mathrm{id}} M \to M/N'$ is zero. It follows that M is torsion.

Conversely, suppose M is torsion. It suffices to show that $\operatorname{Hom}_{\mathcal{A}/\mathcal{T}}(\pi M, \pi M) = 0$. But every $f : \pi M \to \pi N$ is represented by some $f : M' \to N/N'$ for some pair $(M', N') \in I$. If M is torsion, then $(M', N') \leqslant (0, N')$, which means that f factors through the zero map. □

Corollary 2.1.13. *Let $f : M \to N$ be a morphism in $\mathcal{A}$. Then*

1. $\operatorname{Ker}(\pi f) = \pi(\operatorname{Ker} f)$
2. $\operatorname{Coker}(\pi f) = \pi(\operatorname{Coker} f)$
3. *πf is monic if and only if* $\operatorname{Ker} f \in \operatorname{Ob}(\mathcal{T})$
4. *π is epic if and only if* $\operatorname{Coker}(f) \in \operatorname{Ob}(\mathcal{T})$
5. *πf is an isomorphism if and only if* $\operatorname{Ker}(f), \operatorname{Coker}(f)$ *are in* $\operatorname{Ob}(\mathcal{T})$

Theorem 2.1.14. *Let $\mathcal{A}$ be an abelian category. Then $\pi : \mathcal{A}/ \to \mathcal{A}/\mathcal{T}$ is exact.*

Moreover, given any additive exact functor $F : \mathcal{A} \to \mathcal{C}$ such that $F(\mathcal{T}) = 0$, there is a unique functor $\widetilde{F} : \mathcal{A}/\mathcal{T} \to \mathcal{C}$ such that $\widetilde{\pi} = F$.

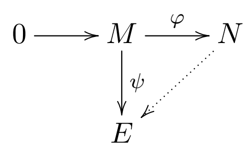
One can check that if $\mathcal{S} \subset \mathcal{T} \subset \mathcal{A}$ are Serre subcategories, then we have $(\mathcal{A}/\mathcal{S})/(\mathcal{T}/\mathcal{S}) \simeq (\mathcal{A}/\mathcal{T})$. Given a short exact sequence $0 \to \mathscr{L} \to \mathscr{M} \to \mathscr{N} \to 0$ in $\mathcal{A}$, there are isomorphisms α, β, γ in $\mathcal{A}$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{L} & \longrightarrow & \mathscr{M} & \longrightarrow & \mathscr{N} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & \pi L & \longrightarrow & \pi M & \longrightarrow & \pi N & \longrightarrow & 0 \end{array}$$

Remark 2.1.15. If $\varphi \in \mathrm{Hom}_{\mathcal{A}/\mathcal{T}}(\pi M, \pi N)$ is represented by $f : M' \to N/N'$ for some $(M', N') \in I$, then let $i : M' \hookrightarrow M$ and $p : N \twoheadrightarrow N/N'$ be the canonical maps. By the Corollary 2.1.13, the morphisms πi and πp are isomorphisms in $\mathcal{A}/\mathcal{T}$. So we can write $\varphi = (\pi p)^{-1} \circ \pi f \circ (\pi i)^{-1}$. Let S be the class of all morphisms in A with kernel and cokernel both in $\mathcal{T}$. Then we have the isomorphism $\mathcal{A}/\mathcal{T} \simeq \mathcal{A}[S^{-1}]$.

2.2 Injective envelopes

Let $\mathcal{A}$ be an abelian category. Recall (see definition 3.1.1 in Chapter 4) that an object E in $\mathcal{A}$ is *injective* if the functor $\mathrm{Hom}(-, E) : \mathcal{A}^\circ \to \mathsf{Ab}$ is exact. Alternatively, arrows lift as in the following diagram:



The category $\mathcal{A}$ has *enough injectives* if for any M in $\mathcal{A}$, there is a monic $M \hookrightarrow E$ with E injective.

For convenience we will think of the category $\mathcal{A}$ as a subcategory of modules over some ring. Mitchell's theorem 1.4.1 in Chapter 4 guaranties that we are not losing generality.

Definition 2.2.1. *Let $M \subset N$ be objects in $\mathcal{A}$. We say that M is* essential *in N if for all $N' \subset M$ with $N' \neq 0$, the intersection $N' \cap M \neq 0$.*

We say that a monic $f : M \to N$ is essential if $\mathrm{Im}(f) \subset N$ is essential.

Theorem 2.2.2 (Eckmann-Schopf,[ES53])**.** *An object E is injective if and only if E has no proper essential extensions.*

Definition 2.2.3. *An* injective envelope *of an object $M \in \mathrm{Ob}(\mathcal{A})$ is any essential injective extension of M.*

Example 2.2.4. Let A be a simple Noetherian hereditary domain (for example, a Dedekind domain or the Weil algebra $k\langle x, y\rangle/([x, y] = 1)$). Goldie's theorem gives existence of the field of fractions $Q = \mathrm{Frac}(A)$, the localization of A at the (Ore) set $A \smallsetminus 0$. It turns out that $A \hookrightarrow Q$ is an injective envelope of A.

Theorem 2.2.5 (Baer,[Bae40](**not sure if the correct reference**))**.** *Let E be an injective envelope of M with respect to the inclusion $i : M \hookrightarrow E$. Then*

1. *for any essential monic $f : M \to N$, there exists monic $g : N \hookrightarrow E$ such that $gf = i$*

2. *for any injective extension $j : M \hookrightarrow E'$, there exists $g : E \to E'$ such that $j = gi$.*

Corollary 2.2.6. *An injective envelope of M is a maximal essential extension of M.*

Corollary 2.2.7. *Given two injective envelopes $i : M \hookrightarrow E$, $i' : M \hookrightarrow E'$, there is an isomorphism $f : E \to E'$ such that $i'f = i$. Hence injective envelopes are unique up to isomorphism, but* not *up to* canonical *isomorphism.*

Remark 2.2.8. The category $\mathcal{A}$ may have enough injectives but *not* enough injective envelopes. For example, in the category $\mathsf{Mod}(k[t])^\circ$, the module $k[t]$ has no injective envelope (because injective envelope in $\mathsf{Mod}(k[t])^\circ$ is the same as projective envelope in $\mathsf{Mod}(k[t])$). On the other hand, Baer proved that any category of modules $\mathsf{Mod}(R)$ has enough injective envelopes.

2.3 Localizing subcategories

The Serre subcategory $\mathcal{T}$ appearing above need not be closed under taking arbitrary (infinite) direct sums. For example, one can consider $\mathsf{Vect}_k^{\mathrm{fd}}$ as a subcategory of Vect_k.

Definition 2.3.1. *Call a Serre subcategory $\mathcal{T} \subset \mathcal{A}$* localizing *if the quotient functor $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{T}$ has a right adjoint.*

One traditionally denotes the adjoint by ω, and call ω the "local section functor." We will see later that this setup captures coherent sheaf cohomology.

Lemma 2.3.2. *For a Serre subcategory $\mathcal{T} \subset \mathcal{A}$, the following are equivalent:*

1. *every $M \in \mathrm{Ob}(\mathcal{A})$ has a largest torsion subobject*
2. *the inclusion $i : \mathcal{T} \hookrightarrow \mathcal{A}$ has a right adjoint τ (called the "torsion functor")*
3. *the direct sum of torsion modules is torsion*

Example 2.3.3. Let $A = k[x_0, \ldots, x_n]$ with the standard grading. Let $\mathcal{A} = \mathsf{GrMod}(A)$ and $\mathcal{T} = \mathsf{Tors}(A)$. Recall we defined $\mathsf{Tails}(A) = \mathcal{A}/\mathcal{T}$. Serre's theorem tells us that $\mathsf{Qcoh}(\mathbb{P}^n_k) \simeq \mathcal{A}/\mathcal{T}$. Geometrically, our adjoint $\omega : \mathsf{Qcoh}(\mathbb{P}^n) \to \mathsf{GrMod}(A)$ is $\mathscr{M} \mapsto \bigoplus_{n\in\mathbb{Z}} \mathrm{H}^0(X, \mathscr{M}(n))$, which has right derived functors $\mathsf{R}^i\omega \simeq \bigoplus_{n\in\mathbb{Z}} \mathrm{H}^i(X, -(n))$. Along the same lines, τ induces the local cohomology $M \mapsto \mathrm{H}^0_{\mathfrak{m}}(M)$, where $\mathfrak{m} \subset A$ is the augmentation ideal.

Theorem 2.3.4. *Assume $\mathcal{A}$ has injective envelopes, and that $\mathcal{T} \subset \mathcal{A}$ is closed under essential extensions (so in particular $\mathcal{T}$ is closed under injective envelopes). Then the following are equivalent:*

1. *$\mathcal{T}$ is a localizing subcategory*
2. *the inclusion functor $\mathcal{T} \to \mathcal{A}$ ha a right adjoint (the torsion functor) $\tau : \mathcal{A} \to \mathcal{T}$*

In this case, $(\mathcal{A}, \mathcal{T})$ is called a stable torsion pair*.*

Corollary 2.3.5. *Under the assumptions of Theorem 2.3.4,*

1. *$\mathcal{A}/\mathcal{T}$ has enough injectives (so both ω and τ have right derived functors)*

2. $\pi\omega \simeq \mathrm{id}_{\mathcal{A}/\mathcal{T}}$ *(counit adjunction)*

3. *there is an exact sequence of functors*

$$0 \longrightarrow \tau \longrightarrow \mathrm{id}_{\mathcal{A}} \longrightarrow \omega\pi \longrightarrow \mathsf{R}^1\tau \longrightarrow 0,$$

and $\mathsf{R}^i\omega = \mathsf{R}^{i+1}\tau$ *for all* $i \geqslant 1$.

Example: projective geometry

Let $A = \bigoplus_{n\geqslant 0} A_n$ be a graded noetherian connected k-algebra. Let $\mathcal{A}$ be the category of graded A-modules (or finitely-generated A-modules). For example, we could have $A = k[x_0, \dots, x_n]$ with the standard grading. Let $\mathcal{T} = \mathrm{Tors}(A)$, which under our assumptions consists of finite-dimensional modules. Recall we defined $\mathsf{Tails}(A)$ and $\mathsf{tails}(A)$ to be the quotients of $\mathsf{GrMod}(A)$ and $\mathsf{grmod}(A)$ by $\mathrm{Tors}(A)$. If A is commutative and generated in degree one, then a theorem of Serre tells us that for $X = \mathrm{Proj}(A)$, we have $\mathsf{Qcoh}(X) \simeq \mathsf{Tails}(X)$ and $\mathsf{coh}(X) \simeq \mathsf{tails}(A)$.

In this setting, $\pi : \mathsf{GrMod}(A) \to \mathsf{Qcoh}(X)$ is the functor $M \mapsto \widetilde{M}$, and $\omega\mathscr{M} = \bigoplus_{n\geq 0} \mathrm{H}^0(X, \mathscr{M}(n))$. We would like to identify the functors τ, $\mathsf{R}^i\tau$, and $\mathsf{R}^i\omega$.

Start with the exact sequence $0 \to A_{\geqslant n} \to A \to A/A_{\geqslant n} \to 0$. For any graded A-module M, applying the functor $\mathrm{Hom}(-, M)$ gives the following exact sequence

$$0 \longrightarrow \mathrm{Hom}(A/A_{\geqslant n}, M) \longrightarrow \mathrm{Hom}(A, M) \longrightarrow \mathrm{Hom}(A_{\geqslant n}, M) \longrightarrow \mathrm{Ext}^1(A/A_{\geqslant n}, M) \longrightarrow 0$$

If we now take $\varinjlim_n$, we get the following isomorphisms:

- $\tau M = \varinjlim \mathrm{Hom}(A/A_{\geqslant n}, M)$;
- $\mathrm{R}^i\tau M = \varinjlim \mathrm{Ext}^i(A/A_{\geqslant n}, M)$;
- $\omega\mathscr{M} = \bigoplus_{n\geq 0} \mathrm{H}^0(X, \mathscr{M}(n))$;
- $\mathrm{R}^i\omega\mathscr{M} = \bigoplus_{n\geq 0} \mathrm{H}^i(X, \mathscr{M}(n))$.

It is a good exercise to derive carefully these isomorphisms.

3 Derived categories

3.1 Definition and basic examples

Let $\mathcal{A}$ be an abelian category. Consider the category $\mathsf{Com}(\mathcal{A})$ of complexes in $\mathcal{A}$ which is also an abelian category (see Theorem 1.3.9 in Chapter 4). Put $S = \mathrm{Qis} \subset \mathrm{Mor}(\mathsf{Com}(\mathcal{A}))$ to be the family of all quasi-isomorphisms.

Definition 3.1.1. *The* derived category $\mathsf{D}(\mathcal{A})$ *is the localization* $\mathsf{D}(\mathcal{A}) = \mathsf{Com}(\mathcal{A})[S^{-1}]$. *Denote* $Q\colon \mathcal{A} \to \mathsf{D}(\mathcal{A})$ *the canonical localization functor.*

Example 3.1.2. Let's call the complex $(C^\bullet, d^\bullet)$ *cyclic* if all $d^i = 0$. Let $\mathsf{Com}^0(\mathcal{A})$ be the full subcategory of $\mathsf{Com}(\mathcal{A})$ consisting of the cyclic complexes. Let $i\colon \mathsf{Com}^0(\mathcal{A}) \hookrightarrow \mathsf{Com}(\mathcal{A})$ be the canonical inclusion. Notice that $\mathsf{Com}^0(\mathcal{A}) \simeq \prod_{n\in\mathbb{Z}} \mathcal{A}[n]$, where $\mathcal{A}$ just denotes a copy of $\mathcal{A}$ that corresponds to $n \in \mathbb{Z}$.

Define the functor $\mathrm{H}^\bullet\colon \mathsf{Com}(\mathcal{A}) \to \mathsf{Com}^0(\mathcal{A})$ in the opposit direction, which sends a complex $(C^\bullet, d^\bullet)$ to the cyclic complex $(\mathrm{H}^\bullet(C), 0)$. Notice that $\mathrm{H}^\bullet$ sends quasi-isomorphisms to isomorphisms.

Hence, by the universal property of localizations, $\mathrm{H}^\bullet$ factors through Q

$$\begin{array}{ccc} \mathsf{Com}(\mathcal{A}) & \xrightarrow{\mathrm{H}^\bullet} & \mathsf{Com}^0(\mathcal{A}) \\ & \searrow \quad \nearrow_{\bar{\mathrm{H}}^\bullet} & \\ & \mathsf{D}(\mathcal{A}) & \end{array}$$

Recall that an abelian category $\mathcal{A}$ is called *semisimple* if every short exact sequence in $\mathcal{A}$ splits. Equivalently, $\mathcal{A}$ is semisimple if $\mathrm{Ext}^i(X,Y) = 0, \forall i > 0$ and for any $X, Y \in \mathcal{A}$. For example, categories Vect of vector spaces and $kG{-}\mathsf{fdMod}$ of finite dimensional representations of a finite group G are semisimple.

Theorem 3.1.3. *If $\mathcal{A}$ is semisimple, then $\bar{\mathrm{H}}^\bullet$ is an equivalence of categories.*

Proof. Define for any complex $(C^\bullet, d) \in \mathsf{Com}(\mathcal{A})$ two morphisms

$$f_C^\bullet\colon (C^\bullet, d) \longrightarrow (\mathrm{H}^\bullet(C), 0)$$
$$g_C^\bullet\colon (\mathrm{H}^\bullet(C), 0) \longrightarrow (C^\bullet, d)$$

as follows.

For $C^\bullet = [\ldots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \longrightarrow \ldots]$ we put $B^n = \mathrm{Im}(d^{n-1})$, $Z^n = \mathrm{Ker}\, d^n$ and $H^n = B^n/Z^n$ for each $n \in \mathbb{Z}$. Then we have the following exam=ct sequences

$$0 \longrightarrow Z^n \longrightarrow C^n \longrightarrow B^{n+1} \longrightarrow 0 \tag{5.1}$$
$$0 \longrightarrow Z^n \longrightarrow C^n \longrightarrow B^{n+1} \longrightarrow 0 \tag{5.2}$$

Since $\mathcal{A}$ is semisimple, for each n we can choose splittings of the above sequences. This will give us $C^n \simeq Z^n \oplus B^{n+1} \simeq B^n \oplus H^n \oplus B^{n+1}$. With these identifications, the map $d^n\colon C^n \to C^{n+1}$ will be given by the map $B^n \oplus H^n \oplus B^{n+1} \to B^{n+1} \oplus H^{n+1} \oplus B^{n+2}$ which sends (b^n, h^n, b^{n+1}) to $(b^{n+1}, 0, 0)$.

Then we can define the maps $f_C^\bullet$ and $g_C^\bullet$ as the canonical projection $B^n \oplus H^n \oplus B^{n+1} \to H^n$ and embedding $H^n \hookrightarrow B^n \oplus H^n \oplus B^{n+1}$ respectively.

We can define the functor $\mathcal{L}\colon \mathsf{Com}^0(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ to be the composition

$$\mathsf{Com}^0(\mathcal{A}) \xrightarrow{i} \mathsf{Com}(\mathcal{A}) \xrightarrow{Q} \mathsf{D}(\mathcal{A})$$

It is a straight forward calculation to show that $\mathcal{L}$ is mutually inverse to $\bar{\mathrm{H}}^\bullet$. Namely, the families of morphisms $\{f_C^\bullet\}_{C\in\mathsf{Com}}$ and $\{g_C^\bullet\}_{C\in\mathsf{Com}}$ defined above induce isomorphisms of functors $f^\bullet\colon \mathrm{id} \xrightarrow{\sim} \mathcal{L}\circ \bar{H}^\bullet$ and $g^\bullet\colon \bar{H}^\bullet\circ\mathcal{L} \xrightarrow{\sim} \mathrm{id}$.

□

Remark 3.1.4. In general, $\mathsf{D}(\mathcal{A})$ captures much more information than just cohomology functor. How much more? is a very delicate question. We call $\mathcal{A}$ and $\mathcal{B}$ *derived equivalent* if $\mathsf{D}(\mathcal{A})\simeq\mathsf{D}(\mathcal{B})$, i.e. their derived categories are equivalent (as triangulated categories).

For example, take $\mathcal{A}=\mathsf{Mod}(A)$ and $\mathcal{B}=\mathsf{Mod}(B)$. Define *global dimension* (which can be ∞) of $\mathcal{A}$ as $gldim(\mathcal{A})=n$ s.t. $\mathrm{Ext}^n(X,Y)\neq 0$ for some $X,Y\in\mathcal{A}$, but $\mathrm{Ext}^{n+1}(X,Y)=0,\forall X,Y\in\mathcal{A}$. Then *gldim* is *not* a derived invariant, but the *finiteness* of *gldim is.*

Unfortunate fact: The class Qis is a saturated class, but it does not satisfy any other Ore conditions.

Hence we can't really conclude anything about the derived category $\mathsf{D}(\mathcal{A})$. We don't even know if it is additive. So we need a bit different construction of $\mathsf{D}(\mathcal{A})$. We will give a construction using the homotopy category $\mathsf{K}(\mathcal{A})$.

3.2 The homotopy category

Definition 3.2.1. *Let* Hot $\subset$ Qis *be the subclass of homotopy equivalences (see 1.0.8 on page 2). Then the* homotopy category $\mathsf{K}(\mathcal{A})$ *is defined as the localization* $\mathsf{K}(\mathcal{A})=\mathsf{Com}(\mathcal{A})[\mathrm{Hot}^{-1}]$. *We denote by* Q_h *the corresponding localization functor.*

This category can be equivalently defined as follows. To fix the notation, if $f,g\colon X\to Y$ are homotopy equivalent morphisms of complexes, we write $f\sim_h g$. Now define the category $\mathsf{K}'(\mathcal{A})$ by $\mathrm{Ob}(\mathsf{K}'(\mathcal{A}))=\mathrm{Ob}(\mathsf{Com}(\mathcal{A}))$, and

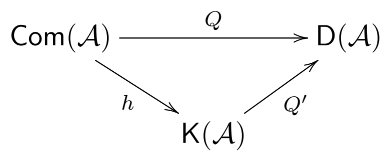
$$\mathrm{Hom}_{\mathsf{K}'(\mathcal{A})}(X,Y)=\mathrm{Hom}_{\mathsf{Com}(\mathcal{A})}(X,Y)/\sim_h$$

Key fact: Nilhomotopic morphisms form a two-sided ideal in complexes. It means, if $f\sim_h 0$ then $g\circ f\sim_h 0$ and $f\circ l\sim_h 0$ for any morphisms g,l that are composable with f.

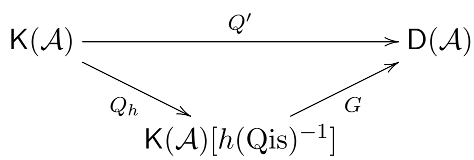
Lemma 3.2.2. *The natural quotient functor* $h\colon\mathsf{Com}(\mathcal{A})\to\mathsf{K}'(\mathcal{A})$ *maps homotopy equivalences to isomorphisms, and so it induces a functor* $\tilde{h}\colon\mathsf{K}(\mathcal{A})\to\mathsf{K}'(\mathcal{A})$. *The induced functor* $\tilde{h}$ *is an equivalence of categories.*

So from now on we will identify $\mathsf{K}(\mathcal{A})$ with $\mathsf{K}'(\mathcal{A})$ via the functor $\bar{h}$. The description $\mathsf{K}'(\mathcal{A})$ of the homotopy category as a quotient turns out to be more convenient than the description $\mathsf{K}(\mathcal{A})$ as a localization. We will use the notation $\mathsf{K}(\mathcal{A})$ for the homotopy category.

Lemma 3.2.3. *1. The localization functor $Q\colon \mathsf{Com}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ factors through the quotient functor $h\colon \mathsf{Com}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$:*

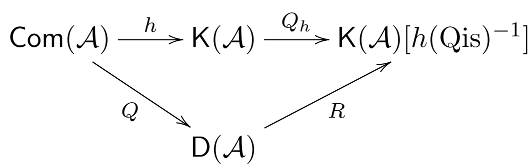


2. Functor Q' factors through the the localization functor Q_h



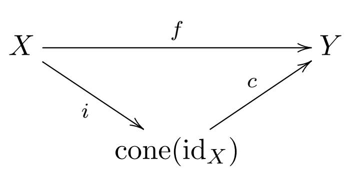
Moreover, the induced functor G is an equivalence of categories.

Proof. Since $Q_h \circ h$ maps Qis to isomorphisms, there exists a functor $F\colon \mathsf{D}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})[h(\mathrm{Qis})^{-1}]$ s.t. $Q_h \circ h \simeq F \circ Q$:



We construct the inverse functor to F as follows. First observe that the functor $Q\colon \mathsf{Com}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ maps all nilhomotopic morphisms f to zero.

Indeed, suppose $f\colon X \to Y$, $X, Y \in \mathsf{Com}(\mathcal{A})$ and suppose $f \sim_h 0$. This means that there is a map $h\colon X \to Y$ of degree -1 s.t. $f = dh + hd$. Consider $\mathrm{cone}(\mathrm{id}_X) = X \oplus X[1]$ and define a map $c^\bullet\colon \mathrm{cone}(\mathrm{id}_X) \to Y$ by $c^n = (f^n, h^{n+1})$. It is a straightforward calculation to verify that since h is a homotopy, the map c will actually be a morphism of complexes, and f factors through it.



Applying the functor Q to this diagram gives $Q(f) = Q(c) \circ Q(i)$. Now, since id_X is obviously a quasi-isomorphism, $\mathrm{cone}(\mathrm{id}_X)$ is acyclic (see 3.2.3 on page 109), and so $Q(\mathrm{cone}(\mathrm{id}_X)) = 0$. Therefore, $Q(f) = 0$.

But then Q must factor through $h\colon \mathsf{Com}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$, since h is universal among all the functors $L\colon \mathsf{Com}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$ such that $L(f) = 0$ whenever $f \sim_h 0$. Hence $Q \simeq Q' \circ h$.

This implies $Q'(h(\text{Qis})) = Q(\text{Qis}) \subset \text{Iso}$. But if Q' sends $h(\text{Qis})$ to isomorphisms, it must factor through Q_h. This implies that $Q' = G \circ Q_h$ for some functor $G\colon \mathsf{K}(\mathcal{A})[h(\text{Qis})^{-1}] \to \mathsf{D}(\mathcal{A})$.

To summarize, we have obtained the following diagram:

$$\begin{array}{ccc}
\mathsf{Com}(\mathcal{A}) & \xrightarrow{h} & \mathsf{K}(\mathcal{A}) \\
{\scriptstyle Q}\downarrow & \overset{Q'}{\swarrow} & \downarrow{\scriptstyle Q_h} \\
\mathsf{D}(\mathcal{A}) & \underset{G}{\overset{F}{\rightleftarrows}} & \mathsf{K}(\mathcal{A})[h(\text{Qis})^{-1}]
\end{array}$$

Hence we have $G \circ F \circ Q \simeq G \circ Q_h \circ h \simeq Q' \circ h \simeq Q$. By the uniqueness of Q we get $G \circ F \simeq \text{id}$. Similarly, using the diagram above we get $F \circ G \circ Q_h \simeq F \circ Q' \simeq Q_h$, and so $F \circ \text{id}$. □

3.3 Verdier theorem

To simplify the notations we will denote the image $h(\text{Qis})$ of Qis under the functor $h\colon \mathsf{Com}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$ again by Qis.

Question (Yuri): Is h universal among all (not necessarily additive) functors $F\colon \mathsf{Com}(\mathcal{A}) \to \mathcal{C}$ s.t. $F(f) = 0_{\mathcal{C}}$ whenever $f \sim_h 0$.

Theorem 3.3.1 (Verdier, [Ver96])**.** *The class of morphisms* Qis *in* $\mathsf{K}(\mathcal{A})$ *is both left and right Ore.*

Proof. We want to check the Ore axioms 1.4.1 and 1.4.2. Saturatedness of the localizing family of morphisms Qis is obvious.

First let's prove the axiom (LO2). Suppose we are given some maps $t\colon W \to X$ and $g\colon W \to Y$, $t \in \text{Qis}$. We want to find $f\colon X \to Z$ and $s\colon Y \to Z$ with $s \in \text{Qis}$.

$$\begin{array}{ccccc}
 & & Z & & \\
 & \overset{f}{\nearrow} & & \overset{s}{\nwarrow} & \\
X & & & & Y \\
 & \underset{t}{\nwarrow} & & \underset{g}{\nearrow} & \\
 & & W & &
\end{array}$$

We are thinking of morphisms in the homotopy category $\mathsf{K}(\mathcal{A})$ as honest morphisms of complexes, since any morphism in $\mathsf{K}(\mathcal{A})$ is represented by some morphism in $\mathsf{Com}(\mathcal{A})$.

Define $\varphi = t \oplus g\colon W \to X \oplus Y$. Take $Z = \text{cone}(\varphi)$. Then we have the following exact sequence

$$W \xrightarrow{\varphi} X \oplus Y \xrightarrow{i_\varphi} \text{cone}(\varphi) \longrightarrow W[1] \tag{5.3}$$

where $i_\varphi \circ \varphi \sim_h 0$ (see 3.2 on page 109). Write $i_\varphi = (-f) \oplus s\colon X \oplus Y \to Z$, where f and s are defined by

$$-f\colon\ X \xrightarrow{i_X} X \oplus Y \xrightarrow{i_\varphi} \operatorname{cone}(\varphi)$$

$$s\colon\ Y \xrightarrow{i_Y} X \oplus Y \xrightarrow{i_\varphi} \operatorname{cone}(\varphi)$$

We claim that Z, f, s are exactly what we are looking for. So first of all we want to prove that the diagram

$$\begin{array}{ccccc} & & Z & & \\ & \overset{f}{\nearrow} & & \overset{s}{\nwarrow} & \\ X & & & & Y \\ & \underset{t}{\nwarrow} & & \underset{g}{\nearrow} & \\ & & W & & \end{array}$$

commutes up to homotopy. We have $0 \sim_h i_\varphi \circ \varphi = (-f, s) \circ (t, g) = -ft + sg$, and so $sg \sim_h ft$.

Next we want to show that s is actually a quasi-isomorphism. To see that, first note that the exact sequence 5.3 induces the following long exact sequence

$$\ldots \longrightarrow \mathrm{H}^k(W) \xrightarrow{\mathrm{H}^k(\varphi)} \mathrm{H}^k(X) \oplus \mathrm{H}^k(Y) \longrightarrow \mathrm{H}^k(Z) \longrightarrow \mathrm{H}^{k+1}(W) \longrightarrow \ldots$$

Since $\varphi = (t, g)$ and t is a quasi-isomorphism, the map $\mathrm{H}^k(\varphi)$ must be injective. Hence $\mathrm{H}^k(Z) \to \mathrm{H}^{k+1}(W)$ is the zero map. Thus the long exact sequence splits into short exact sequences

$$0 \longrightarrow \mathrm{H}^k(W) \longrightarrow \mathrm{H}^k(X) \oplus \mathrm{H}^k(Y) \longrightarrow \mathrm{H}^k(Z) \longrightarrow 0$$

This follows immediately from the definitions that exactness of such short exact sequence is equivalent to the square

$$\begin{array}{ccc} \mathrm{H}^k(W) & \xrightarrow{\mathrm{H}^k(t)} & \mathrm{H}^k(X) \\ {\scriptstyle \mathrm{H}^k(g)}\downarrow & & \downarrow{\scriptstyle \mathrm{H}^k(f)} \\ \mathrm{H}^k(Y) & \xrightarrow{\mathrm{H}^k(s)} & \mathrm{H}^k(Z) \end{array}$$

being cartesian and cocartesian. Then since $\mathrm{H}^k(t)$ is an isomorphism, $\mathrm{H}^k(s)$ also must be an isomorphism. So the map s is a quasi-isomorphism, and we have proved the axiom (LO2). □

Chapter 6

Triangulated categories

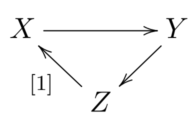
1 The basics

1.1 Definitions

Definition 1.1.1. *A* triangulated category $\mathcal{D}$ *is an additive category equipped with extra structure:*

- *an auto-equivalence* $[1]\colon \mathcal{D} \to \mathcal{D}$*, which we write* $X \mapsto X[1]$*, and call the* shift (or suspension) functor
- *a class* $\mathcal{E}$ *of diagrams of the form* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$*, called* exact (or distinguished) triangles

Write



for such a distinguished triangle.

The class $\mathcal{E}$ *is required to satisfy the following axioms.*

TR1 (a) For all X *in* $\mathcal{D}$*, the triangle* $0 \to X \xrightarrow{1} X \to 0[1]$ *is exact;*

(b) The class $\mathcal{E}$ *is closed under isomorphisms of triangles;*

(c) Every morphism $u : X \to Y$ *in* $\mathcal{D}$ *can be completed to an exact triangle in* $\mathcal{E}$*:*

$$X \xrightarrow{u} Y \overset{\exists v}{\dashrightarrow} W \overset{\exists w}{\dashrightarrow} X[1]\ .$$

TR2 The class $\mathcal{E}$ *is closed under shifts in the sense that if* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ *is in* $\mathcal{E}$*, then so is* $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u(1)} Y[1]$*.*

TR3 Given a commutative diagram

$$\begin{CD}
X @>u>> Y @>v>> Z @>w>> X[1] \\
@VfVV @VgVV @V\exists hVV @Vf[1]VV \\
\widetilde{X} @>\widetilde{u}>> \widetilde{Y} @>\widetilde{v}>> \widetilde{Z} @>\widetilde{w}>> \widetilde{X}[1]
\end{CD}$$

there exists $h\colon Z \to \widetilde{Z}$ *making the rest of the diagram commute. Note that* h *is not functorial.*

TR4 (Octahedron axiom). Given two exact triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$, $Y \xrightarrow{f} \widetilde{Y} \xrightarrow{g} W \xrightarrow{h} Y[1]$, *we can construct the following commutative diagram*

$$\begin{CD}
X @>u>> Y @>v>> Z @>w>> X[1] \\
@V\mathrm{id}VV @VfVV @V\exists pVV @V\mathrm{id}[1]VV \\
X @>\exists\widetilde{u}>> \widetilde{Y} @>\exists\widetilde{v}>> \widetilde{Z} @>w>> X[1] \\
@. @VgVV @V\exists qVV @. \\
@. W @>\mathrm{id}_W>> W @. \\
@. @VhVV @V\exists rVV @. \\
@. Y[1] @>v[1]>> Z[1]. @.
\end{CD}$$

such that rows and columns are exact.

1.2 Examples of triangulated categories

Example 1.2.1. Let $\mathcal{D} = \mathsf{Vect}_k$ be the category of vector spaces over a field k. Put the shift functor [1] to be just $[1] = \mathrm{id}_{\mathcal{D}}$. The class $\mathcal{E}$ of exact triangles is defined by the following rule:

$$\mathcal{E} = \{U \oplus V \xrightarrow{u} V \oplus W \xrightarrow{v} W \oplus U \xrightarrow{w} U \oplus V\}$$

for any triple U, V, W of vector spaces. The maps u, v, w are

$$\begin{aligned}
u &= U \oplus V \twoheadrightarrow V \hookrightarrow V \oplus W \\
v &= V \oplus W \twoheadrightarrow W \hookrightarrow W \oplus U \\
w &= W \oplus U \twoheadrightarrow U \hookrightarrow U \oplus V
\end{aligned}$$

It is a good exercise to check that this actually gives Vect_k the structure of a triangulated category.

Example 1.2.2. The standard examples are $\mathsf{K}^*(\mathcal{A})$ and $\mathsf{D}^*(\mathcal{A})$, for $\mathcal{A}$ an abelian category and $* \in \{\varnothing, b, +, -\}$. In these examples, $[1] : C^\bullet \mapsto C^\bullet$ is the usual shift of complexes. The exact triangles are diagrams isomorphic to

$$X \xrightarrow{u} Y \xrightarrow{i_u} \operatorname{cone}(u) \xrightarrow{p_u} X[1]$$

We will postpone verification that this actually is a triangulated structure to the next section.

Example: vector bundles on projective spaces

This example is due to [BGG78], and is exposited in [OSS11]. Let k be a field of characteristic not 2. Let E be a fixed E-vector space of dimension $n+1$. Let $\Lambda = \bigwedge_k^\bullet(E) = \bigoplus_{i=0}^{n+1} \bigwedge^i(E)$ be the $\mathbb{Z}$-graded exterior algebra of E. Let $\mathsf{M}(\Lambda)$ be the category of graded left Λ-modules, and let $\mathsf{M}^b(\Lambda)$ be the full subcategory of graded modules which are finite-dimensional over k. Let $\mathcal{F} \subset \mathsf{M}^b(\Lambda)$ be the full subcategory of free modules over Λ. It turns out that here $\mathcal{F}$ actually coincides with both subcategories of projective and injective modules.

Let's call $f : V \to V'$ in $\mathsf{M}^b(\Lambda)$ *equivalent to zero* if $f = V \to F \to V'$ for some $F \in \mathrm{Ob}(\mathcal{F})$. Obviously the class $\mathcal{I}$ of morphisms in $\mathsf{M}^b(\Lambda)$ which are equivalent to zero is a two-sided ideal. Define the *stable module category* $\overline{\mathsf{M}}^b(\Lambda)$ to be $\overline{\mathsf{M}}^b(\Lambda) = \mathsf{M}^b(\Lambda)/\mathcal{I}$. Objects of $\overline{\mathsf{M}}^b(\Lambda)$ are the same as those in $\mathsf{M}^b(\Lambda)$, and hom-sets are

$$\mathrm{Hom}_{\overline{\mathsf{M}}^b(\Lambda)}(X, Y) = \mathrm{Hom}_\Lambda(X, Y)/\mathcal{I}.$$

Theorem 1.2.3 (Beilinson, Gelfand, Gelfand)**.** *There is a natural structure of a triangulated category on $\overline{\mathsf{M}}^b(\Lambda)$ in which the shift functor* $[1]$ *is defined by*

$$V \mapsto (\Lambda(-n) \otimes_k V)/i(V)(-n) \qquad (n = \dim E - 1),$$

where $W(m) = \bigoplus_{i \in \mathbb{Z}} W^{i-m}$, and $i(V) = \Lambda^{n+1}(E) \otimes V \subset \Lambda \otimes V$.

I what follows we will try to explain where this triangulated structure comes from.

Observe that every graded Λ-module V can be viewed as a family of complexes

$$\mathscr{L}_e(V) = \cdots \longrightarrow V^{i-1} \xrightarrow{d_e^{i-1}} V^i \xrightarrow{d_e^i} V^{i+1} \longrightarrow \cdots$$

parameterized by $e \in E$. The fact that $e^2 = 0$ in Λ implies that $d_e(v) = e \cdot v$ is a differential. Notice that if $\widetilde{e} = \lambda e$ for $\lambda \in k^\times$, then $\mathscr{L}_{\widetilde{e}}(V) \simeq \mathscr{L}_e(V)$. Thus we have an operation $V \mapsto \{\mathscr{L}_e(V)\}_{e \in \mathbb{P}(E)}$. Let's re-interpret this using algebraic geometry.

Call a complex of quasi-coherent sheaves on $\mathbb{P}(E)$ *rigid* if it has the form

$$\mathscr{L} = \cdots \to V^i \otimes \mathscr{O}_{\mathbb{P}}(i) \to V^{i+1} \otimes \mathscr{O}_{\mathbb{P}}(i+1) \to \cdots.$$

Let Rig be the full subcategory of $\mathsf{Com}(\mathsf{Qcoh}(\mathbb{P}))$ consisting of rigid complexes. Note that the $\{\mathscr{L}_e(V)\}_{e \in \mathbb{P}(E)}$ induces a complex $\mathscr{L}$ in Rig.

In the opposite direction, given a rigid complex $\mathscr{L}$, define $V(\mathscr{L}) \in \mathsf{M}(\Lambda)$ by the rule

$$V(\mathscr{L}) = \bigoplus_{i\in\mathbb{Z}} V(\mathscr{L})^i = \bigoplus_{i\in\mathbb{Z}} \Gamma\left(\mathbb{P}(E), \mathscr{L}^i(-i)\right).$$

We need to give $V(\mathscr{L})$ the structure of a Λ-module. Recall that $\bigoplus_{i\in\mathbb{Z}} \Gamma(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(i))$ is canonically isomorphic to $\mathrm{Sym}_k^\bullet(E^\vee)$. Consider a map

$$a = \Gamma(\mathbb{P}, d^i_{\mathscr{L}}(-i)) : V(\mathscr{L})^i \to V(\mathscr{L})^{i+1} \otimes \Gamma(\mathbb{P}, \mathscr{O}(1)) \simeq V^{i+1} \otimes E^\vee.$$

This allows us to give $V(\mathscr{L})$ a Λ-module structure by

$$e \cdot v = (-1)^i (\mathrm{id} \otimes s_e) a(v)$$

where $s_e : E^\vee \to k$ is $\ell \mapsto \ell(e)$, $e \in E$ and $v \in V(\mathscr{L})^i$.

It is a straightforward verification that $d^{i+1}_{\mathscr{L}} \circ dd^i_{\mathscr{L}} = 0$ implies that $e^2 v = 0$. So we get a graded Λ-module.

Lemma 1.2.4. *Two functors defined above by $V \mapsto \{\mathscr{L}_e(V)\}_{e\in\mathbb{P}(E)}$ and $\mathscr{L} \mapsto V(\mathscr{L})$ are inverse to each other, and thus give the equivalence of categories $\mathsf{Rig} \simeq \mathsf{M}(\Lambda)$.*

We call a rigid complex $\mathscr{L}$ *finite* if $\mathscr{L}^i = 0$ for $|i| >> 0$ and $\dim(\mathscr{L}^i) < \infty$. Denote by Rig^f the full subcategory of Rig of finite complexes.

Lemma 1.2.5. *The equivalence from Lemma 1.2.4 restricts to the equivalence $\mathsf{Rig}^f \simeq \mathsf{M}^b(\Lambda)$.*

Consider now the functor $\Phi \colon \mathsf{M}^b(\Lambda) \xrightarrow{\sim} \mathsf{Rig}^f \to \mathsf{D}^b(\mathsf{coh}(\mathbb{P}(E)))$. Be the Theorem 1.2.3 it factors through $\overline{\mathsf{M}}^b(\Lambda)$.

Theorem 1.2.6. *Functor Φ induces an equivalence of triangulated categories $\overline{\mathsf{M}}^b(\Lambda) \xrightarrow{\sim} \mathsf{D}^b(\mathsf{coh}(\mathbb{P}(E)))$.*

So the "twisted" triangulated structure on $\overline{\mathsf{M}}^b(\Lambda)$ comes from the usual triangulated structure on $\mathsf{D}^b(\mathsf{coh}(\mathbb{P}(E)))$ when being induced by the functor Φ.

1.3 Basic properties of triangulated categories

Recall that a triangulated category consists of an additive category $\mathcal{D}$ together with an autoequivalence [1] and a class $\mathcal{E}$ of "exact triangles" satisfying some axioms appearing in Definition 1.1.1. We are interested in developing some basic consequences of these axioms.

For any triangulated category $\mathcal{D}$, there is (by definition) a quasi-inverse $[-1] \colon \mathcal{D} \to \mathcal{D}$ to [1]. For all $X \in \mathcal{D}$, we have natural isomorphisms $X[1][-1] \simeq X \simeq X[-1][1]$. Replacing $\mathcal{D}$ by an equivalent category, we may assume [1] and $[-1]$ are inverses "on the nose," i.e. $X[1][-1] = X = X[-1][1]$. More precisely, we have the following lemma.

Lemma 1.3.1. *Let $\mathcal{D}$ be a category with an auto-equivalence $[1] : \mathcal{D} \to \mathcal{D}$. Then there exists a category $\widetilde{\mathcal{D}}$ with an automorphism $\widetilde{[1]} : \widetilde{\mathcal{D}} \to \widetilde{\mathcal{D}}$ and $F : \mathcal{D} \to \widetilde{\mathcal{D}}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{[1]} & \mathcal{D} \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ \widetilde{\mathcal{D}} & \xrightarrow{\widetilde{[1]}} & \widetilde{\mathcal{D}} \end{array}$$

Moreover, if $\mathcal{D}$ is triangulated, so is $\widetilde{\mathcal{D}}$, and F is an equivalence of triangulated categories.

Proof. We construct $\widetilde{\mathcal{D}}$ directly. It has as objects sequences $(X_n, \varphi_n)_{n\in\mathbb{Z}}$ where $X_n \in \mathrm{Ob}(\mathcal{D})$ and $\varphi_n : X_n[1] \xrightarrow{\sim} X_{n+1}$. We define $\mathrm{Hom}_{\widetilde{\mathcal{D}}}((X_n, \varphi_n)_n, (Y_n, \psi_n)_n)$ to be the set of tuples $(f_n \in \mathrm{Hom}_{\mathcal{D}}(X_n, Y_n))_n$ such that for all n, the following diagram commutes:

$$\begin{array}{ccc} X_n[1] & \xrightarrow{f_n[1]} & Y_n[1] \\ \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \psi_n} \\ X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1} \end{array}$$

The shift automorphism in $\widetilde{\mathcal{D}}$ is defined by $(X_n, \varphi_n)_{n\in\mathbb{Z}}\widetilde{[1]} = (X_{n+1}, \varphi_{n+1})_{n\in\mathbb{Z}}$.

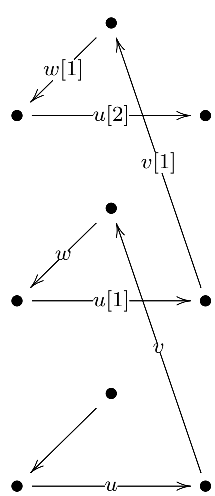
Define $F^{-1} : \widetilde{\mathcal{D}} \to \mathcal{D}$ by $(X_n, \varphi_n) \mapsto X_0$. It is a direct calculation to check that this indeed defines an inverse to F, and that the category $\widetilde{\mathcal{D}}$ satisfies the required properties. □

In light of this lemma, we will henceforth assume that the shift functors on our triangulated categories are automorphisms.

Using the automorphisms [1], [−1] and the axiom TR2, we can extend every exact triangle to a *helix*

$$\cdots \to X[-1] \to Y[-1] \to Z[-1] \to X \to X \to Y \to Z \to X[1] \to \cdots$$

Another way to draw this "helix" is



A morphism of exact triangles extends to a "double helix," and such a "double helix" is uniquely determined (by TR3) up to isomorphism by any of the two arrows in the exact triangle.

For an integer $n \geqslant 0$, let $[n]$ be the n-fold composition of $[1]$, and for $n \leqslant 0$, let $[n]$ be the n-fold composition of $[-1]$. Also, to simplify notation we write $(X, Y) = \operatorname{Hom}_{\mathcal{D}}(X, Y)$.

Proposition 1.3.2. *Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ be an exact triangle. Then for all $U \in \operatorname{Ob}(\mathcal{D})$, the following two sequences of abelian groups are exact:*

$$\cdots \to (U, Z[-1]) \longrightarrow (U, X) \xrightarrow{u_*} (U, Y) \xrightarrow{v_*} (U, Z) \xrightarrow{w_*} (U, X[1]) \longrightarrow (U, Y[1]) \longrightarrow \cdots$$

$$\cdots \longrightarrow (Y[1], U) \longrightarrow (X[1], U) \xrightarrow{w^*} (Z, U) \xrightarrow{v^*} (Y, U) \xrightarrow{u^*} (X, U) \longrightarrow (Z[-1], U) \to \cdots$$

Proof. It suffices to prove that the top sequence is exact at (U, Y). First we show that $u_* v_* = 0$. For $f : U \to X$, define $g = u_* f = fu$, and consider the following diagram.

$$\begin{array}{ccccccc} U & = & U & \longrightarrow & 0 & \longrightarrow & U[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \vdots{\scriptstyle \exists h} & & \downarrow{\scriptstyle f[1]} \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \end{array}$$

The first row is exact by TR1 and TR2. By TR3, there exists $h : 0 \to Z$ such that $vg = h \circ 0 = 0$, which implies $v_* u_* f = 0$ for all f.

Now assume $v_* g = 0$ for some $g : U \to Y$ and consider the following diagram with exact rows:

$$\begin{array}{ccccccc} U & \longrightarrow & 0 & \longrightarrow & U[1] & \xrightarrow{-\mathrm{id}[1]} & U[1] \\ \downarrow{\scriptstyle g} & & \downarrow & & \vdots{\scriptstyle \exists \widetilde{f}} & & \downarrow{\scriptstyle g[1]} \\ Y & \xrightarrow{v} & Z & \longrightarrow & X[1] & \xrightarrow{-u[1]} & Y[1] \end{array}$$

By TR3, there exists $\widetilde{f} : U[1] \to X[1]$ such that $-g[1] = -u[1] \circ \widetilde{f}$. Applying $[-1]$, we see that $g = u \circ \widetilde{f}[-1]$, hence g is in the image of u_*. This proves exactness of the first sequence. For the second one the proof is similar. $\square$

Corollary 1.3.3 (5-lemma)**.** *If the morphisms f, g in axiom TR3 are both isomorphisms, then so is h.*

Proof. Fix some extension h, and choose an arbitrary $U \in \operatorname{Ob}(\mathcal{D})$. Consider the following sequence (notation as in Proposition 1.3.2):

$$\begin{array}{ccccccccc} (U, X) & \xrightarrow{u_*} & (U, Y) & \xrightarrow{v_*} & (U, Z) & \xrightarrow{w_*} & (U, X[1]) & \xrightarrow{-u[1]_*} & (U, Y[1]) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle f[1]_*} & & \downarrow{\scriptstyle g[1]_*} \\ (U, \widetilde{X}) & \xrightarrow{\widetilde{u}_*} & (U, \widetilde{Y}) & \xrightarrow{\widetilde{v}_*} & (U, \widetilde{Z}) & \xrightarrow{\widetilde{w}_*} & (U, \widetilde{X}[1]) & \xrightarrow{\widetilde{u}[1]_*} & (U, \widetilde{Y}[1]) \end{array}$$

By Proposition 1.3.2, both rows are exact, so by the usual 5-Lemma for abelian groups, $h_* : (U, Z) \to (U, \widetilde{Z})$ is an isomorphism for all U. By the Yoneda Lemma, $h : Z \to \widetilde{Z}$ is also an isomorphism. □

Corollary 1.3.4. *In any exact triangle* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$*, we must have*

$$\begin{aligned} vu &= 0 \\ wv &= 0 \\ u[1]w &= 0 \end{aligned}$$

Proof. The proof is exactly as in that of Corollary 1.3.3. One sees that $v_* u_* = 0$ relative to all $U \in \mathrm{Ob}(\mathcal{D})$, whence $vu = 0$ by the Yoneda Lemma. The other parts are similar. □

Corollary 1.3.5. *If we apply TR1 to* $u : X \to Y$*, then the exact triangle completing* u *is determined up to isomorphism.*

Proof. Apply Proposition 1.3.2 to $(f, g) = (\mathrm{id}_X, \mathrm{id}_Y)$ as in

$$\begin{array}{ccccc} X & \xrightarrow{u} & Y & \longrightarrow & \cdots \\ \| & & \| & & \\ X & \xrightarrow{u} & Y & \longrightarrow & \cdots \end{array}$$

□

Corollary 1.3.6. *Let* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ *and* $\widetilde{X} \xrightarrow{\widetilde{u}} \widetilde{Y} \xrightarrow{\widetilde{v}} \widetilde{Z} \xrightarrow{\widetilde{w}} \widetilde{X}[1]$ *be exact triangles. Assume there exists* $g : Y \to \widetilde{Y}$ *such that* $\widetilde{v}gu = 0$*. Then there exists* f *and* h *making the following diagram commute:*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \downarrow{\scriptstyle \exists f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \exists h} & & \downarrow{\scriptstyle f[1]} \\ \widetilde{X} & \xrightarrow{\widetilde{u}} & \widetilde{Y} & \xrightarrow{\widetilde{v}} & \widetilde{Z} & \xrightarrow{\widetilde{w}} & \widetilde{X}[1] \end{array}$$

Moreover, if $\mathrm{Hom}_{\mathcal{D}}(X, \widetilde{Z}[1]) = 0$*, then* f *and* h *are uniquely determined b* g*.*

Proof. Consider $gu : X \to \widetilde{Y}$. Since $\widetilde{v}_*(gu) = 0$, Proposition 1.3.2 applied to $U = \widetilde{X}$ yields $f : X \to \widetilde{X}$ such that $gu = \widetilde{u}_*(f) = \widetilde{u} \circ f$. Similarly one shows that h exists. The uniqueness of f and h follows from the long exact sequence. □

Remark 1.3.7. For any $u : X \to Y$, the axioms TR1 and TR3 yield an exact triangle $X \xrightarrow{u} Y \to Z \to X[1]$. In analogy to the situation with categories of complexes, we write $Z = \mathrm{cone}(f)$. We could even hope that cone : $\mathsf{Mor}(\mathcal{D}) \to \mathcal{D}$ is a functor. Unfortunately, in general one does not always have $\mathrm{cone}(u \circ v) = \mathrm{cone}(u) \circ \mathrm{cone}(v)$. In other words, for arbitrary triangulated categories, the "cone construction" is *not* functorial. Non-functoriality of cone causes a lot of inconvenience, and later we will see possibilities how to deal with that.

2 Further properties

2.1 Abstract cone and octahedron axiom

Let $\mathcal{D}$ be a triangulated category. If $u : X \to Y$ is a morphism in $\mathcal{D}$, then TR1 shows that $X \to Y$ can be extended to an exact triangle $X \xrightarrow{u} Y \xrightarrow{v} C(u) \xrightarrow{w} X[1]$. Moreover, u determines $C(u)$ uniquely (up to isomorphism). We call $C(u)$ an *abstract cone* of u. The axiom TR3 tells us that $C(-)$ behaves "almost functorially." Define $C : \mathsf{Mor}(\mathcal{D}) \to \mathcal{D}$ by $(X \xrightarrow{u} Y) \mapsto C(u)$. For $(f, g) : (X \xrightarrow{u} Y) \to (\widetilde{X} \xrightarrow{\widetilde{u}} \widetilde{Y})$, put $C(f, g) = h$, where h makes the following diagram commute:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \vdots{\scriptstyle \exists h} & & \downarrow{\scriptstyle f[1]} \\ \widetilde{X} & \xrightarrow{\widetilde{u}} & \widetilde{Y} & \xrightarrow{\widetilde{v}} & \widetilde{Z} & \xrightarrow{\widetilde{w}} & \widetilde{X}[1] \end{array}$$

The problem is that $C(u_2 \circ u_1) = C(u_2) \circ C(u_1)$ does not follow from the axiomatics. What can we say about $C(u_2 \circ u_1)$ just using the axioms for a triangulated category – in particular, TR4?

Suppose we are given two morphisms $u_1 : X \to Y$ and $u_2 : Y \to Z$. We can complete these to exact triangles $X \xrightarrow{u_1} Y \xrightarrow{v_1} C(u_1) \xrightarrow{w_1} X[1]$ and $Y \xrightarrow{u_2} Z \xrightarrow{v_2} C(u_2) \xrightarrow{w_2} Y[1]$. Define w to be the composite $C(u_2) \xrightarrow{w_2} Y[1] \xrightarrow{v_1[1]} C(u_1)[1]$.

Lemma 2.1.1. $C(u_2 \circ u_1) \simeq C\left(C(u_2) \xrightarrow{w} C(u_1)[1]\right)[-1]$.

Proof. This is a consequence of Verdier's octahedron axiom. The statement is equivalent to the existence of an exact triangle

$$C(u_1) \to C(u_2 \circ u_1) \to C(u_2) \to C(u_1)[1].$$

One shows that such an exact triangle exists via a direct application of TR4. Simply consider the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u_1} & Y & \xrightarrow{v_1} & C(u_1) & \longrightarrow & X[1] \\ \| & & \downarrow{\scriptstyle u_2} & & \vdots{\scriptstyle \exists} & & \| \\ X & \xrightarrow{u_2 u_1} & Z & \longrightarrow & C(u_2 u_1) & \longrightarrow & X[1] \\ & & \downarrow & & \downarrow & & \\ & & C(u_2) & = & C(u_2) & & \\ & & \downarrow & & \downarrow & & \\ & & Y[1] & \xrightarrow{v_1[1]} & C(u_2)[1] & & \end{array}$$

□

Corollary 2.1.2. *In* $\mathsf{D}(\mathcal{A})$*, any* $X \xrightarrow{u_1} Y \xrightarrow{u_2} Z$ *gives rise to*

$$\mathrm{cone}(u_2 \circ u_1) = \mathrm{cone}\left(\mathrm{cone}(u_2) \to \mathrm{cone}(u_1)[1]\right)[-1].$$

In the special case when u_1, u_2 *are embeddings, the lemma is just the second isomorphism theorem:* $Z/Y \simeq (Z/X)/(Y/X)$*.*

2.2 The homotopy category is triangulated (need proofs!!)

The point of this section is to prove the following theorem.

Theorem 2.2.1. *Let* $\mathcal{A}$ *be an abelian category. Then* $\mathsf{K}(\mathcal{A})$ *is triangulated.*

One puts $[1] : \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$ by $(X[1])^i = X^{i+1}$ and $d^n_{X[1]} = (-1)^n d^n_X$. We say that a diagram $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is exact in $\mathsf{K}(\mathcal{A})$ if and only if it is isomorphic to a diagram of the form

$$X \xrightarrow{u} Y \xrightarrow{i_u} \mathrm{cone}(u) \xrightarrow{p_u} X[1].$$

Lemma 2.2.2. *Suppose* $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ *is an exact sequence in* $\mathsf{Com}(\mathcal{A})$ *such that for every* n*, the sequence* $0 \to X^n \to Y^n \to Z^n \to 0$ *splits (this is* not *the same as* $0 \to X \to Y \to Z \to 0$ *splitting). Let* $s^n : Z^n \to Y^n$ *and* $p^n : Y^n \to X^n$ *be the right (resp. left) inverses of* v^n *(resp.* u^n*). Define* $w : Z \to X[1]$ *by* $w^n = -p^{n+1} \circ d^n_Y$*. Then* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ *is an exact triangle in* $\mathsf{K}(\mathcal{A})$*.*

Lemma 2.2.3. *Every exact triangle in* $\mathsf{K}(\mathcal{A})$ *is isomorphic to one as in Lemma 2.2.2.*

Lemma 2.2.4. *The axioms of a triangulated category hold for triangles as in Lemma 2.2.2.*

2.3 Localization of triangulated categories

Let $\mathcal{D}$ be a triangulated category and $S \subset \mathrm{Mor}(\mathcal{D})$. Let $\mathcal{D}[S^{-1}]$ be the (abstract) localization of $\mathcal{D}$ at S. We know that if S is Ore, then $\mathcal{D}[S^{-1}]$ is additive. We are interested in extra conditions on S that force $\mathcal{D}[S^{-1}]$ to be triangulated.

Verdier found such conditions. Assume that S, in addition to being Ore, satisfies the following two properties:

O4 S is closed under $[1]$, i.e. $s \in S$ if and only if $s[1] \in S$.

O5 If $f, g \in S$ are as in TR3, then h can be chosen to be in S.

Proposition 2.3.1. *If* S *satisfies O1-O5, then* $\mathcal{D}[S^{-1}]$ *has a natural triangulated structure.*

Proof. Recall that if S is Ore, then morphisms in $\mathcal{D}[S^{-1}]$ can be represented by equivalence classes of "left S-fractions." It suffices to define the shift functor on these fractions. Put $(s^{-1} \circ f)[1] = s[1]^{-1} \circ f[1]$. Say that a triangle in $\mathcal{D}[S^{-1}]$ is exact if and only if it is isomorphic to the image of an exact triangle in $\mathcal{D}$.

All that remains is to check the axioms TR3 and TR4. This is pretty tedious direct verification. □

Theorem 2.3.2. *Let $\mathcal{A}$ be an abelian category. Then the derived category $\mathsf{D}(\mathcal{A})$ is triangulated.*

Proof. First we use the fact that $\mathsf{D}(\mathcal{A}) \simeq \mathsf{K}(\mathcal{A})[\mathrm{Qis}^{-1}]$. Next, check that Qis in $\mathsf{K}(\mathcal{A})$ satisfies O1-O5. The axiom O4 is obvious because $\mathrm{H}^k(S[1]) = \mathrm{H}^{k+1}(S)$. On the other hand, showing O5 requires some use of TR4. We will omit this part. $\square$

2.4 Exact functors

Definition 2.4.1. *If $\mathcal{D}, \mathcal{D}'$ are triangulated categories and $F : \mathcal{D} \to \mathcal{D}'$ is additive, then F is called* exact *(or sometimes* triangulated *or even* triangular*) if F respects the triangulated structures, i.e. if*

1. *there exists $\theta : F \circ [1] \xrightarrow{\sim} [1] \circ F$*
2. *F sends distinguished triangles $\mathcal{E}_{\mathcal{D}}$ in $\mathcal{D}$ to distinguished triangles $\mathcal{E}_{\mathcal{D}'}$ in $\mathcal{D}'$.*

Lemma 2.4.2. *Any additive functor $F : \mathcal{A} \to \mathcal{B}$ between* abelian *categories induces an exact functor $F : \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{B})$.*

Proof. This is trivial – F preserves the cone construction. $\square$

Lemma 2.4.3. *If $F : \mathcal{D} \to \mathcal{D}'$ is exact and $G : \mathcal{D}' \to \mathcal{D}$ is an adjoint (either left or right) to F, then G must also be exact.*

Now we want to see why exact sequences give rise to exact triangles. Recall first that for $f : X \to Y$ is a morphism in $\mathsf{Com}(\mathcal{A})$ we defined $\mathrm{cone}(f) = X[1] \oplus Y$ and $\mathrm{cyl}(f) = X \oplus X[1] \oplus Y$, with appropriate twisted differentials, see subsection 3.2.

Lemma 2.4.4. *For any $f : X \to Y$ in $\mathsf{Com}(\mathcal{A})$, define $\beta : \mathrm{cyl}(f) \to Y$ by $\beta(x^k, x^{k+1}, y^k) = f(x^k) + yk$. Then there exists a commutative diagram*

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & Y & \longrightarrow & \mathrm{cone}(f) & \longrightarrow & X[1] \longrightarrow 0 \\
 & & & & \downarrow & & \| & & \\
0 & \longrightarrow & X & \longrightarrow & \mathrm{cyl}(f) & \xrightarrow{\pi} & \mathrm{cone}(f) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \beta} & & & & \\
 & & X & \xrightarrow{f} & Y & & & &
\end{array}$$

such that the rows are exact and α, β are homotopy equivalences. Moreover, $\beta\alpha = \mathrm{id}_Y$ and $\alpha\beta_h \sim \mathrm{id}_{\mathrm{cyl}(f)}$.

Corollary 2.4.5. *Every exact triangle in $\mathsf{K}(\mathcal{A})$ (and also $\mathsf{D}(\mathcal{A})$) is isomorphic to one of the form*

$$X \xrightarrow{i} \mathrm{cyl}(f) \xrightarrow{\pi} \mathrm{cone}(f) \xrightarrow{p_f} X[1]$$

Proof. Just use the commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & \operatorname{cone}(f) & \longrightarrow & X[1] \\
\| & & \uparrow \wr & & \| & & \| \\
X & \longrightarrow & \operatorname{cyl}(Y) & \longrightarrow & \operatorname{cone}(f) & \longrightarrow & X[1]
\end{array}$$

□

Proposition 2.4.6. *Every short exact sequence in* $\mathsf{Com}(\mathcal{A})$ *can be completed to an exact triangle in* $\mathsf{D}(\mathcal{A})$*. More precisely, every short exact sequence of complexes* $0 \to X \to Y \to Z \to 0$ *is quasi-isomorphic to*

$$X \xrightarrow{i_f} \operatorname{cyl}(f) \xrightarrow{\pi} \operatorname{cone}(f) \xrightarrow{p_f} X[1].$$

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
 & & \| & & \uparrow \beta & & \uparrow \gamma & & \\
0 & \longrightarrow & X & \longrightarrow & \operatorname{cyl}(f) & \longrightarrow & \operatorname{cone}(f) & \longrightarrow & 0
\end{array}$$

Define $\gamma : \operatorname{cone}(f) \to Z$ by $\gamma(x^{k+1}, y^k) := g(y^k)$. One checks that the diagram above commutes in $\mathsf{Com}(\mathcal{A})$, and that γ is a quasi-isomorphism. Indeed, since g is surjective, so is γ, and $\operatorname{Ker}(\gamma) = X[1] \oplus \operatorname{Ker}(g) = X[1] \oplus \operatorname{Im}(f) \simeq X[1] \oplus X$. If we show that the kernel of γ is acyclic, it will imply that $\mathrm{H}^\bullet(\gamma)$ is an isomorphism. Now, $\operatorname{Ker}(\gamma)$ is acyclic because one can exhibit an explicit homotopy $\mathrm{id}_{\operatorname{Ker}(\gamma)} \sim_h 0$. The homotopy $h : \operatorname{Ker}(\gamma) \to \operatorname{Ker}(\gamma)[-1]$ is given by $(x^{k+1}, x^k) \mapsto (x^k, 0)$. □

As a conclusion of the results above, we might forget about short exact sequences and work instead with exact triangles.

2.5 Verdier quotients

Let $\mathcal{D}$ be a triangulated category, $\mathcal{N} \subset \mathcal{D}$ a subcategory.

Definition 2.5.1. *The subcategory* $\mathcal{N}$ *is called a* triangulated *subcategory of* $\mathcal{D}$ *if the triangulated structure of* $\mathcal{D}$ *induces one on* $\mathcal{N}$ *(in the obvious sense), and if the inclusion functor* $\mathcal{N} \hookrightarrow \mathcal{D}$ *is exact.*

Lemma 2.5.2. *Assume* $\mathcal{N} \subset \mathcal{D}$ *is strictly full (i.e. closed under isomorphisms). Then* $\mathcal{N}$ *is a triangulated subcategory if and only if* $\mathcal{N}$ *is closed under* $[1]_{\mathcal{D}}$ *and taking cones in* $\mathcal{D}$*.*

Sometimes one calls $\mathcal{N}$ “thick” or “épaisse”.

Example 2.5.3. Let $\mathcal{N} = \mathsf{K}^*(\mathcal{A})$, where $* \in \{-, +, b\}$. This is a triangulated subcategory of the full derived category $\mathsf{K}(\mathcal{A})$.

Definition 2.5.4. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{N}$ a triangulated subcategory. The Verdier quotient of $\mathcal{D}$ by $\mathcal{N}$ is a pair $(\mathcal{D}/\mathcal{N}, Q : \mathcal{D} \to \mathcal{D}/\mathcal{N})$, where $\mathcal{D}/\mathcal{N}$ is triangulated, $Q : \mathcal{D} \to \mathcal{D}/\mathcal{N}$ is exact, such that*

1. $Q(\mathcal{N}) = 0$
2. *Q is universal among exact functors which kill $\mathcal{N}$.*

Definition 2.5.5. *Given $(\mathcal{D}, \mathcal{N})$ as above, define $S(\mathcal{N})$ to be the class of morphisms $s : X \to Y$ in $\mathcal{D}$ such that in any exact triangle $X \xrightarrow{s} Y \to Z \to X[1]$, we have $Z \in \mathrm{Ob}(\mathcal{N})$.*

Theorem 2.5.6 (Verdier)**.** *The class $S(\mathcal{N})$ satisfies Ore conditions O1-O3 and is compatible with the triangulated structure of $\mathcal{D}$ (i.e. it satisfies O4-O5). The localization $(Q : \mathcal{D} \to \mathcal{D}[S^{-1}]$ is a quotient $\mathcal{D}/\mathcal{N}$.*

Example 2.5.7. Recall that one way to define $\mathsf{D}(\mathcal{A})$ is as the localization of $\mathsf{K}(\mathcal{A})$ with respect to quasi-isomorphisms. One could also let $\mathcal{D} = \mathsf{K}(\mathcal{A})$, and let $\mathcal{N}$ be the full subcategory consisting of acyclic complexes. In that case, $S(\mathcal{N}) = \mathrm{Qis}$, so $\mathsf{D}(\mathcal{A}) = \mathsf{K}(\mathcal{A})/\{\text{acyclic}\}$.

Example 2.5.8 (Singularity category)**.** Let $\mathcal{A} = \mathsf{mod}(\mathcal{A})$, or more generally $\mathcal{A} = \mathsf{coh}(X)$ for X a projective variety. Consider the subcategory $\mathsf{Perf}(\mathcal{A}) \subset \mathsf{Com}(\mathcal{A})$ consisting of "perfect complexes", i.e. complexes isomorphic to $\cdots \to P^i \to P^{i+1} \to \cdots$ where $P^i = 0$ for $|i| \gg 0$, and the P^i are locally free sheaves of finite rank. If A is a regular ring (or X is a smooth variety), then it is well-known that $\mathsf{Perf}(\mathcal{A}) = \mathsf{D}^b(\mathcal{A})$. On the other hand, for $A = k[x]/x^2$ and $\mathcal{A} = \mathsf{Mod}(\mathcal{A})$, one calls $\mathsf{D}^{\mathrm{sing}}(\mathcal{A}) = \mathsf{D}(A)/\mathsf{Perf}(\mathcal{A})$ the *singularity category* of $k[x]/x^2$. For our example, $\mathsf{D}^{\mathrm{sing}}(k[x]/x^2)$ is the category of vector spaces with a previously described triangulated structure, see example 1.2.1.

2.6 Exact categories

Exact categories were created by Quillen in [Qui73]. Let $\mathcal{A}$ be an abelian category, $\mathcal{B} \subset \mathcal{A}$ a full additive subcategory. We assume that $\mathcal{B}$ is closed under extensions in $\mathcal{A}$. In other words, if $0 \to X \to Y \to Z \to 0$ is exact in $\mathcal{A}$ and $X, Z \in \mathcal{B}$, then $Y \in \mathcal{B}$.

Definition 2.6.1. *An* exact category *is a pair $(\mathcal{B}, \mathcal{E})$ where $\mathcal{E}$ consists of diagrams $X \to Y \to Z$ such that $0 \to X \to Y \to Z \to 0$ is exact in some ambient abelian category $\mathcal{A}$.*

Remark 2.6.2. Quillen defined exact categories axiomatically. Given an exact category $(\mathcal{B}, \mathcal{E})$, we can always canonically construct $\mathcal{A}$ as a certain subcategory of $\mathsf{Fun}^{\mathrm{add}}, \mathcal{B}^\circ, \mathsf{Ab})$.

Example 2.6.3. Let $\mathcal{B} = \mathcal{A}$, with $\mathcal{E}$ the class of all short exact sequences. Then $\mathcal{A}$ is exact.

Example 2.6.4. Let $\mathcal{A} = \mathsf{Mod}(A)$ for A some associative unital ring. The subcategory $\mathcal{B}$ of projective / flat / free / injective etc. A-modules, is exact.

Example 2.6.5. Let $\mathcal{B}$ be any additive category. Then $\mathcal{B}$ is exact if we let $\mathcal{E}$ consist of split exact sequences $X \to X \oplus Y \to Y$.

Example 2.6.6. Let $\mathcal{A}$ be an abelian category, $\mathcal{B} = \mathsf{Com}(\mathcal{A})$. Let $\mathcal{E}$ consist of $X \xrightarrow{i} Y \xrightarrow{p} Z$ such that for all n, the exact sequence $0 \to X^n \xrightarrow{i^n} Y^n \xrightarrow{p^n} Z^n \to 0$ splits.

Let $(\mathcal{B}, \mathcal{E})$ be an exact category.

Definition 2.6.7. *An object $I \in \mathrm{Ob}(\mathcal{B})$ is called* $\mathcal{E}$-injective *if for all $X \to Y \to Z$ in $\mathcal{E}$, the sequence*

$$\mathrm{Hom}_{\mathcal{B}}(Y, I) \to \mathrm{Hom}_{\mathcal{B}}(X, I) \to 0$$

is exact. Equivalently, I is $\mathcal{E}$-injective if any $I \to X \to Y$ in $\mathcal{E}$ splits.

Dually, $P \in \mathrm{Ob}(\mathcal{B})$ is $\mathcal{E}$-projective if whenever $X \to Y \to Z$ is in $\mathcal{E}$, then

$$\mathrm{Hom}_{\mathcal{B}}(P, Y) \to \mathrm{Hom}_{\mathcal{B}}(P, Z) \to 0$$

is exact. Equivalently, all $X \to Y \to P$ in $\mathcal{E}$ split in $\mathcal{B}$.

Definition 2.6.8. *We say that $\mathcal{B}$ has* enough injectives *if for every $X \in \mathrm{Ob}(\mathcal{B})$, there is a diagram of the form $X \xrightarrow{i} IX \xrightarrow{p} SX$ in $\mathcal{E}$, with IX injective. Similarly we say that $\mathcal{B}$ has* enough projectives *if all $X \in \mathrm{Ob}(\mathcal{B})$, there is $SX \to PX \to X$ in $\mathcal{E}$ with PX projective.*

Definition 2.6.9. *An exact category $(\mathcal{B}, \mathcal{E})$ is called* Frobenius *if*

1. *$\mathcal{B}$ has enough $\mathcal{E}$-injectives and $\mathcal{E}$-projectives*
2. *$\mathcal{E}$-injectives and $\mathcal{E}$-projectives coincide*

Example 2.6.10. Let A be an associative unital ring, $\mathcal{A} = \mathsf{Mod}(A)$, and $\mathcal{E}$ the class of exact sequences. The condition 1 always holds, but condition 2 does not hold in general. But if, for example, $A = k[G]$ for G a finite group and k characteristic zero, then $\mathsf{Mod}(k[G])$ is Frobenius.

Example 2.6.11. Let $A = \bigwedge^{\bullet}(E)$ for E a finite-dimensional k-vector space. Then $\mathcal{A} = \mathsf{M}^b(A)$ is Frobenius.

Example 2.6.12. Let $\mathcal{B} = \mathsf{Com}(\mathcal{A})$ for an abelian category $\mathcal{A}$. Let $\mathcal{E}$ be the class of term-wise-split exact sequences. For $X \in \mathrm{Ob}(\mathcal{B})$, define IX by $(IX)^n = X^n \oplus X^{n+1}$, with differential $d^n_{IX}(x^n, x^{n+1}) = (x^{n+1}, 0)$. Define SX by $(SX)^n = X^{n+1}$ with $d^n_{SX} = -d^{n+1}_X$. We now define $i_X : X \to IX$ and $p_X : IX \to SX$. Put $i_X(x^n) = (x^n, d^n_X x^n)$ and $p_X(x^n, x^{n+1}) = -d^n_x x^n + x^{n+1}$.

First one checks that IX is injective in $(\mathcal{B}, \mathcal{E})$. Next, one shows that i_X splits if and only if $X \sim_h 0$. So an object $Z \in \mathrm{Ob}(\mathcal{B})$ is $\mathcal{E}$-injective if and only if $Z \sim_h 0$. Moreover, IX is also $\mathcal{E}$-projective, so $(\mathsf{Com}(\mathcal{A}), \mathcal{E})$ is a Frobenius category.

Definition 2.6.13. *Let $\mathcal{B}$ be a Frobenius category, and let $X, Y \in \mathrm{Ob}(\mathcal{B})$. Define $I(X,Y)$ to be the set of $f \in \mathrm{Hom}_{\mathcal{B}}(X,Y)$ such that there exists a factorization $f = X \xrightarrow{g} Z \xrightarrow{h} Y$, and Z is projective (hence also injective).*

Definition 2.6.14. *The* stable category *of $\mathcal{B}$, denoted $\underline{\mathcal{B}}$, is the quotient $\mathcal{B}/I$. Objects in $\underline{\mathcal{B}}$ are objects of $\mathcal{B}$, and*

$$\mathrm{Hom}_{\underline{\mathcal{B}}}(X,Y) = \mathrm{Hom}_{\mathcal{B}}(X,Y)/I(X,Y).$$

Example 2.6.15. Let $\mathcal{B} = (\mathsf{Com}(\mathcal{A}), \mathcal{E})$ for $\mathcal{A}$ an abelian category. Then $\underline{\mathcal{B}} = \mathsf{K}(\mathcal{A})$.

Theorem 2.6.16. *The stable category of any Frobenius category has a canonical triangulated structure.*

Proof. First we define the suspension functor in $\underline{\mathcal{B}}$. Given $X \in \mathrm{Ob}(\mathcal{B})$ and "injective resolutions" $X \to I \to Y$ and $X \to I' \to Y'$, there is an extension of id_X to a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & I & \longrightarrow & Y \\ \| & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ X & \longrightarrow & I' & \longrightarrow & Y' \end{array}$$

where v is not unique "on the nose," but any two v are the same in $\mathrm{Hom}_{\underline{\mathcal{B}}}(Y, Y')$. This allows us to define the suspension functor $S : \underline{\mathcal{B}} \to \underline{\mathcal{B}}$ by $X \mapsto SX$ and $X \xrightarrow{u} Y \mapsto SX \xrightarrow{Su} SY$. One has to check that a) this is a well-defined functor, and that b) S is an equivalence of categories.

We need to define exact triangles in $\underline{\mathcal{B}}$. For $u : X \to Y$, consider

$$\begin{array}{ccccc} X & \xrightarrow{i} & IX & \longrightarrow & SX \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle t} & & \| \\ Y & \xrightarrow{v} & C & \xrightarrow{\exists! w} & SX \end{array}$$

with the leftmost square co-cartesian. There exists a unique $w : C \to SX$ making the diagram commute. Exact triangles in $\underline{\mathcal{B}}$ are exactly those of the form $X \to Y \to C \to SX$ as above. □

Let $(\mathcal{B}, \mathcal{E})$ be an exact category such that $\mathcal{B}$ has enough injectives. Let $\mathcal{I}_{\mathcal{B}}$ be the class of injectives in $\mathcal{B}$. Recall that we defined the *stable category* $\underline{\mathcal{B}}$, whose objects are objects in $\mathcal{B}$, and has morphisms $\mathrm{Mor}(\mathcal{B})/\mathcal{I}$, where $\mathcal{I}$ consists of morphisms which factor through injectives. The category $\underline{\mathcal{B}}$ is additive, but not exact.

For every $\in \mathrm{Ob}(\mathcal{B})$, choose $SX \in \mathrm{Ob}(\underline{\mathcal{B}})$ such that there is $X \to IX \to SX$ in $\mathcal{E}$ with IX in $\mathcal{I}_{\mathcal{B}}$. Given

$$\begin{array}{ccccc} X & \xrightarrow{i} & I & \xrightarrow{p} & Y \\ \| & & \downarrow{\scriptstyle \exists u} & & \downarrow{\scriptstyle \exists v} \\ X & \xrightarrow{i'} & I' & \xrightarrow{p'} & Y' \end{array}$$

the injectivity of I' yields u making the diagram commute. Exactness gives v making the diagram commute. Suppose we also have extensions $\widetilde{u}$ and $\widetilde{v}$ of $\mathrm{id}_X : X \to X$.

Lemma 2.6.17. *Given two extensions (u, v), $(\widetilde{u}, \widetilde{v})$, we have $v = \widetilde{v}$ in $\mathrm{Hom}_{\underline{\mathcal{B}}}(Y, Y')$.*

Proof. By commutativity of the diagram, we get $ui = i' = \widetilde{u}i$, whence $(u - \widetilde{u})i = 0$. Similarly $(\widetilde{v} - v)p = 0$. The first tells us that $\mathrm{Ker}(u - \widetilde{u}) \supset \mathrm{Im}(i) = \mathrm{Ker}(p)$, whence $u - \widetilde{u}$ factors through $I/\mathrm{Ker}(p)$ as in

$$\begin{array}{ccc} I/\mathrm{Ker}(p) & \xrightarrow{\overline{p}} & Y \\ \downarrow{\scriptstyle u-\widetilde{u}} & & \downarrow{\scriptstyle v-\widetilde{v}} \\ I' & \xrightarrow{p'} & Y \end{array}$$

We have $v - \widetilde{v} = p'(u - \widetilde{u})\overline{p}^{-1}$, whence the result. □

Corollary 2.6.18. *The assignment $X \mapsto SX$ extends to an endofunctor $S : \underline{\mathcal{B}} \to \underline{\mathcal{B}}$.*

Proof. Suppose we have $f : X \to Y$. Then as above, we have an extension

$$\begin{array}{ccccc} X & \longrightarrow & IX & \longrightarrow & SX \\ \downarrow{\scriptstyle f} & & \vdots\,{\scriptstyle \exists g} & & \vdots\,{\scriptstyle \exists h} \\ Y & \longrightarrow & IY & \longrightarrow & SY \end{array}$$

As above, one sees that $\overline{h}$ is independent of the choice of g. □

The functor S is called the *suspension functor*. Given $X \xrightarrow{u} Y \xrightarrow{v} Z$ in $\mathcal{E}$, we can consider

$$\begin{array}{ccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z \\ \| & & \vdots & & \vdots\,{\scriptstyle \exists w} \\ X & \xrightarrow{i_X} & IX & \xrightarrow{p_X} & SX \end{array}$$

Definition 2.6.19. *A triangle $X \to Y \to Z \to SX$ in $\underline{\mathcal{B}}$ is called* exact *if it is isomorphic to one of the form $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$ as above.*

Note that this defines a functor from the category of diagrams $\mathcal{E}$ to the "category of S-triangles." Denote by $\underline{\mathcal{E}}$ the full image of this functor.

Theorem 2.6.20. *The triple $(\underline{\mathcal{B}}, S, \underline{\mathcal{E}})$ is a* suspended category, *in the sense that all axioms TR0-TR4 hold, except that S need not be an equivalence. If $\mathcal{B}$ is Frobenius, then S is an equivalence.*

Proof sketch. Assume $\mathcal{B}$ is Frobenius. Define $(\mathcal{I})$ to be the category of all (unbounded) acyclic complexes with terms in $\mathcal{I}$. Morphisms in $(\mathcal{I})$ are homotopy-classes of morphisms in $\mathsf{Com}(\mathcal{B})$. We claim that $\underline{\mathcal{B}} \simeq \mathsf{K}(\mathcal{I})$. The corresponding functors are defined are defined as follows. First $\alpha : \mathsf{K}(\mathcal{I}) \to \underline{\mathcal{B}}$ sends X to $\operatorname{Ker}(X^0 \xrightarrow{d^0} X^1)$, and $\beta : \underline{\mathcal{B}} \to \mathsf{K}(\mathcal{I})$ sends X to a complex

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \to I^0 \to I^1 \to I^2 \to \cdots,$$

where $P^\bullet \xrightarrow{\varepsilon_P} X$ is a projectives resolution, $X \xrightarrow{\varepsilon_I} I^0$, and $P^0 \to I^0$ is $\varepsilon_P \circ \varepsilon_I$. It is easy to check that $\alpha \circ [1] \simeq S \circ \alpha$, and that α sends exact triangles to exact triangles. □

Example 2.6.21 (Tate cohomology)**.** Let G be a finite group. Let k be a commutative ring, $A = k[G]$, and $\mathcal{B} = \mathsf{mod}(A)$. This is a Frobenius category. Define

$$\widehat{\mathrm{H}}^i(G, M) = \operatorname{Hom}_{\underline{\mathcal{B}}}(k, S^i M)$$

This is the *i-th Tate cohomology of G with coefficients in M.*

3 t-structures and the recollement

3.1 Motivation and definition

The same triangulated categories can be identified with the derived categories of *completely different* abelian categories. In other words, a triangulated category $\mathcal{D}$ can be equivalent to $\mathsf{D}(\mathcal{A})$ and $\mathsf{D}(\mathcal{B})$ for unrelated abelian categories $\mathcal{A}, \mathcal{B}$. We would like to have some kind of extra structure that allows us to recover $\mathcal{A}$ and $\mathcal{B}$ from $\mathcal{D}$. More generally, we would like to be able to "catch" abelian subcategories of a triangulated category $\mathcal{D}$. Our main example is $\mathcal{D} = \mathsf{Mod}(\mathscr{D}_X)$, where X is a variety and $\mathscr{D}_X$ is the sheaf of differential operators on X.

Example 3.1.1. Let $\mathcal{A}$ be an abelian category, $\mathcal{D} = \mathsf{D}^*(\mathcal{A})$ for $* \in \{\varnothing, +, -, b\}$. Define

$$\mathsf{D}^{\geqslant n}(\mathcal{A}) = \{X \in \mathcal{D} : \mathrm{H}^i(X) = 0 \text{ for all } i < n\}$$
$$\mathsf{D}^{\leqslant n}(\mathcal{A}) = \{X \in \mathcal{D} : \mathrm{H}^i(X) = 0 \text{ for all } i > n\}.$$

The natural "inclusion" $i : \mathcal{A} \to \mathcal{D}$ sending X to $0 \to X \to 0$ is fully faithful, and $\operatorname{Im}(i) = \mathsf{D}^{\leqslant 0}(\mathcal{A}) \cap \mathsf{D}^{\geqslant 0}(\mathcal{A})$. The general notion of a t-structure is an axiomatization of this situation.

Definition 3.1.2. *Let $\mathcal{D}$ be a triangulated category. A* t-structure *on $\mathcal{D}$ is a pair $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ of strictly full subcategories satisfying*

TS1 $\mathcal{D}^{\leqslant 0} \subset \mathcal{D}^{\leqslant 1}$ *and* $\mathcal{D}^{\geqslant 1} \subset \mathcal{D}^{\geqslant 0}$

TS2 For all $X \in \mathrm{Ob}(\mathcal{D}^{\leqslant 0})$ and $Y \in \mathrm{Ob}(\mathcal{D}^{\geqslant 1})$, $\operatorname{Hom}_{\mathcal{D}}(X, Y) = 0$.

TS3 For all $X \in \mathrm{Ob}(\mathcal{D})$, there is an exact triangle $A \to X \to B \to A[1]$ in $\mathcal{D}$ with $A \in \mathrm{Ob}(\mathcal{D}^{\leqslant 0})$ and $B \in \mathrm{Ob}(\mathcal{D}^{\geqslant 1})$.

Here, we are using the standard notation

$$\mathcal{D}^{\leqslant n} = \mathcal{D}^{\leqslant 0}[-n]$$
$$\mathcal{D}^{\geqslant n} = \mathcal{D}^{\geqslant 0}[-n].$$

3.2 The core of a t-structure and truncation functors

Definition 3.2.1. *Let $\mathcal{D}$ be a triangulated category with t-structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$. The* core *of this t-structure is the full subcategory $\mathcal{D}^{\leqslant 0} \cap \mathcal{D}^{\geqslant 0}$. We will sometimes denote it by $\mathcal{D}^{\heartsuit}$.*

Lemma 3.2.2. *The core of the standard t-structure on $\mathsf{D}^*(\mathcal{A})$ is $\mathcal{A}$.*

Theorem 3.2.3. *The core of any t-structure is an abelian category.*

During the rest of this subsection we will sketch the main part of the proof. The proof (which follows the original paper [BBD82]) is based on the following key lemma.

Lemma 3.2.4. *Let $\mathcal{D}$ be a triangulated category equipped with a t-structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$.*

1. *For each $n \in \mathbb{Z}$, the inclusion functors*

$$i_{\leqslant n} : \mathcal{D}^{\leqslant n} \to \mathcal{D}$$
$$i_{\geqslant n} : \mathcal{D}^{\geqslant n} \to \mathcal{D}$$

   have right (resp. left) adjoints

$$\tau_{\leqslant n} : \mathcal{D} \to \mathcal{D}^{\leqslant n}$$
$$\tau_{\geqslant n} : \mathcal{D} \to \mathcal{D}^{\geqslant n}$$

   called truncation functors.

2. *For all $X \in \mathrm{Ob}(\mathcal{D})$, there is an exact triangle of the form*

$$\tau_{\leqslant 0}(X) \to X \to \tau_{\geqslant 1}(X) \to \tau_{\leqslant 0}(X)[1].$$

   Moreover, any two triangles $A \to X \to B \to A[1]$ with $A \in \mathrm{Ob}(\mathcal{D}^{\leqslant 0})$ and $B \in \mathrm{Ob}(\mathcal{D}^{\geqslant 1})$ are canonically isomorphic.

Proof. Let's prove the existence of $\tau_{\leqslant 0}$ and $\tau_{\geqslant 1}$. The proofs for general n are similar. Using the axioms for a t-structure, for all $X \in \mathrm{Ob}(\mathcal{D})$, we can choose $A \to X \to B \to A[1]$ with $A \in \mathcal{D}^{\leqslant 0}$ and $B \in \mathcal{D}^{\geqslant 1}$. Define

$$\tau_{\geqslant 0}(X) = A$$
$$\tau_{\geqslant 1}(X) = B$$

Given $f : X \to Y$ in $\mathrm{Mor}(\mathcal{D})$, let's show that the $A \to X \xrightarrow{f} Y$ in the following diagram factors *uniquely* through A':

$$\begin{array}{ccccccccc}
 & & A & \longrightarrow & X & \longrightarrow & B & \longrightarrow & A[1] \\
 & & \vdots\,\tilde{f} & & \downarrow f & & \vdots & & \vdots \\
B'[-1] & \longrightarrow & A' & \longrightarrow & Y & \longrightarrow & B' & \longrightarrow & A'[1]
\end{array}$$

Applying functor $\mathrm{Hom}_{\mathcal{D}}(A,-)$ we get a long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{D}}(A, B'[-1]) \to \mathrm{Hom}_{\mathcal{D}}(A, A') \to \mathrm{Hom}_{\mathcal{D}}(A, Y) \to \mathrm{Hom}_{\mathcal{D}}(A, B') \to \dots$$

By the axiom TS2 we have $\mathrm{Hom}_{\mathcal{D}}(A, B') = 0$ and $\mathrm{Hom}_{\mathcal{D}}(A, B'[-1]) = 0$. The uniqueness of $\widetilde{f}$ is now clear. Thus we can define $\tau_{\geqslant 0}(f) = \widetilde{f}$. A similar argument shows that an extension of f to $B \to B'$ is unique. Our argument gives isomorphisms of functors:

$$\mathrm{Hom}_{\mathcal{D}^{\leqslant 0}}(A, \tau_{\leqslant 0}(Y)) = \mathrm{Hom}_{\mathcal{D}}(i_{\leqslant 0}A, Y)$$

hence the adjunction. □

Start with the case $\mathcal{D} = \mathsf{D}^*(\mathcal{A})$. We have $\tau_{\leqslant n}X = 0$ if and only if the adjunction morphism $X \to \tau_{\geqslant n+1}X$ is an isomorphism. Similarly, $\tau_{\geqslant n}X = 0$ if and only if $\tau_{\leqslant n-1}X \to X$ is an isomorphism.

If $m \leqslant n$, there are natural isomorphisms

$$\begin{aligned} \tau_{\leqslant m} &\xrightarrow{\sim} \tau_{\leqslant n} \circ \tau_{\leqslant m} \\ \tau_{\geqslant n} &\xrightarrow{\sim} \tau_{\geqslant n} \circ \tau_{\geqslant m} \\ \tau_{\geqslant m} \circ \tau_{\leqslant n} &\xrightarrow{\sim} \tau_{\leqslant n} \circ \tau_{\geqslant m} \end{aligned}$$

we write $\tau_{[m,n]}$ for the last functor.

We can now use these relations to construct *kernels and cokernels* in the core $\mathcal{A} = \mathcal{D}^{\heartsuit}$. Let $f : X \to Y$ be a morphism in $\mathcal{A} \subset \mathcal{D}$. Extend this to an exact triangle $X \xrightarrow{f} Y \to Z \to X[1]$ in $\mathcal{D}$. As before, we denote $Z = \mathrm{cone}(f)$, and call Z the *(abstract) cone* of f. Recall that the cone construction is *not* functorial in this setting, even though the object $Z = \mathrm{cone}(f)$ is unique up to a (non-canonical) isomorphism. Define

$$\begin{aligned} K &= \tau_{\leqslant -1}(Z)[-1] \\ C &= \tau_{\geqslant 0}(Z) \end{aligned}$$

There are exact triangles

$$K \xrightarrow{k} Z[-1] \longrightarrow X$$

$$Y \longrightarrow Z \xrightarrow{c} C$$

One checks that K is the categorical kernel of f, and that C is the categorical cokernel of f. Moreover, $\mathrm{Coker}(k) = \mathrm{Ker}(c)$, so $\mathcal{A}$ is an abelian category.

3.3 Cohomological functors

Recall that if $\mathcal{D} = \mathsf{D}^*(\mathcal{A})$, then for all $i \in \mathbb{Z}$, there are cohomology functors $\mathrm{H}^i : \mathcal{D} \to \mathcal{A}$, defined by $\mathrm{H}^i(X) = \mathrm{H}^0(X[i])$. The functor $\mathrm{H}^0(-)$ is just $X \mapsto \tau_{\geqslant 0}\tau_{\leqslant 0}X = \tau_{[0,0]}X$.

Definition 3.3.1. *For a general triangulated category $\mathcal{D}$ with t-structure and $\mathcal{A} = \mathcal{D}^{\heartsuit}$, define $\mathrm{H}^0 : \mathcal{D} \to \mathcal{A}$ by $X \mapsto \tau_{[0,0]}X$ and $\mathrm{H}^i = \mathrm{H}^0 \circ [i]$.*

Definition 3.3.2. *A t-structure is called* non-degenerate *if*

$$\bigcap_{n\in\mathbb{Z}} \mathcal{D}^{\geqslant n} = \bigcap_{n\in\mathbb{Z}} \mathcal{D}^{\leqslant n} = 0.$$

Definition 3.3.3. *Functor $F : \mathcal{D} \to \mathcal{A}$ from a triangulated category to an abelian category is* cohomological *if whenever $X \to Y \to Z \to X[1]$ is exact in $\mathcal{D}$, the obvious long sequence*

$$\dots \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow F(X[1]) \longrightarrow \dots$$

is exact.

Theorem 3.3.4. *1. For any t-structure, H^0 is a cohomological functor.*

2. If the t-structure is non-degenerate, then $f : X \to Y$ in $\mathcal{D}$ is an isomorphism if and only if $\mathrm{H}^i(f) : \mathrm{H}^i(X) \to \mathrm{H}^i(Y)$ is an isomorphism for all i.

3. we have

$$\begin{aligned} \mathrm{Ob}(\mathcal{D}^{\leqslant n}) &= \{X \in \mathrm{Ob}(\mathcal{D}) : \mathrm{H}^i(X) = 0 \text{ for all } i > n\} \\ \mathrm{Ob}(\mathcal{D}^{\geqslant n}) &= \{X \in \mathrm{Ob}(\mathcal{D}) : \mathrm{H}^i(X) = 0 \text{ for all } i < n\} \end{aligned}$$

3.4 t-exact functors

Suppose we have two triangulated categories $\mathcal{D}, \widetilde{\mathcal{D}}$ with t-structures, and $F \colon \mathcal{D} \to \widetilde{\mathcal{D}}$ is an exact (i.e. triangulated) functor.

Definition 3.4.1. *Functor F is called* left t-exact *if $F(\mathcal{D}^{\geqslant 0}) \subseteq \widetilde{\mathcal{D}}^{\geqslant 0}$. Similarly, F is* right t-exact *if $F(\mathcal{D}^{\leqslant 0}) \subseteq \widetilde{\mathcal{D}}^{\leqslant 0}$.*

Note that if F is left t-exact, then the functor $\mathrm{H}^0 \circ F \colon \mathcal{A} \to \widetilde{\mathcal{A}}$ between two cores of t-structures is left exact in the sense of abelian categories.

Example 3.4.2. Suppose $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ are two abelian categories. Take $\mathcal{D} = \mathsf{D}^*(\mathcal{A})$ and $\widetilde{\mathcal{D}} = \mathsf{D}^*(\widetilde{\mathcal{A}})$. Take any left exact functor φ
colon$\mathcal{A} \to \widetilde{\mathcal{A}}$. If both categories $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ have enough injectives, functor φ has total right derived functor $F = \mathsf{R}\varphi$ fitting into a diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\varphi} & \widetilde{\mathcal{A}} \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathcal{D} & \xrightarrow{F=\mathsf{R}\varphi} & \widetilde{\mathcal{D}} \end{array}$$

Then we can recover the original functor φ by only knowing F via $\varphi = \mathrm{H}^0 \circ F$.

3.5 Derived category of t-structure

Question: Take a triangulated category $\mathcal{D}$ with a t-structure, take its core $\mathcal{A}$, and take its derived category $\mathsf{D}(\mathcal{A})$. How are $\mathcal{D}$ and $\mathsf{D}(\mathcal{A})$ related?

Answer: In general there is no relation between them, unless we have some extra conditions.

Indeed, suppose we have some functor $F\colon \mathsf{Com}(\mathcal{A}) \to \mathcal{D}$ such that it restricts to the embedding $\mathcal{A} \hookrightarrow \mathcal{D}$. Then for a complex of the form $C = [0 \longrightarrow X \xrightarrow{f} Y \longrightarrow 0]$ we should have $F(C) = \mathrm{cone}_{cD}(f)$. But the cone construction is not functorial in general. So there is no such functor F.

We will now look at a situation when such F does exist and we will give a criterion when F is an equivalence of categories.

Definition 3.5.1. *A t-structure is called* bounded *if*

1. *it's non-degenerate (see definition 3.3.2);*
2. *for any $X \in \mathrm{Ob}(\mathcal{D})$ we have $\mathrm{H}^i(X) = 0$ for all but finitely many i's (see definition 3.3.1).*

Example 3.5.2. If $\mathcal{A}$ is abelian, then $\mathcal{D} = \mathsf{D}^b(\mathcal{A})$ with the standard t-structure is bounded. On the other hand $\mathcal{D}(\mathcal{A})$ is not bounded.

Definition 3.5.3. *For $X, Y \in \mathrm{Ob}(\mathcal{A}) = \mathrm{Ob}(\mathcal{D}^{\heartsuit})$ define*

$$\mathrm{Ext}^n_{\mathcal{D}}(X, Y) := \mathrm{Hom}_{\mathcal{D}}(X, Y[n]).$$

There a natural map

$$\mathrm{Ext}^i_{\mathcal{D}}(X, Y) \times \mathrm{Ext}^j_{\mathcal{D}}(Y, Z) \to \mathrm{Ext}^{i+j}_{\mathcal{D}}(X, Z) \tag{6.1}$$

called the Yoneda product. To define it, notice that

$$\mathrm{Ext}^j_{\mathcal{D}}(Y, Z) = \mathrm{Hom}_{\mathcal{D}}(Y, Z[j]) \simeq \mathrm{Hom}_{\mathcal{D}}(Y[i], Z[i+j])$$

Then the Yoneda product is just given by the composition map

$$\mathrm{Hom}_{\mathcal{D}}(X, Y[i]) \times \mathrm{Hom}_{\mathcal{D}}(Y[i], Z[i+j]) \to \mathrm{Hom}_{\mathcal{D}}(X, Z[i+j]).$$

Remark 3.5.4. If $X = Y = Z$, the construction above turns $\mathrm{Ext}^{\bullet}_{\mathcal{D}}(X)$ into a ring (algebra) called *Youneda algebra.*

Theorem 3.5.5 ([BBD82])**.** *Suppose a triangulated category $\mathcal{D}$ is equipped with* bounded *t-structure, and suppose $\mathcal{A} = \mathcal{D}^{\heartsuit}$ is its core. Assume that $F\colon \mathsf{D}^b(\mathcal{A}) \to \mathcal{D}$ is a t-exact functor.*

Then F is an equivalence of triangulated categories if and only if $\mathrm{Ext}^{\bullet}_{\mathcal{D}}$ is generated by $\mathrm{Ext}^1_{\mathcal{D}}$ in the sense that $\forall \alpha \in \mathrm{Ext}^i(X, Y)$, $\forall X, Y \in \mathrm{Ob}(\mathcal{A})$, α can be written as a linear combination of "monomials" $\beta_1 \circ \cdots \circ b_i$, with $\beta_j \in \mathrm{Ext}^1_{\mathcal{D}}(X_j, X_{j+1})$, where $X_1 = X$ and $X_{i+1} = Y$.

3.6 Gluing t-structures

Question: How can we get non-trivial t-structures?

One way to do that is to glue new t-structures from old ones. Let's see how we can do that in more detail.

Let $\mathcal{D}$ be a triangulated category, and $\mathcal{N} \subset \mathcal{D}$ be a triangulated (thick) subcategory. We call

$$\mathcal{N} \xrightarrow{i} \mathcal{D} \xrightarrow{Q} \mathcal{D}/\mathcal{N} \tag{6.2}$$

an exact triple of triangulated categories. Call $\mathcal{E} = \mathcal{D}/\mathcal{N}$.

Definition 3.6.1. *Suppose $\mathcal{N}, \mathcal{D}, \mathcal{E}$ have t-structures. They are called* compatible *if both i and Q are t-exact functors. Then we also say that such triple is a* t-exact triple*.*

Notice that if the exact triple 6.2 is t-exact, then t-structures on $\mathcal{N}$ and $\mathcal{E}$ are uniquely determined by the t-structure on $\mathcal{D}$:

$$\mathcal{N}^{\geqslant 0} = \mathcal{N} \cap \mathcal{D}^{\geqslant 0} \qquad \mathcal{N}^{\leqslant 0} = \mathcal{N} \cap \mathcal{D}^{\leqslant 0}$$
$$\mathcal{E}^{\geqslant 0} = Q(\mathcal{D}^{\geqslant 0}) \qquad \mathcal{E}^{\geqslant 0} = Q(\mathcal{D}^{\geqslant 0})$$

Theorem 3.6.2 ([BBD82])**.** *Assume the triple 6.2 is t-exact. Define*

$$^{\perp}\left(\mathcal{N}^{>0}\right) = \{X \in \mathrm{Ob}(\mathcal{D}) \mid \mathrm{Hom}_{\mathcal{D}}(X, Y) = 0,\ \forall Y \in \mathcal{N}^{>0}\}$$
$$\left(\mathcal{N}^{>0}\right)^{\perp} = \{X \in \mathrm{Ob}(\mathcal{D}) \mid \mathrm{Hom}_{\mathcal{D}}(Y, X) = 0,\ \forall Y \in \mathcal{N}^{<0}\}$$

Then the t-structure on $\mathcal{D}$ is given by

$$\mathcal{D}^{\leqslant 0} = Q^{-1}\left(\mathcal{E}^{\leqslant 0}\right) \cap^{\perp} \left(\mathcal{N}^{>0}\right)$$
$$\mathcal{D}^{\geqslant 0} = Q^{-1}\left(\mathcal{E}^{\geqslant 0}\right) \cap \left(\mathcal{N}^{<0}\right)^{\perp}$$

Corollary 3.6.3. *If $\mathcal{N} \to \mathcal{D} \to \mathcal{E}$ is an exact triple, then for any t-structures on $\mathcal{N}$ and $\mathcal{E}$ there exists* at most one *t-structure on $\mathcal{D}$ making the sequence $\mathcal{N} \to \mathcal{D} \to \mathcal{E}$ t-exact.*

Theorem 3.6.4 (Recollement)**.** *Let $\mathcal{N} \to \mathcal{D} \to \mathcal{E}$ be an exact triple. Assume that $i\colon \mathcal{N} \to \mathcal{D}$ has both left and right adjoints (equivalently, $Q\colon \mathcal{D} \to \mathcal{E}$ has both left and right adjoints). Then for any t-structures on $\mathcal{N}$ and $\mathcal{E}$ there exist unique t-structure on $\mathcal{D}$, compatible with the given ones.*

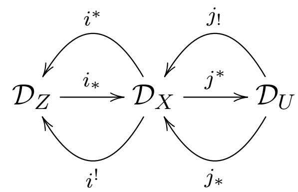
3.7 Examples of gluing

Classical (topological) recollement

Suppose X is a topological space, $Z \subset X$ is a closed subset, and $U = X \setminus Z$ is its complement. Denote by $i\colon Z \hookrightarrow X$ and $j\colon U \to X$ the natural inclusions.

Consider the categories $\mathcal{A}_X, \mathcal{A}_Z, \mathcal{A}_U$ of abelian sheaves (i.e. sheaves of abelian groups) on the spaces X, Z, U respectively. Denote by $\mathcal{D}_X = \mathsf{D}^b(\mathcal{A}_X)$, $\mathcal{D}_Z = \mathsf{D}^b(\mathcal{A}_Z)$ and $\mathcal{D}_U = \mathsf{D}^b(\mathcal{A}_U)$

the correspondent derived categories. Then we have the following diagram (see subsection 2.4 in the Chapter 3)



Let us list some properties of these functors.

1. We have adjoint triples $(i^*, I_* = i_!, i^!)$ and $(j_!, j^! = j^*, j_*)$.
2. Also $i^! j_* = 0$, and so $j^* i_* = 0$ and $j^* j_! = 0$.

In this case theorem 3.6.2 says that

$$\mathcal{D}_X^{\leqslant 0} = \left\{ \mathcal{F} \in \mathcal{D}_X \mid j^*\mathcal{F} \in \mathcal{D}_U^{\leqslant 0} \text{ and } i^* \in \mathcal{D}_Z^{\leqslant 0} \right\}$$

Similarly,

$$\mathcal{D}_X^{\geqslant 0} = \left\{ \mathcal{F} \in \mathcal{D}_X \mid j^*\mathcal{F} \in \mathcal{D}_U^{\geqslant 0} \text{ and } i^! \in \mathcal{D}_Z^{\geqslant 0} \right\}$$

Fix *perversity* $p(Z) = n$ and $p(U) = m$ (this is just a pair of integers associated to each of the sets Z and U). Consider shifted t-structures, $\mathcal{A}_Z[n] := \mathcal{D}_Z^{\leqslant n} \cap \mathcal{D}_Z^{\geqslant n}$ and $\mathcal{A}_U[m] := \mathcal{D}_U^{\leqslant m} \cap \mathcal{D}_U^{\geqslant m}$. Then the gluing theorem allow us to glue these shifted t-structures into a t-structure on $\mathcal{D}_X$ called *perverse* t-structure with perversity p. We denote $\mathcal{D}_X$ equipped with such a t-structure by ${}^p\mathcal{D}_X$. It's core ${}^p\mathcal{A} = ({}^p\mathcal{D}_X)^\heartsuit$ is called *perverse core.*

Abstract blow-down

Suppose X is a smooth surface, $x \in X$ is a closed point, and $\alpha\colon \widetilde{X} \to X$ is the blow-up of X at x with the exceptional curve $\mathcal{L} = \alpha^{-1}(x) \subset X$. We would like to know if it is possible to recover the category $\mathsf{coh}(X)$ of coherent sheaves on X from the category $\mathsf{coh}(\widetilde{X})$ of coherent sheaves on $\widetilde{X}$.

Denote by $\mathcal{F} = \mathcal{O}_{\mathcal{L}} \in \mathsf{coh}(\widetilde{X})$ the structure sheaf of the curve $\mathcal{L}$. Denote

$$\mathcal{N} = \{\mathcal{G} \in \mathcal{D}^b(\mathsf{coh}(\widetilde{X})) \mid \mathsf{R}\operatorname{Hom}_{\mathcal{D}(\mathsf{coh}(\widetilde{X}))}(\mathcal{G}, \mathcal{F}) = 0\}.$$

Theorem 3.7.1 (Bondal-Orlov,[BO95])**.** *There is an equivalence of triangulated categories* $\mathcal{D}^b(\mathsf{coh}(\widetilde{X}))/\mathcal{N} \simeq \mathcal{D}^b(\mathsf{coh}(X))$.

So we know that we can recover the derived category of coherent sheaves on X, but we want to recover the category $\mathsf{coh}(X)$ itself. The problem is that it sits inside $\mathcal{D}^b(\mathsf{coh}(X))$ in a twisted way, i.e. it does not coincide with the core of $\mathcal{D}^b(\mathsf{coh}(X))$ with the standard t-structure, but rather with a twisted t-structure.

Define subcategories of coherent sheaves $\mathfrak{T} = \{T \in \mathsf{coh}(\widetilde{X}) \mid \operatorname{Hom}_{\mathsf{coh}}(T, \mathcal{F}) = 0\}$ and $\mathfrak{F} = \{F \in \mathsf{coh}(\widetilde{X}) \mid \operatorname{Hom}_{\mathsf{coh}}(F, T) = 0,\ \forall T \in \mathfrak{T}\}$.

Define "perverse" (or "twisted") t-structure on $\mathcal{D} := \mathcal{D}^b(\mathsf{coh}(\widetilde{X}))$ by putting ${}^p\mathcal{D}^{\leqslant 0} = \{B \in \mathcal{D}^{\leqslant 0} \mid \mathrm{H}^0(B) \in \mathfrak{T}\}$ and ${}^p\mathcal{D}^{\geqslant 0} = \{B \in \mathcal{D}^{\geqslant -1} \mid \mathrm{H}^{-1}(B) \in \mathfrak{F}\}$. Let ${}^p\mathcal{A} ={}^p \mathcal{D}^{\leqslant 0} \cap^p \mathcal{D}^{\geqslant 0}$ be the core of this perverse t-structure. Then we have the following theorem.

Theorem 3.7.2. *If we denote $Q\colon \mathcal{D} \to \mathcal{D}/\mathcal{N} \simeq \mathcal{D}^b(\mathsf{coh}(X))$ (using notations above) the canonical quotient functor, then we have equivalence of categories $Q({}^p\mathcal{A}) \simeq \mathsf{coh}(X)$.*

3.8 Recollement for triangulated categories

We have already breafly discussed the ideas below in the subsection 2.4 of the Chapter 3, and also in the example 3.7 above. Here we will look at it more carefully.

Definition 3.8.1. *Given three triangulated categories $\mathcal{D}', \mathcal{D}, \mathcal{D}''$ (notice: we do not mention any t-structures at all) we say that $\mathcal{D}$ is* recollement *(or* gluing*) with respect to $\mathcal{D}'$ and $\mathcal{D}''$ if we have six functors*

$$\xymatrix{\mathcal{D}_Z \ar[r]^{i_*} & \mathcal{D}_X \ar@/_2pc/[l]_{i^*} \ar@/^2pc/[l]^{i^!} \ar[r]^{j^*} & \mathcal{D}_U \ar@/_2pc/[l]_{j_!} \ar@/^2pc/[l]^{j_*}} \tag{6.3}$$

satisfying the following properties

R1 $(i^*, I_* = i_!, i^!)$ *and* $(j_!, j^! = j^*, j_*)$ *are adjoint triples;*

R2 $i^! j_* = 0$ *(it is equivalent by the adjunction to $j^* i_* = 0$);*

R3 *The following adjunction morphisms are isomorphisms:*

$$i^* i_* \xrightarrow{\sim} \mathrm{id}_{\mathcal{D}'} \xrightarrow{\sim} i^* i_!$$
$$j^* j_* \xrightarrow{\sim} \mathrm{id}_{\mathcal{D}''} \xrightarrow{\sim} j^* j_!$$

R4 *There are exact triangles*

$$i_* i^! X \to X \to j_* j^! X \to i_* i^! X[1]$$
$$j_! j^* X \to X \to i_! i^* X \to j_! j^* X[1]$$

Remark 3.8.2. Notice that the axiom R3 implies that functors $i_*, i_!$, $j_*, j_!$ are full embeddings.

3.9 Example: topological recollement

We have already discussed this example when we were considering gluing theorems for t-structures, see subsection 3.7.

Recall that we have the following setup. We have a topological space X, $Z \subset X$ is a closed subset, and $U = X \setminus Z$ is its complement, and $i\colon Z \hookrightarrow X$ and $j\colon U \to X$ are the natural inclusions.

Consider the categories $\mathcal{A}_X, \mathcal{A}_Z, \mathcal{A}_U$ of abelian sheaves on the spaces X, Z, U respectively. We denote by $\mathcal{D} = \mathsf{D}^b(\mathcal{A}_X)$, $\mathcal{D}' = \mathsf{D}^b(\mathcal{A}_Z)$ and $\mathcal{D}'' = \mathsf{D}^b(\mathcal{A}_U)$ the correspondent derived categories.

Then we will have the recollement diagram (6.3) (where we need to replace functors $j_!$ and j_* by $\mathsf{R}j_!$ and $\mathsf{R}j_*$ respectively).

We need to say a few words what these functors from the diagram (6.3) actually are. Functors $i_*, \mathsf{R}j_*$ are push-forwards of sheaves, and $i*, j^*$ are pull-backs.

Recall that a continuous map $f\colon X \to Y$ is called *proper* if for any compact subset $K \subset Y$ the set $f^{-1}(K) \subset X$ is also compact. For any such map f we can define functor $f_!$ as follows. For any sheaf $\mathcal{F} \in \mathcal{A}_X$ we define $f_!\mathcal{F} \subseteq f_*\mathcal{F}$ to be the sub-sheaf of the push-forward sheaf, such that

$$f_!\mathcal{F}(U) = \{s \in \mathcal{F}(f^{-1}) \mid f|_{\mathrm{supp}(s)}\colon \mathrm{supp}(s) \hookrightarrow U \text{ is proper}\}$$

Notice that if f is a closed embedding, then $f_! = f_*$. If f is open embedding, then $f_!$ is "extension by zero" functor.

Later we will need the following table

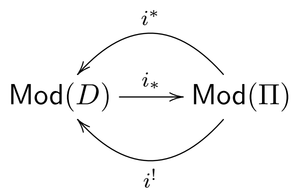
Functor	Exactness	Acyclic classes	Total derived	Classical derived
Γ	left	injective, flabby	$\mathsf{R}\Gamma$	$\mathrm{H}^i(X, -)$ (sheaf cohomology)
f_*	left	inj., flabby	$\mathsf{R}f_*$	$\mathsf{R}^i f_*$
$f_!$	left	inj., flabby, soft	$\mathsf{R}f_!$	$\mathsf{R}^i f_!$
f^*	—	—	—	—
Hom	left	inj.	$\mathsf{R}\,\mathrm{Hom}$	Ext^i
$\mathcal{H}om$	left	inj.	$\mathsf{R}\mathcal{H}om$	$\mathcal{E}xt^i$
$\otimes$	right	flat	$\overset{\mathsf{L}}{\otimes}$	$\mathcal{T}or_i$

Here $\mathsf{R}\mathcal{H}om(\mathcal{F}, \mathcal{G}) := \underline{\mathcal{H}om}^\bullet(\mathcal{F}, \mathcal{I}^\bullet)$ for an injective resolution $\mathcal{G} \xrightarrow{\sim} \mathcal{I}^\bullet$. Moreover, $\mathcal{T}or_i(\mathcal{F}, \mathcal{G}) := \mathrm{H}^{-i}(\mathcal{F} \otimes \mathcal{P}^\bullet)$ for a flat resolution $\mathcal{P}^\bullet \xrightarrow{\sim} \mathcal{G}$.

3.10 Example: algebraic recollement

Let Π be a k-algebra, and assume that it has finite global dimension. Recall that *global dimension* is defined to be the supremum over all Π-modules M of lengths of minimal projective resolutions of M. Assuma also that we fixed a 2-sided ideal $I \subset \Pi$ and denote by D the quotient $D = \Pi/I$. Denote by i the canonical projection $\Pi \twoheadrightarrow \Pi/I = D$.

Then we will obtain the following triple of functors

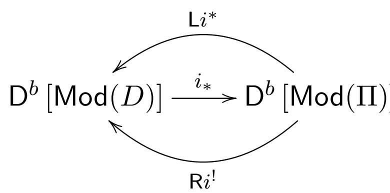


Let's describe what the functors above are. First of all, i_* denotes the restriction of scalars, given by $i_*M = M_D \otimes_D D_\Pi$. Using the usual tensor-hom adjunction, we get

$$\mathrm{Hom}_\Pi(M_D \otimes_D D_\Pi, N_\Pi) \simeq \mathrm{Hom}_D(M, \mathrm{Hom}({}_D D_\Pi, N_\Pi)).$$

Thus, using this adjunction we can define the right adjoint functor $i^!$ to the functor i_* simply by $i^! = \mathrm{Hom}_\Pi({}_D D_\Pi, -)$.

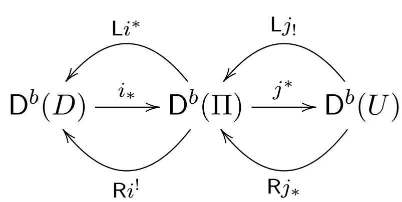
From the other hand, we might as well define the restriction of scalars i_* as $i_* = \mathrm{Hom}_D({}_\Pi D_D, M_D)$. Again, using the standard tensor-hom adjunction we get the left adjoint functor i^* given by $i^* = - \otimes_\Pi D$. So we've got an adjoint triple $(i^*, i_*, i^!)$. We can pass to the derived categories to obtain the adjoint triple



In general, without any extra-assumptions, we can't extend to full recollement situation. Assume from now on that the ideal I was also *idempotent*, i.e. it is generated by an idempotent e, $I = \Pi e \Pi$ where $e^2 = e$.

Define an algebra $U = e\pi e$. Notice that the unit in U is given by e. Moreover, notice that $U \simeq \mathrm{End}_\Pi(e\Pi)$. Assume also the U has finite global dimension.

Then we indeed have all 6 functors from the recollement. Namely, we have the following diagram



In this diagram, functors $\mathsf{R}i^!, i_*$ and $\mathsf{L}i^*$ were defined above. Moreover, functor j^* is given by $j^*M = e\pi \otimes_\Pi M = eM$, functor $\mathsf{L}j_!$ is defined as $\mathsf{L}j_!N = \Pi e \overset{\mathsf{L}}{\otimes}_U N$. Finally, we define $\mathsf{R}j_*N = \mathsf{R}\,\mathrm{Hom}_U(e\Pi, N)$.

Remark 3.10.1. The functor $\mathsf{Mod}(\Pi) \to \mathsf{Mod}(e\pi e)$ given by $M \mapsto eM$ is Morita equivalence if and only if $\Pi e \Pi = \Pi$.

For an application of the above general construction see [BCE08]. The following example describes the situation the paper is dealing with.

Example 3.10.2. Suppose X is a smooth variety, G is a finite group acting on X. Denote by $\mathcal{D}(X)$ the ring of differential operators on X, and consider the algebra $\Pi = \mathcal{D}(X) \rtimes G$. Define an idempotent $e = \frac{1}{|G|} \sum_{g \in G} g$. Note that $\mathsf{Mod}(\mathcal{D}(X) \rtimes G) \simeq \mathsf{Mod}(\mathcal{D}(X)^G)$. this observation is very useful when one is interested in some invariants of $\mathcal{D}(X)^G$ which are invariant under Morita equivalence. Indeed, algebra $\mathcal{D}(X) \rtimes G$ is much easier to understand than $\mathcal{D}(X)^G$.

Chapter 7

Applications of homological algebra

1 Perverse sheaves and quivers

In this section we will discuss application of the gluing theorems for t-structures. First we will discuss abstract recollement, and then we will start talking about perverse sheaves. Our main references on the subject are [BBD82], [Rei10], [GMV96] and [MV88].

1.1 Stratifications

Let X be a topological space.

Definition 1.1.1. *A* stratification $\mathcal{S} = \{S_i\}_{i\in I}$ *of X is a finite decomposition of X into a disjoint union of nonempty, locally closed strata S_i.*

In other words, $X = \bigcup S_i$, the S_i are pairwise disjoint, and each $\overline{S_i}$ is a union of strata, i.e. $\overline{S_i} = \bigcup_{j\in J} S_j$ for some $J \subset I$.

Remark 1.1.2. Any complex-analytic or algebraic variety X over $\mathbb{C}$ admits a stratification with all S_i nonsingular, and such that the following "equidimensionality condition" holds: for any $p, q \in S_i$, there is a diffeomorphism σ of X, preserving all strata, such that $\sigma(p) = q$.

Remark 1.1.3. If $\mathcal{S}$ is a stratification of X, then $\mathcal{S}$ admits a natural partial order whereby $S_i \leqslant S_j$ if and only if $S_i \subset \overline{S_j}$. In this partial order, S_i is minimal if and only if S_i is closed, ad S_i is maximal if and only if S_i is open.

Example 1.1.4. Let $Z \subset X$ be a closed nonempty subspace, $U = X \smallsetminus Z$. Let $\mathcal{S} = \{S_0 = Z, S_1 = U\}$. We have $S_0 \leqslant S_1$, and clearly X is the disjoint union of S_0 and S_1.

Example 1.1.5 (coordinate stratification)**.** Let $X = \mathbb{C}^n$, and define

$$S_I = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_i = 0 \text{ if and only if } i \in I\}.$$

Then $\mathcal{S} = \{S_I : I \in 2^{\{1,\ldots,n\}}\}$ is a stratification of $\mathbb{C}^n$. If $n = 1$, then our stratification is $\{0, \mathbb{C}^\times\}$.

Example 1.1.6 (Schubert stratification)**.** Let G be a complex semisimple algebraic group, $B \subset G$ a Borel subgroup, and $X = G/B$ the flag variety. Then X admits a stratification $\{S_w : w \in W\}$, where the S_w are *Schubert cells.*

Example 1.1.7 (MacPherson-Vilonen, see [MV88])**.** Let $n \leqslant m$ be integers. Put $X = \mathbb{C}^2$, $S_0 = 0$, $S_1 = \{(x, y) : y^n = x^m\} \smallsetminus S_0$, and $S_2 = \mathbb{C}^2 \smallsetminus (S_0 \cup S_2)$.

1.2 Perversity function and perverse sheaves

Definition 1.2.1. *Let* $(X, \mathcal{S})$ *be a stratified space. A* perversity *is a function* $p : \mathcal{S} \to \mathbb{Z}$*.*

Given $(\mathcal{S}, p)$ and $S \in \mathcal{S}$, write $i_S : S \hookrightarrow X$ for the inclusion. Write $\mathsf{D}(X) = \mathsf{D}(\mathsf{Sh}_X)$. We define two full subcategories of $\mathsf{D}(X)$:

$$
\begin{aligned}
{}^p\mathsf{D}^{\leqslant 0}(X) &= \{\mathscr{F}^\bullet \in \mathsf{D}(X) : \mathrm{H}^n(i_S^* \mathscr{F}) = 0 \text{ for all } S \in \mathcal{S} \text{ and } n > p(S)\} \\
{}^p\mathsf{D}^{\geqslant 0}(X) &= \{\mathscr{F}^\bullet \in \mathsf{D}(X) : \mathrm{H}^n(i_S^! \mathscr{F}) = 0 \text{ for all } S \in \mathcal{S} \text{ and } n < p(s)\}.
\end{aligned}
$$

Equivalently,

$$
\begin{aligned}
{}^p\mathsf{D}^{\leqslant 0}(X) &= \{\mathscr{F}^\bullet \in \mathsf{D}(X) : i_S^* \mathscr{F} \in \mathsf{D}(X)^{\leqslant p(S)} \text{ for all } S \in \mathcal{S}\} \\
{}^p\mathsf{D}^{\geqslant 0}(X) &= \{\mathscr{F}^\bullet \in \mathsf{D}(X) : i_S^! \mathscr{F} \in \mathsf{D}(X)^{\geqslant p(S)} \text{ for all } S \in \mathcal{S}\}
\end{aligned}
$$

Theorem 1.2.2. *For any* $(\mathcal{S}, p)$*, the pair* $({}^p\mathsf{D}^{\leqslant 0}, {}^p\mathsf{D}^{\geqslant 0})$ *is a t-structure on* $\mathsf{D}(X)$*, which induces a t-structure on* $\mathsf{D}^*(X)$ *for all* $* \in \{b, +, -\}$*.*

Definition 1.2.3. *We put* $\mathsf{Perv}^p(X) = {}^p\mathsf{D}^{\leqslant 0}(X) \cap {}^p\mathsf{D}^{\geqslant 0}(X)$*; this is the (abelian) category of* p-perverse sheaves on X*.*

If $p = 0$, then $\mathsf{Perv}^p(X) = \mathsf{Sh}_X$. We can replace $\mathbb{Z}$ by any constant sheaf $\mathscr{O}$ of rings on X, and replace Sh_X by the category of sheaves of $\mathscr{O}$-modules. We write $\mathsf{Perv}^p(S, \mathscr{O})$ for the resulting category.

The most common setting is where each S_i is over $\mathbb{C}$, and $p = -\frac{1}{2}\dim_{\mathbb{R}} S = \dim_{\mathbb{C}} S$. This is called *middle perversity.*

1.3 Middle extension

We will define a functor $j_{!*}$, sometimes called *middle extension*, or also *Goresky-MacPherson extension.*

Theorem 1.3.1 (GM)**.** *Let* $Z \xrightarrow{i} X \xleftarrow{j} U$ *be the open-closed stratification. Let* $(\mathsf{D}_U^{\leqslant 0}, \mathsf{D}_U^{\geqslant 0})$ *and* $(\mathsf{D}_Z^{\leqslant 0}, \mathsf{D}_Z^{\geqslant 0})$ *be t-structures on* $\mathsf{D}^b(U)$ *and* $\mathsf{D}^b(Z)$*. Let* $\mathcal{A}_U = \mathsf{D}_U^{\heartsuit}$ *and* $\mathcal{A}_Z = \mathsf{D}_Z^{\heartsuit}$ *be the corresponding cores. Then for any* $\mathscr{F}^\bullet \in \mathcal{A}_U$ *there exists a unique* $\mathscr{G}^\bullet \in \mathcal{A}_X$ *such that*

1. $j^* \mathscr{G}^\bullet = \mathscr{F}^\bullet$
2. $i^* \mathscr{G} \in \mathsf{D}_Z^{\leqslant -1}$

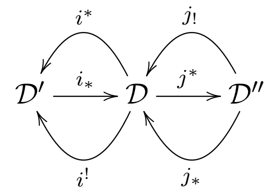
3. $i^!\mathscr{G} \in \mathsf{D}_Z^{\geqslant 1}$.

Proof. Recall that $j^* j_* \simeq \mathrm{id}_U$ and $(j_!, j^*, j_*)$ is an adjoint triple. Thus for any $\mathscr{F}_1, \mathscr{F}_2 \in \mathsf{D}_U$,

$$\begin{aligned} \mathrm{Hom}_{\mathsf{D}}(j_! \mathscr{F}_1, j_* \mathscr{F}_2) &= \mathrm{Hom}_{\mathsf{D}_U}(\mathscr{F}_1, j^* j_* \mathscr{F}_2) \\ &\simeq \mathrm{Hom}_{\mathsf{D}_U}(\mathscr{F}_1, \mathscr{F}_2). \end{aligned}$$

Taking $\mathscr{F} = \mathscr{F}_1 = \mathscr{F}_2$, the identity morphism on $\mathscr{F}$ gives a morphism $N : j_! \mathscr{F} \xrightarrow{N\mathscr{F}} j_* \mathscr{F}$. In other words, we have a natural transformation $N : j_! \to j_*$. Put $j_{!*} = \mathrm{Im}(N : j_! \to j_*)$. The functor $j_{!*}$ has the desired properties. □

Recall the general situation of recollement. We have three triangulated categories with six functors



Here, $(i_!, j^! = j^*, j_*)$ is an adjoint triple, and we require that $j^* j_* \simeq \mathrm{id}_{\mathcal{D}''} \simeq j^* j_!$.

For any $F, G \in \mathrm{Ob}(\mathcal{D}')$, we have

$$\mathrm{Hom}_{\mathcal{D}}(j_! F, j_* G) \simeq \mathrm{Hom}_{\mathcal{D}''}(F, j^* j_* G) \simeq \mathrm{Hom}_{\mathcal{D}''}(F, G).$$

Setting $F = G$, we get for any $F \in \mathrm{Ob}(\mathcal{D}'')$ a morphism $N(F) : j_! F \to j_* F$. This yields a natural transformation (called the *norm*) $N : j_! \to j_*$.

Assume $\mathcal{D}', \mathcal{D}, \mathcal{D}''$ are derived categories of abelian categories.

Definition 1.3.2. *The (Goresky-MacPherson) middle extension is* $j_{!*} = \mathrm{Im}(N : j_! \to j_*)$.

So $j_{!*}$ is a functor $\mathcal{D}'' \to \mathcal{D}$. [**Note that it is not clear *a priori* if this makes sense – the image may not exist.**]

Definition 1.3.3. *Let $(X, \mathcal{S})$ be a stratified space with perversity function $p : \mathcal{S} \to \mathbb{Z}$. Let R be a noetherian ring, and $\mathcal{D} = \mathsf{D}(X)$ be the derived category of sheaves of R-modules on X. Let $i_S : S \hookrightarrow X$ be the inclusion for $S \in \mathcal{S}$.*

$$\begin{aligned} {}^p\mathsf{D}^{\leqslant 0}(X) &= \{\mathscr{F}^\bullet \in \mathsf{D}(X) : i_S^* \mathscr{F} \in \mathsf{D}^{\leqslant p(S)}(X) \text{ for all } S \in \mathcal{S}\} \\ {}^p\mathsf{D}^{\geqslant 0}(X) &= \{\mathscr{F}^\bullet \in \mathsf{D}(X) : i_S^! \mathscr{F} \in \mathsf{D}^{\geqslant p(S)}(X) \text{ for all } S \in \mathcal{S}\}. \end{aligned}$$

Let $\mathsf{Perv}^p(\mathcal{S})$ be the core of this t-structure.

Let U be a (union of) open strata, $j : U \hookrightarrow X$. Then we have functors $\mathsf{R}j_*, j_! : \mathsf{D}(U) \to \mathsf{D}(X)$.

Lemma 1.3.4. *For any perversity p, the functor $\mathsf{R}j_*$ (resp. $j_!$) is left t-exact (resp. right t-exact) with respect to the p-perverse structure $({}^p\mathsf{D}^{\leqslant 0}(X), {}^p\mathsf{D}^{\geqslant 0}(X))$.*

Hence, if $\mathscr{P} \in \mathsf{Perv}^p(U)$ is a perverse sheaf, then $\mathsf{R}j_*(\mathscr{P}) \in \mathrm{Ob}({}^p\mathsf{D}(X)^{\geqslant 0})$ and $j_!(\mathscr{P}) \in \mathrm{Ob}({}^p\mathsf{D}(X)^{\leqslant 0})$.

Definition 1.3.5. *Let the situation be as above. Define functors* ${}^pj_*, {}^pj_! : \mathsf{Perv}^p(U) \to \mathsf{Perv}^p(X)$ *by*

$$
\begin{aligned}
{}^pj_*(\mathscr{P}) &= {}^p\tau_{\leqslant 0}\left(\mathsf{R}j_*\mathscr{P}\right) \simeq {}^p\tau_{[0,0]}\left(\mathsf{R}j_*\mathscr{P}\right) \\
{}^pj_!(\mathscr{P}) &= {}^p\tau_{\geqslant 0}\left(j_!\mathscr{P}\right) \simeq {}^p\tau_{[0,0]}\left(j_!\mathscr{P}\right).
\end{aligned}
$$

Here ${}^p\tau_{\leqslant 0}$ and ${}^p\tau_{\geqslant 0}$ are the truncation functors with respect to the p-perverse structure. The norm transform $N : j_! \to \mathsf{R}j_*$ induces a "perverse norm" ${}^pj_! \to {}^pj_*$, which a natural transformation between functors $\mathsf{Perv}(p, U) \to \mathsf{Perv}(p, X)$.

Definition 1.3.6. *The (perverse) Goresky-MacPherson extension is*

$$
{}^pj_{!*}(\mathscr{P}) = \mathrm{Im}({}^pj_!(\mathscr{P}) \to {}^pj_*(\mathscr{P})).
$$

Proposition 1.3.7. *For any union of open strata* U*, if* $\mathscr{P} \in \mathsf{Perv}^p(U)$*, then* $\mathscr{F} = {}^pj_{!*}(\mathscr{P}) \in \mathsf{Perv}^p(X)$*, as an extension of* $\mathscr{P}$ *to* X *(i.e.* $j^*\mathscr{F} = \mathscr{P}$*), is characterized uniquely by the following conditions:*

$$
\begin{aligned}
\mathrm{H}^n(i_S^*\mathscr{F}) &= 0 \qquad \text{for all } n \geqslant p(s) \\
\mathrm{H}^n(i_S^!\mathscr{F}) &= 0 \qquad \text{for all } n \leqslant p(s).
\end{aligned}
$$

for all $S \subset X \setminus U$*.*

1.4 Constructible complexes

Let $(X, \mathcal{S})$ be a stratified space. We assume that all strata are "good," i.e. equidimensional, with t each $i_S : S \hookrightarrow X$ having finite cohomological dimension. Let R be a noetherian ring.

Definition 1.4.1. *A sheaf* $\mathscr{F} \in \mathsf{Sh}(X, R)$ *is* constructible *with respect to stratification* $\mathcal{S}$ *if* $i_S^*\mathscr{F} =: \mathscr{F}|_S$ *is locally constant (i.e. a local system) for each* $S \in \mathcal{S}$*. A complex* $\mathscr{F}^\bullet \in \mathsf{Com}(\mathsf{Sh}_X(R))$ *is (cohomologically) constructible if each* $\mathrm{H}^n(\mathscr{F})$ *is a local system.*

Definition 1.4.2. *We let* $\mathsf{D}_{\mathcal{S}}(X, R) \subset \mathsf{D}(X, R)$ *be the full subcategory of constructible complexes with respect to* $\mathcal{S}$*. Let* $\mathsf{D}_c(X, R) = \bigcup_{\mathcal{S}} \mathsf{D}_{\mathcal{S}}(X, R)$*, where* $\mathcal{S}$ *ranges over all good stratifications of* X*.*

One reason to introduce $\mathsf{D}_c(X, R)$ is Verdier duality. Recall that for $f : X \to \{*\}$ the unique map, we define $D_X^\bullet = f^!R$; this is called the *dualizing complex*.

Theorem 1.4.3 (Poincaré-Verdier duality)**.** *Let* $R = k$ *be a field. Then the dualizing complex* D_X *is in* $\mathsf{D}_c(X, k)$*. If we define* $\mathscr{D} = \mathsf{R}\mathscr{H}\!om(-, D_X)$*, then* $\mathscr{D}_X : \mathsf{D}_c(X, k) \to \mathsf{D}_c(X, k)^\circ$ *is an equivalence of categories.*

From now on, by p-perverse sheaves we mean the category

$$\mathsf{Perv}_c^p(X,R) = \mathsf{D}_c(X,R)^\heartsuit = {}^p\mathsf{D}_c^{\leqslant 0}(X,R) \cap {}^p\mathsf{D}_c^{\geqslant 0}(X,R)$$

There is a duality theory for perverse sheaves. Define $p^* : \mathcal{S} \to \mathbb{Z}$ by $p^*(S) = -p(S) - \dim_{\mathbb{R}}(S)$. Notice that

$$\begin{aligned} \mathrm{H}^n(i_S^{*\text{ or }!}\mathscr{F}) &= 0 \qquad n > p(S) \\ \mathrm{H}^n(i_S^{*\text{ or }!}\mathscr{D}_X\mathscr{F}) &= 0 \qquad n > p^*(S). \end{aligned}$$

Again, assume $R = k$ is a field. Then the duality functor $\mathscr{D}_X : \mathsf{Perv}_c^p(X,k) \to \mathsf{Perv}_c^p(X,k)^\circ$ is an equivalence of categories. Assume all $S \in \mathcal{S}$ are even-dimensional, and put $p_{1/2}(S) = -\frac{1}{2}\dim_{\mathbb{R}}(S)$. This is known as the *middle perversity*. Poincaré-Verdier duality restricts to an autoduality on $\mathsf{Perv}_c^{p_{1/2}}(X,k)$. In general, we get a duality between $\mathsf{Perv}_c^p(X,k)$ and $\mathsf{Perv}_c^{p^*}(X,k)$.

Proposition 1.4.4. *Let $p = p_{1/2}$. Let $j : U \hookrightarrow X$ be the inclusion of an open stratum. Let $\mathscr{P} \in \mathsf{Perv}_c^p(X,k)$ be autodual. Then $\mathscr{F} = {}^p j_{!*}\mathscr{P}$ is the unique autodual p-perverse extension of $\mathscr{P}$ to $\mathscr{F}$ such that $\mathrm{H}^n(i_S^*\mathscr{F}) = 0$ for all $S \subset X \smallsetminus U$.*

1.5 Refining stratifications

Fix a space X. Assume that a stratification $\mathcal{T}$ is obtained from $\mathcal{S}$ by "refining strata" in that each $S \in \mathcal{S}$ is a union of strata in $\mathcal{T}$. There is a natural inclusion $\mathsf{D}_{\mathcal{S}}(X,R) \hookrightarrow \mathsf{D}_{\mathcal{T}}(X,R)$. Moreover, we assume that $\mathcal{S}$ and $\mathcal{T}$ come with compatible perversities p, q, i.e.

$$p(S) \leqslant q(T) \leqslant p(S) + \dim S - \dim T$$

for all $S \in \mathcal{S}$, $T \in \mathcal{T}$ with $T \subset S$.

Theorem 1.5.1. *The t-structure on $\mathsf{D}_{\mathcal{T}}(X,R)$ of perversity q induces the t-structure of perversity p on $\mathsf{D}_{\mathcal{S}}(X,R)$, and an exact embedding of abelian categories $\mathsf{Perv}_c^p(X) \hookrightarrow \mathsf{Perv}_c^q(X)$. For any $j : U \hookrightarrow X$, the corresponding functors ${}^q j_*$, ${}^q j_!$, ${}^q j_{*!}$ are compatible with those for p, in the sense that e.g. the following diagram commutes:*

$$\begin{array}{ccc} \mathsf{Perv}_c^p(X) & \longrightarrow & \mathsf{Perv}_c^q(X) \\ \downarrow {}^p j^* & & \downarrow {}^q j^* \\ \mathsf{Perv}_c^p(U) & \longrightarrow & \mathsf{Perv}_c^q(U) \end{array}$$

Example 1.5.2. Let X be a complex manifold, $R = \mathbb{C}$, and $\mathcal{S}$ a stratification of X by nonsingular complex submanifolds. Let $p : \mathcal{S} \to \mathbb{Z}$ be a perversity such that $p(S)$ only depends on $2\dim_{\mathbb{C}}(S)$. We also require that p satisfy

$$0 \leqslant p(n) - p(m) \leqslant m - n \qquad n \leqslant m. \tag{$*$}$$

Using Theorem 1.5.1, define a t-structure of perversity p on $\mathsf{D}_c^b(X,\mathbb{C})$. Let $\mathsf{Perv}_c^p(X)$ be the core of this t-structure.

Proposition 1.5.3. *For $\mathscr{F}^\bullet \in \mathsf{D}_b^c(X,\mathbb{C})$. The following are equivalent:*

1. *$\mathscr{F}^\bullet \in \mathsf{Perv}_c^p(X)$*

2. *All irreducible submanifolds $S \subset X$ contain a Zariski-analytic-open subset $U \xrightarrow{j} S$, such that*

$$\begin{aligned} \mathrm{H}^i(\mathsf{R}j^*\mathscr{F}) &= 0 \qquad i > p(\dim_{\mathbb{R}} S) \\ \mathrm{H}^i(j^!\mathscr{F}) &= 0 \qquad i < p(S). \end{aligned}$$

Theorem 1.5.4. *Under the assumption $(*)$, the category $\mathsf{Perv}_{\mathcal{S}}^c(X) = \mathsf{Perv}^p(X) \cap \mathsf{D}_{\mathcal{S}}(X,\mathbb{C})$ is Artinian (and Noetherian) and its simple objects are of the form*

$$\mathscr{L}(S,\mathscr{E}) = ({}^p i_S)_{*!}\mathscr{E}[p(S)]$$

where $\mathscr{E}$ is a locally constant sheaf on $S \in \mathcal{S}$. In particular, if all S are simply connected, then there is a bijection between simple objects and strata in $\mathcal{S}$.

In general, the category $\mathsf{Perv}^p(X)$ is *not* Artinian, but it is Noetherian. If $p = p_{1/2}$, then autoduality + Noetherian $\Rightarrow$ Artinian. For this reason, one often restricts to the (finite-length) category $\mathsf{Perv}^{p_{1/2}}(X)$.

1.6 Gluing

Start with an easy stratification $Z \hookrightarrow X \hookleftarrow U$, $Z = X \smallsetminus U$. Let $i : Z \to X$ and $j : U \to X$ be the inclusions. We will assume that U is a union of open strata, so that p induces perversities on U and Z. A good example is $\{0\} \hookrightarrow \mathbb{C} \hookleftarrow \mathbb{C}^\times$, $k = \mathbb{C}^{\mathrm{an}}$. In this example, one typically puts $p(\{0\}) = 0 = p(\mathbb{C}^\times)$. We would like to Describe the category $\mathsf{Perv}_c^p(X,k)$ in terms of $\mathsf{Perv}_c^p(Z,k)$ and $\mathsf{Perv}_c^p(U,k)$. Also, given $\mathscr{G}^\bullet \in \mathsf{Perv}^p(U)$ and $\mathscr{H}^\bullet \in \mathsf{Perv}^p(Z)$, we would like conditions on $\mathscr{G}$ and $\mathscr{H}$ that guarantee the existence of $\mathscr{F}^\bullet \in \mathsf{Perv}^p(X)$ such that

$$\begin{aligned} i^*\mathscr{F} &\simeq \mathscr{H} \\ j^*\mathscr{F} &\simeq \mathscr{G}. \end{aligned}$$

For the first question, we have the following theorem.

Theorem 1.6.1. *The t-structures of perversity p on $\mathsf{D}_c^b(U,\mathbb{C})$, $\mathsf{D}_c^b(Z,\mathbb{C})$ and $\mathsf{D}^b(X,\mathbb{C})$ are related by the general recollement construction.*

See Section 1.3 for the general notion of recollement.

Theorem 1.6.2. $\mathsf{Perv}_c^{p_{1/2}}(\mathbb{C},\mathbb{C}^{an})$ *is equivalent to the category of quadruples $(V, W, A : V \leftrightarrows W : B)$ such that $AB+1$ and $BA+1$ are invertible.*

This equivalence comes from the functors "vanishing cycles" and "nearby cycles."

This corresponds to the quiver $Q = \bullet \to \bullet$. From this, we can form the *deformed preprojective algebra* $\Lambda^{q,\hbar}(Q)$, and $\mathsf{Perv}_c^{p_{1/2}}(\mathbb{C},\mathbb{C}^{an}) \simeq \Lambda^{q,\hbar}(Q)$-$\mathsf{Mod}$.

2 Matrix factorizations

2.1 Introduction

Matrix factorization was first introduced by Eisenbud [Eis80] to describe the stable homological features of modules over a hypersurface singularity. The ideas and phenomena described in that paper was later formalized in modern language ([Orl04] [Buc86]) as exact equivalences between the triangulated categories

$$[\mathsf{MF}(R,w)] \simeq \underline{\mathsf{MCM}}(S) \simeq \mathsf{D}_{\mathrm{sg}}(S),$$

where

1. R is a regular commutative ring with finite Krull dimension, and $w \in R$ is a non-unit non-zerodivisor.

2. The category $[\mathsf{MF}(R,w)]$ is the homotopy category associated to the DG category of matrix factorizations

3. The category $\underline{\mathsf{MCM}}(S)$ is the stable category associated to the Frobenius category of maximal Cohen-Macaulay modules over the (Gorenstein) ring S

4. The category $\mathsf{D}_{\mathrm{sg}}(S)$ is the singularity category of $X = Spec(S)$, defined by the Verdier quotient $\mathsf{D}_{\mathsf{sg}}(X) = \mathsf{D}^b(\mathsf{coh}\, X)/\mathsf{perf}(X)$

By its construction, The category $[\mathsf{MF}(R,w)]$ has a natural enrichment $\mathsf{MF}(R,w)$, which is a $\mathbb{Z}/2$-graded DG category. (i.e., its morphism are 2-periodic complexes). Dycherhoff[Dyc] then set out to study this DG-category, viewing it as a noncommutative space in the sense of [KKP08]. Indeed, showing that a suitable enlargement of $MF(R,w)$ has a compact generator, Dycherhoff has shown that $\mathsf{MF}(R,w)$ is derived Morita equivalent to a DG-algebra, in the sense of Toen [Toë].

Therefore, the noncommutative space in question is DG-affine, and we can use that to compute the Hochschild cohomology of this DG-category, which can be thought of as Hodge theory on the noncommutative space.

2.2 The classical exact equivalences

In this section, we describe the three triangulated categories mentioned in the introduction, and show that they are exact equivalent.

Definition 2.2.1. *Let R be a regular domain with finite Krull dimension, and $w \in R$ is a nonzero non-unit. The category $\mathsf{MF}(R,w)$ of* matrix factorization *is defined by:*

1. *An object of this category consists of a $\mathbb{Z}/2$-graded R-module $P = P^0 \oplus P^1$ with two R-linear maps $d^0 : P^0 \to P^1$ and $d^1 : P^1 \to P^0$ satisfying $d^1 d^0 = w \cdot id_{P^0}$ and*

$d^0 d^1 = w \cdot id_{P^1}$. We will often call such an object a matrix factorization, and denote it by

$$\left[P^1 \underset{d^0}{\overset{d^1}{\rightleftarrows}} P^0 \right].$$

2. *Between two objects (P, d_P) and (Q, d_Q) there is a $\mathbb{Z}/2$-graded morphism complex $\underline{\mathrm{Hom}}(P, Q)$ defined as the set of R-linear homomorphism from P to Q with obvious grading. This complex has the differentials $d^0 : \underline{\mathrm{Hom}}^0(P, Q) \to \underline{\mathrm{Hom}}^1(P, Q)$ and $d^1 : \underline{\mathrm{Hom}}^1(P, Q) \to \underline{\mathrm{Hom}}^0(P, Q)$ both defined by the formula $df = d_Q \circ f - (-1)^{|f|} f \circ d_P$.*

3. *Composition of morphism complexes is defined by composition of the R-module maps.*

It is then straightforward to check that this is a $\mathbb{Z}/2$-graded DG-category. That is, its morphism complexes actually satisfy $d^2 = 0$, and its composition maps $\underline{\mathrm{Hom}}(Q, R) \otimes \underline{\mathrm{Hom}}(P, Q) \to \underline{\mathrm{Hom}}(P, R)$ are chain maps.

Given any $\mathbb{Z}/2$-graded DG-category $\mathcal{C}$, we can define the category $[\mathcal{C}] = \mathrm{H}^0(\mathcal{C})$ by

$$\mathrm{Ob}([\mathcal{C}]) = \mathrm{Ob}(\mathcal{C})$$
$$\mathrm{Hom}_{[\mathcal{C}]}(X, Y) = \mathrm{H}^0\left(\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)\right).$$

In our case $\mathcal{C} = \mathsf{MF}(R, w)$ and we see that $[\mathcal{C}]$ is the category of chain maps between matrix factorizations modulo homotopy.

Given objects X and Y in $\mathsf{MF}(R, w)$, and a chain map $f : X \to Y$, (i.e. $f \in Z^0(\underline{\mathrm{Hom}}(P, Q))$), we can define the cone of f in exactly the same way as the standard cone construction of chain complexes. The collection of triangles isomorphic to the associated standard triangles will make $[\mathcal{C}]$ into a triangulated category. For details, see [Orl04].

Definition 2.2.2. *Let X be a Noetherian scheme of finite Krull dimension which is either affine or is quasiprojective over a field. Then we define its singularity category as the Verdier quotient*

$$\mathsf{D}_{\mathrm{sg}}(X) = \mathsf{D}^b(\mathsf{coh}\, X)/\mathsf{perf}(X),$$

where $\mathsf{perf}(X)$ denotes the triangulated subcategory of $\mathsf{D}^b(\mathsf{coh}\, X)$ consisting of complexes that are isomorphic in $\mathsf{D}^b(\mathsf{coh}\, X)$ to a bounded complex of locally free sheaves.

The following proposition explains why it is called the singularity category.

Proposition 2.2.3. *X is nonsingular if and only if $\mathsf{D}^b(\mathsf{coh}\, X) = \mathsf{perf}(X)$.*

Proof. By Serre's theorem, $\mathsf{coh}(X)$ has enough locally free sheaves. Thus, for any bounded complex $M^\bullet$ of coherent sheaves, there is a quasi-isomorphism $P^\bullet \to M^\bullet$ from a bounded above complex of locally free sheaves. Then, $\mathrm{Coker}(d_P^{-n})$ is always locally free if and only if the local ring at every stalk has finite global dimension, if and only if X is regular. □

Proposition 2.2.4. *Let S be a Gorenstein ring of Krull dimension n. (i.e., S is a Noetherian commutative ring and every localization $S_{\mathfrak{p}}$ at a prime is a Gorenstein local ring.) Let M be a finitely generated S-module. Then the following statements are equivalent:*

1. *M is a high enough syzygy. i.e., there exists an exact sequence*

$$0 \to M \to P^{-n} \to \cdots \to P^{-1} \to N \to 0$$

   for some other finitely generated S-module N, where P^i are finitely generated projective modules over S.

2. *For any prime $\mathfrak{p} \supset ann(M)$, the localization $M_{\mathfrak{p}}$ satisfies* $\operatorname{depth}(M_{\mathfrak{p}}) = \dim(S_{\mathfrak{p}})$

3. *$ext^i(M, S) = 0$ for all $i > 0$*

4. *There is a quasi-isomorphism $M \to P^\bullet$ from M to a complex of finitely generated projective modules bounded below at 0, i.e.*

$$P^\bullet = 0 \to P^0 \to P^1 \to \cdots$$

Proof. $1 \Rightarrow 2$. One can show that, in fact, any regular sequence of $S_{\mathfrak{p}}$ is actually regular in $M_{\mathfrak{p}}$ [Eis80, Lemma 0.1]. Since $S_{\mathfrak{p}}$ is Cohen-Macaulay, we have our claim.

$2 \Rightarrow 3$. For any prime $\mathfrak{p} \supset \operatorname{ann}(M)$, let $(a_1, \ldots, a_d)$ be a maximal regular sequence of $M_{\mathfrak{p}}$, then apply the long exact sequence of
$ext(-, S_{\mathfrak{p}})$

$$0 \longrightarrow M_{\mathfrak{p}}/(a_1, \ldots, a_{i-1}) \xrightarrow{a_d} M_{\mathfrak{p}}/(a_1, \ldots, a_{i-1}) \longrightarrow M_{\mathfrak{p}}/(a_1, \ldots, a_i) \longrightarrow 0.$$

Use the Nakayama lemma to prove inductively that

$$\begin{aligned} ext^{d+i}(M_{\mathfrak{p}}/(a_1, \ldots, a_d), S_{\mathfrak{p}}) = 0 &\Rightarrow \\ ext^{d+i-1}(M_{\mathfrak{p}}/(a_1, \ldots, a_{d-1}), S_{\mathfrak{p}}) = 0 & \\ &\Rightarrow \cdots \\ &\Rightarrow \\ ext^i(M_{\mathfrak{p}}, S_{\mathfrak{p}}) = 0. & \end{aligned}$$

$3 \Rightarrow 4$. Use the fact that $\mathsf{R}\operatorname{Hom}(-, S)$ is a duality functor on $\mathsf{D}^b(S)$, we see that such a functor restricts to a duality functor on the class of finitely generated modules such that 3 holds, and is simply the functor $(-)^\vee$ there. Thus, we can obtain a free resolution of $M^\vee$, and then apply $(-)^\vee$ again.

$4 \Rightarrow 1$. This is obvious. □

Definition 2.2.5. *We call a module satisfying any of the equivalent conditions a* maximal Cohen-Macaulay module *over S, and they form a full subcategory $\mathsf{MCM}(S)$ of $\mathsf{Mod}(S)$.*

Proposition 2.2.6. *The category $MCM(S)$ can be made into an exact category by endowing it with sequences that are short exact in $\mathsf{Mod}(S)$, Then, an object is projective $\Leftrightarrow$ injective $\Leftrightarrow$ projective as an S-module.*

Proof. By Proposition 2.2.4, a maximal Cohen-Macaulay module has enough projective module in both sides. Hence, if M is either injective or projective in the exact category $\mathsf{MCM}(S)$, then it must be a direct summand of a projective S-module, and hence is itself a projective S-module. The other direction is easy. $\square$

Proposition 2.2.6 tells us that MCM is a *Frobenius category* in the sense of [Kela]. Thus we can form its associated stable category.

Definition 2.2.7. *Write $\underline{\mathsf{MCM}}(S)$ for the stable category associated to the Frobenius category $\mathsf{MCM}(S)$.*

Now, let R be a regular Noetherian domain of finite Krull dimension n, and $w \in R$ is a nonzero non-unit. Let $S = R/w$.

We claim that the triangulated categories $[\mathsf{MF}(R, w)]$, $\underline{\mathsf{MCM}}(S)$ and $\mathsf{D}_{\mathrm{sg}}(\operatorname{Spec} S)$ are exact-equivalent.

Definition 2.2.8. *Given an object X in $\mathsf{MF}(R, w)$, we denote by $\overline{X}^\bullet$ the complex of S-modules*

$$\ldots \to \overline{X^0} \to \overline{X^1} \to \overline{X^0} \to 0$$

where the rightmost nontrivial module is at degree 0.

Proposition 2.2.9. *For any object X in $\mathsf{MF}(R, w)$, the complex $\overline{X}^\bullet$ is exact except possibly at degree 0.*

Proof. This is a straightforward diagram chasing, merely using the fact that multiplication by w is injective on finitely generated projective modules over the regular ring R. $\square$

Given an object X in $\mathsf{MF}(R, w)$, we note that $\mathrm{Im}(d^1) \supset \mathrm{Im}(d^1 d^0) = \mathrm{Im}(w)$, and hence $\mathrm{Coker}(d^1)$ is an S-module.

We know that $\mathrm{Coker}(d^1)$ is an arbitrarily high syzygy, and is therefore a maximal Cohen-Macaulay module over S. One can show that the construction $X \mapsto \mathrm{Coker}(d^1)$ actually induces a functor

$$\mathrm{Coker} : [\mathsf{MF}(R, w)] \to \underline{\mathsf{MCM}}(S).$$

Theorem 2.2.10. *The functor* $\mathrm{Coker} : [\mathsf{MF}(R, w)] \to \underline{\mathsf{MCM}}(S)$ *is an equivalence of categories.*

Proof. The fact that Coker is fully-faithful is a diagram chasing exercise. (It is useful to note that a map of S-modules factor through a projective iff it factors through a free module)

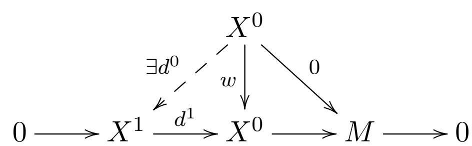
To show that Coker is essentially surjective, suppose we are given a maximal Cohen-Macaulay module M, then apply the Auslander-Buchsbaum formula to the localizations of M at primes $\mathfrak{p} \supset \mathrm{ann}(M)$, considered as an $R_{\mathfrak{p}}$-module, we have:

$$\mathrm{pd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \mathrm{depth}(R_{\mathfrak{p}}) - \mathrm{depth}(M_{\mathfrak{p}}) = m - (m-1) = 1.$$

Hence, there is an exact sequence

$$0 \to X^1 \xrightarrow{d^1} X^0 \to M \to 0$$

where X^1 and X^0 are projective R-module. Then use projectivity of X^0 to show that $w : X^0 \to X^0$ lifts to an R-linear map $d^0 : X^0 \to X^1$, as in the following commutative diagram:



i.e., we have

$$d^1 \circ d^0 = w$$

Hence, we have

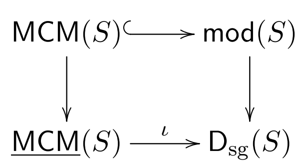
$$d^1 \circ d^0 \circ d^1 = w \circ d^1 = d^1 \circ w$$

Then we can use the injectivity of d^1 to show that $d^0 \circ d^1 = w$ as well. This shows essential surjectivity. $\square$

Now, we shall describe the functor

$$\iota : \underline{\mathsf{MCM}}(S) \to \mathsf{D}_{\mathrm{sg}}(S).$$

By definition, $\mathsf{MCM}(S)$ is a full subcategory of $\mathsf{mod}(S)$. We claim that the inclusion $\mathsf{MCM}(S) \hookrightarrow \mathsf{mod}(S)$ induces ι as follows:



Indeed, since every projective S-module is a perfect complex in $\mathsf{D}_{\mathrm{sg}}(S)$, hence every morphism that factors through a projective S-module is zero in $\mathsf{D}_{\mathrm{sg}}(S)$

Proposition 2.2.11. *The functor* $\iota : \underline{\mathsf{MCM}}(S) \to \mathsf{D}_{\mathrm{sg}}(S)$ *is an equivalence of categories.*

Proof. See [Orl04, Prop 1.21] for a proof that ι is fully faithful.

To show that ι is essentially surjective, suppose we are given a bounded complex $M^\bullet$ of S-modules

$$M^\bullet = [0 \to M^{-e} \to \ldots \to M^0 \to 0]$$

which we assume, without loss of generality, to be bounded above at degree 0.

Then we can find a projective resolution $P^\bullet \xrightarrow{\sim} M^\bullet$, where

$$P^\bullet = [\ldots \to P^{-e} \xrightarrow{d^{-e}} \ldots \xrightarrow{d^{-1}} P^0 \to 0]$$

By the characterization of Proposition 2.2.4(1), we see that $\mathrm{Im}(d^{-e-n-1})$ is maximal Cohen-Macaulay since it is the n-th syzygy of $\mathrm{Im}(d^{-e-1})$. Then the good truncation

$$\tau_{\geq -e-n-1}(P^\bullet) = [0 \to \mathrm{Im}(d^{-e-n-1}) \to P^{-e-n} \to \cdots \to P^0 \to 0]$$

is still quasi-isomorphic to $M^\bullet$. Moreover, in $\mathsf{D}^b(S)$, we have exact triangles

$$\sigma_{\geq -e-n}(P^\bullet) \to \tau_{\geq -e-n-1}(P^\bullet) \to \mathrm{Im}(d^{-e-n-1})[e+n+1] \to \sigma_{\geq -e-n}(P^\bullet)[1]$$

which translate to an equivalence

$$\tau_{\geq -e-n-1}(P^\bullet) \to \mathrm{Im}(d^{-e-n-1})[e+n+1]$$

in $\mathsf{D}_{\mathrm{sg}}(S)$. This shows that $M^\bullet \simeq \mathrm{Im}(d^{-e-n-1})[e+n+1]$ in $\mathsf{D}_{\mathrm{sg}}(S)$.

Use the characterization of Proposition2.2.4(4), we can "syzygy down" the MCM module $\mathrm{Im}(d^{-e-n-1})$, and apply the same argument as above, to show that

$$\mathrm{Im}(d^{-e-n-1})[e+n+1] \simeq N$$

in $\mathsf{D}_{\mathrm{sg}}(S)$ for some MCM module N. This shows essential surjectivity. □

Theorem 2.2.12. *The equivalences* Coker *and* ι *are exact equivalences.*

Proof. The exactness of ι is straight-forward checking.

For the exactness of Coker, it will be easier to show that $\iota \circ$ *coker* is exact. Indeed, by Proposition2.2.9, this composition is just the functor

$$X \mapsto \overline{X}.$$

To show that $\iota \circ \mathrm{Coker}$ is exact, it suffices to note that $\overline{\Sigma X}$ and $\overline{X}[1]$ only differ by a projective module; and that $\overline{\mathrm{cone}(f)}$ and $\mathrm{cone}(\overline{f})$ also only differ by a projective module.

More precisely, we have maps in $\mathsf{D}^b(S)$:

$$
\begin{array}{ccccccc}
 & & \overline{X} & = & \overline{X} & & \\
 & & \downarrow f & & \downarrow f & & \\
 & & \overline{Y} & = & \overline{Y} & & \\
 & & \downarrow & & \downarrow & & \\
X^1 & \longrightarrow & \overline{\mathrm{cone}(f)} & \longrightarrow & \mathrm{cone}(\overline{f}) & \longrightarrow & X^1[1] \\
\| & & \downarrow & & \downarrow & & \| \\
X^1 & \longrightarrow & \overline{\Sigma X} & \longrightarrow & \overline{X}[1] & \longrightarrow & X^1[1]
\end{array}
$$

where the first long column is the image under Coker of an exact triangle in $[\mathsf{MF}(R, w)]$; the second long column is an exact triangle in $\mathsf{D}^b(S)$; the horizontal maps are also exact triangles in $\mathsf{D}^b(S)$ This diagram, when read in $\mathsf{D}_{\mathrm{sg}}(S)$, shows that $\iota \circ \mathrm{Coker}$ is exact. $\square$

2.3 The stabilization of the residue field

In the next several sections, we will survey the results from [Dyc], showing that a suitable enlargement of the triangulated category $[\mathsf{MF}(R, w)]$ has a compact generator in the case of an isolated singularity.

Let R be a regular domain of finite Krull dimension, $w \in R$ a nonzero nonunit, and $S = R/w$. Let $(x_1, ..., x_m)$ is a regular sequence in R. Suppose w is in the ideal $(x_1, ..., x_n)$, then $L := R/(x_1, ..., x_m)$ is an S-module. We will obtain a description of the object in $[\mathsf{MF}(R, w)]$ that correspond to L in $\mathsf{D}_{\mathrm{sg}}(S)$.

Let V be the free R-module $V = Re_1 \oplus \ldots \oplus Re_m$, and suppose V has homological degree -1. It is easy to see that on the graded commutative algebra $\bigwedge^\bullet V$, there is a unique DG-algebra structure with differential $\partial(e_i) = x_i$ of degree $+1$. (Here, $e_i \in V = \bigwedge^1 V$ and $x_i \in R = \bigwedge^0 V$) We denote this differential by s_0, which is a degree $+1$ differential on the graded complex $\bigwedge^\bullet V$:

$$0 \to \bigwedge^m V \xrightarrow{s_0} \ldots \xrightarrow{s_0} \bigwedge^1 V \xrightarrow{s_0} \bigwedge^0 V \to 0$$

Moreover, since w is in the ideal $(x_1, ..., x_n)$, we may suppose $w = \sum w_i x_i$. Then we can define a degree -1 map $s_1 : \bigwedge^i V \to \bigwedge^{i+1} V$ by

$$s_1(x) = (\Sigma w_i e_i) \wedge (x),$$

So we have a sequence

$$0 \leftarrow \bigwedge^m V \xleftarrow{s_1} \ldots \xleftarrow{s_1} \bigwedge^1 V \xleftarrow{s_0} \bigwedge^0 V \leftarrow .$$

Note that we have:

$$\begin{aligned} s_0 s_0 &= 0 \\ s_1 s_1 &= 1 \\ s_0 s_1(x) &= s_0((\Sigma w_i e_i) \wedge (x)) \\ &= s_0(\Sigma w_i e_i) \wedge (x) - (\Sigma w_i e_i) \wedge (s_0(x)) \qquad \text{by Leibniz rule of } s_0 \\ &= w \cdot x - s_1 s_0(x) \end{aligned}$$

So $s_0 s_1 + s_1 s_0 = w \cdot \mathrm{id}$ on $\bigwedge^\bullet V$.

Thus, letting $s = s_0 + s_1$, we have $s \circ s = w \cdot \mathrm{id}$, and hence

$$L^{\mathrm{stab}} = \left[\bigwedge^{\mathrm{odd}} V \underset{s}{\overset{s}{\rightleftarrows}} \bigwedge^{\mathrm{even}} V \right].$$

is a matrix factorization in $\mathsf{MF}(R, w)$.

Proposition 2.3.1. *For any complete intersection module L as described above, $L^{\mathrm{stab}} \in \mathsf{MF}(R, w)$ is the object that corresponds to $L \in \mathsf{D}_{\mathrm{sg}}(S)$ under the exact equivalence $\mathsf{MF}(R, w) \simeq \mathsf{D}_{\mathrm{sg}}(S)$.*

Proof. We refer the reader to [Dyc] for a proof. □

In particular, if R is a regular local ring of dimension n, then its maximal ideal $\mathfrak{m}$ is generated by a regular sequence $\mathfrak{m} = (x_1, ..., x_n)$. Suppose $w \in \mathfrak{m}$ is nonzero, and let $S = R/w$

Hence, its residue field $k = R/\mathfrak{m}$, as an S-module, has stabilization the matrix factorization

$$k^{\mathrm{stab}} = \left[\bigwedge^{\mathrm{odd}} V \rightleftarrows \bigwedge^{\mathrm{even}} V \right].$$

This description will be useful in showing that the k^{stab} is a compact generator of $[\mathsf{MF}^\infty(R, w)]$, to be defined in the next subsection.

2.4 An enlargement of the matrix factorization category

Definition 2.4.1. *Let R be a regular domain with finite Krull dimension, and $w \in R$ is a nonzero non-unit. We define $\mathsf{MF}^\infty(R, w)$ as the category whose objects are matrix factorization*

$$[P^1 \rightleftarrows P^0]$$

as in Definition 2.2.1, but this time we do not *require P^0 and P^1 to be finitely generated. Morphism complexes are defined exactly as in Definition 2.2.1.*

Proposition 2.4.2. *Let R, w, S be as above.*

1. *$\mathsf{MF}^\infty(R, w)$ is a $\mathbb{Z}/2$-graded DG category, and $\mathsf{MF}(R, w) \subset \mathsf{MF}^\infty(R, w)$ is a full DG-subcategory.*

2. $\mathsf{MF}^{\infty}(R,w)$ *has arbitrary coproducts.*

3. $[\mathsf{MF}^{\infty}(R,w)]$ *is a triangulated category with arbitrary coproducts.*

4. $[\mathsf{MF}^{\infty}(R,w)] \xrightarrow{\text{Coker}} \underline{\mathsf{Mod}}(S)$ *is fully faithful.*

5. *Every object in* $[\mathsf{MF}(R,w)]$ *is compact in* $[\mathsf{MF}^{\infty}(R,w)]$.

See Definition 2.4.3 for the precise definition of a compact object.

Proof. 1-4. These are either obvious, or have proofs exactly as for $[\mathsf{MF}(R,w)]$.

5. Simply note that since the R-modules P^0 and P^1 in an object of $[MF(R,w)]$ are finitely generated projective modules, the Hom complexes $\underline{\mathrm{Hom}}^{\bullet}_{\mathsf{MF}(R,w)}(P,-)$ already commute with arbitrary coproducts. The same therefore holds after we take $\mathrm{H}^0(-)$ of the Hom complexes. □

Definition 2.4.3. *Let $\mathcal{T}$ be a triangulated category. We call an object $X \in \mathcal{T}$* compact *if it commutes with arbitrary coproducts. That is, whenever the coproduct of a family$\{Y_\alpha\}$ of objects $\mathcal{T}$ exists, the obvious map*

$$\coprod \mathrm{Hom}_{\mathcal{T}}(X, Y_\alpha) \to \mathrm{Hom}_{\mathcal{T}}(X, \coprod Y_\alpha)$$

is an isomorphism, where the coproduct on the left is taken in the category of abelian groups.

Definition 2.4.4. *Let $\mathcal{T}$ be a triangulated category. We call an object $X \in \mathcal{T}$ a* generator *if the smallest strictly full triangulated subcategory of $\mathcal{T}$ containing X is $\mathcal{T}$ itself.*

Proposition 2.4.5. *Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts, and $X \in \mathcal{T}$ be a compact object. Then X is a generator if and only if for all $Y \in \mathcal{T}$,*

$$\mathrm{Hom}_{\mathcal{T}}(X[i], Y) = 0 \text{ for all } i \in \mathbb{Z} \qquad \Rightarrow \qquad Y = 0 \text{ in } \mathcal{T}.$$

Sketch of proof. This is an application of the technique of Bousfield localization. Let $\mathcal{S}$ be the smallest strictly full triangulated subcategory of $\mathcal{T}$ containing X, and denote by $\mathcal{S}^{\perp}$ its right orthogonal, defined as the full subcategory of $\mathcal{T}$ consisting of objects Y such that $Hom_{\mathcal{T}}(Z,Y) = 0$ for all $Y \in \mathcal{S}$. In this context, we are trying to prove that $\mathcal{S}^{\perp} = 0 \Rightarrow \mathcal{T} = \mathcal{S}$. To prove this, we use the assumption that $\mathcal{T}$ has arbitrary coproducts to conclude (using Brown representability) that $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ has a right adjoint. This will imply that $\mathcal{S}^{\perp} \to \mathcal{T}/\mathcal{S}$ is an exact equivalence. Thus $\mathcal{S}^{\perp} = 0 \Rightarrow \mathcal{T}/\mathcal{S} = 0 \Rightarrow \mathcal{T} = \mathcal{S}$. For details of the argument, we refer the reader to [Nee]. □

We state, without proof, the result in Dyckerhoff's paper [Dyc] which is regarded by him as the main result of that paper.

Definition 2.4.6. *Let R be essentially of finite type over k, equidimensional of dimension n, and the sheaf of Kahler differentials $\Omega_{R/k}$ is locally free of rank n. We say that (R, w) has isolated singularity at a rational point if $Spec(S/(\partial_1 w, ..., \partial_n w))$ is supported at a closed point $\mathfrak{m} \in Spec(R)$ with residue field $R/\mathfrak{m} = k$.*

In such cases, R is regular at $\mathfrak{m}$, and the same argument as in the previous section shows that the residue field $k = R/\mathfrak{m}$ has a free resolution which stabilizes to the reduction mod w of a matrix factorization, and we denote $k^{stab} \in [MF(R, w)]$ as that matrix factorization.

Of course, if k is a perfect field, then R is regular, and $k^{stab} \in [MF(R, w)]$ is the object that correspond to $k = R/\mathfrak{m} \in D_{Sg}(S)$ under the equivalence $[MF(R, w)] \cong D_{Sg}(S)$

Theorem 2.4.7. *Suppose that (R, w) has isolated singularity at a rational point, then k^{stab} is a compact generator of $[MF^{\infty}(R, w)]$*

2.5 Homotopy theory of DG categories

We present some part of the homotopy theory of DG-categories as developed in [Toë]. We will later discuss the way Dyckerhoff applied it to the study of the Hochschild cohomology of the matrix factorization DG-category. We begin by collecting some basic definitons about DG-categories and their homotopy theory. The exposition is brief, and the reader is advised to consult [Toë] and [Kelb] as needed.

Definition 2.5.1. *Let $\mathcal{C}, \mathcal{D}$ be ($\mathbb{Z}/2$-graded) DG-categories. A DG-functor $F : \mathcal{C} \to \mathcal{D}$ is an* quasi-equivalence *if*

1. *For any two objects $X, Y \in \mathcal{C}$, the cochain map*

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{D}}(FX, FY)$$

is a quasi-isomorphism.

2. *The induced map $[F] : [\mathcal{C}] \to [\mathcal{D}]$ is an equivalence of categories.*

We wish to identify DG categories which are quasi-equivalence. In other words, we will eventually be interested in the category $\mathsf{Ho}(\mathsf{DGCat})$ of the category of small DG categories localized at the class of quasi-equivalence. In order to describe such a category, we use the theory of model category. The reader is referred to [DS95] for an exposition of model categories.

Theorem 2.5.2. *There is a model category structure on the category of all (small) ($\mathbb{Z}/2$-graded) DG-categories such that:*

1. *The weak equivalences are quasi-equivalences.*

2. *The fibrations are DG-functors $f : \mathcal{T} \to \mathcal{S}$ that satisfy the following two conditions:*

(a) *Surjectivity on Hom complexes: For any objects $x, y \in \mathcal{T}$, the map of complexes*

$$\underline{\mathrm{Hom}}_{\mathcal{T}}(x,y) \to \underline{\mathrm{Hom}}_{\mathcal{S}}(f(x), f(y))$$

is surjective.

(b) *The induced map $[f] : [\mathcal{T}] \to [\mathcal{S}]$ lifts isomorphisms. That is, if $y' \to f(x)$ is an isomorphism in $[\mathcal{S}]$, then there exists an isomorphism $x' \to x$ that maps to that given isomorphism.*

Proof. See the paper [Tab05]. □

In order to clarify the definitions that will follow, we first make the following definitions. Give $k[u^{\pm}]$ the structure of a DG-algebra with $\deg(u) = 2$. THe category of all modules over $k[u^{\pm}]$ is denoted by $\mathsf{C}(k[u^{\pm}])$. Its objects are simply 2-periodic k-complexes, and morphisms are 2-periodic cochain maps. Let $\mathsf{C}(k)$ be the category of all k-complexes. There are obvious DG-enrichments of these categories, which we denote by $\mathsf{C}_{\mathrm{dg}}(k[u^{\pm}])$ and $\mathsf{C}_{\mathrm{dg}}(k)$, respectively. Thus $\mathsf{C}_{\mathrm{dg}}(k[u^{\pm}])$ is a $\mathbb{Z}/2$-graded DG-category such that $Z^0(\mathsf{C}_{\mathrm{dg}}(k[u^{\pm}])) = \mathsf{C}(k[u^{\pm}])$, and $\mathsf{C}_{\mathrm{dg}}(k)$ is a DG-category with $Z^0(\mathsf{C}_{\mathrm{dg}}(k)) = \mathsf{C}(k)$.

Write dgcat_k for the category of al (small) DG-categories over k. Similarly, write $\mathsf{dgcat}_{k[u^{\pm}]}$ for the category of all (small) $\mathbb{Z}/2$-graded DG-categories over k. The reason for the notation $\mathsf{dgcat}_{k[u^{\pm}]}$ is that we can think of a $\mathbb{Z}/2$-graded DG-category as a category enriched over the monoidal category $\mathsf{C}(k[u^{\pm}])$. We will write $\mathsf{Ho}(\mathsf{dgcat}_k)$ and $\mathsf{Ho}(\mathsf{dgcat}_{k[u^{\pm}]})$ for the respective homotopy categories associated to these two model categories.

Definition 2.5.3. *Given a DG categories $\mathcal{T}$, a (left) $\mathcal{T}$-module is a DG functor $\mathcal{T} \to \mathsf{C}_{\mathrm{dg}}(k)$. Similarly, given a $\mathbb{Z}/2$-graded DG-category $\mathcal{T}$, a (left) $\mathcal{T}$-module is a DG functor $\mathcal{T} \to \mathsf{C}_{\mathrm{dg}}(k[u^{\pm}])$*

For the rest of this subsection, statements and definitions will be given for DG-categories only. There is obvious counterpart in the $\mathbb{Z}/2$-graded case which we shall not repeat.

Example 2.5.4. If $x \in \mathrm{Ob}(\mathcal{T})$, then we have a DG-functor

$$\begin{aligned} h_x : \mathcal{T}^{\circ} &\to \mathsf{C}_{\mathrm{dg}}(k) \\ y &\mapsto \underline{\mathrm{Hom}}_{\mathcal{T}}(y,x). \end{aligned}$$

Note that this is really a DG functor because the map

$$\underline{\mathrm{Hom}}_{\mathcal{T}}(y,z) \to \underline{\mathrm{Hom}}_{\mathsf{C}_{\mathrm{dg}}(k)}(\underline{\mathrm{Hom}}_{\mathcal{T}}(z,x), \underline{\mathrm{Hom}}_{\mathcal{T}}(z,x))$$

is a map of complexes. Thus, h_x is a left $\mathcal{T}$-module.

Given two DG functors $F, G : \mathcal{T} \to \mathcal{S}$ between two DG-categories. It is straightforward to define the notion of a natural transformation of given homogenous degree from F to G; it is likewise straightforward to define the differential of such natural transformation. Then the class of such natural transformations naturally form a complex. (cf [Kelb] for details.)

By the previous paragraph, the class of all DG functors $\mathcal{T} \to \mathcal{S}$ can be given the structure of a DG-category, which we denote by $\underline{\mathrm{Hom}}(\mathcal{T}, \mathcal{S})$. In particular, the class of all $\mathcal{T}$-modules is naturally a DG-category, which we denote by $\mathsf{mod}(\mathcal{T})$, i.e.

$$\mathsf{mod}(\mathcal{T}) = \underline{\mathrm{Hom}}(\mathcal{T}, \mathsf{C}_{\mathrm{dg}}(k)).$$

A map between $\mathcal{T}$-modules is defined to be a morphism in $Z^0(\mathsf{mod}(\mathcal{T}))$. By definition, this means a natural transformation between two DG functors from $\mathcal{T}$ to $\mathsf{C}_{\mathrm{dg}}(k)$ which is required to be degree zero and commute with differentials.

Theorem 2.5.5. *The DG-category* $\mathsf{mod}(\mathcal{T})$ *has the structure of a model category, where the fibrations and weak equivalences are defined levelwise using the model structure on* $\mathsf{C}(k[u^{\pm}])$.

We denote by $\mathsf{Int}(\mathsf{mod}(\mathcal{T}))$ the full subcategory of fibrant cofibrant objects in this model category. Write

$$\widehat{\mathcal{T}} = \mathsf{Int}(\mathsf{mod}(\mathcal{T}^{\circ})).$$

An object in $\widehat{\mathcal{T}}$ is called *perfect* or *compact* if it is perfect or compact as an object in the triangulated category $[\widehat{\mathcal{T}}] \simeq \mathsf{D}(\mathcal{T}^{\circ})$, and we denote by $\widehat{\mathcal{T}}_{\mathrm{pf}}$ the full subcategory of perfect objects of $\widehat{\mathcal{T}}$.

It is easy to check that the image of the Yoneda functor $h : \mathcal{T} \to \mathsf{mod}(\mathcal{T})$ actually lies in $\widehat{\mathcal{T}}_{\mathrm{pf}}$, i.e. h induces a functor $\mathcal{T} \to \widehat{\mathcal{T}}_{\mathrm{pf}}$. Moreover, there is an enriched Yoneda lemma which asserts that this functor is always fully faithful.

Definition 2.5.6. *A DG category $\mathcal{T}$ is called* triangulated *if the DG functor*

$$h : \mathcal{T} \to \widehat{\mathcal{T}}_{\mathrm{pf}}$$

is a quasi-equivalence. The DG category $\widehat{\mathcal{T}}_{\mathrm{pf}}$ is called the triangulated hull *of $\mathcal{T}$.*

For any $\mathsf{C}(k)$-model category $\mathcal{M}$ (i.e., a model category with $C(k)$-enrichment which is compatible with its model structure), we have

$$[\mathsf{Int}(\mathcal{M})] \simeq \mathsf{Ho}(\mathcal{M}).$$

In particular, for $\mathcal{M} = \mathsf{mod}(\mathcal{T}^{\circ})$, we have

$$[\widehat{\mathcal{T}}] \simeq \mathsf{D}(\mathcal{T}^{\circ})$$
$$[\widehat{\mathcal{T}}]_{\mathrm{pf}} \simeq \mathsf{D}_{\mathrm{perf}}(\mathcal{T}^{\circ}).$$

For more details, refer to [Toë11].

If $\mathcal{T}$ is a triangulated DG category, it is clear that the followings hold:

1. For any object $x \in \mathcal{T}$, the $\mathcal{T}$-module $h_x[1]$ is represented by an object, say $x[1]$.

2. For any morphism $x \xrightarrow{f} y$ in $Z^0(\mathcal{T})$, the $\mathcal{T}$-module $\mathrm{cone}(h_x \to h_y)$ is quasi-representable, (i.e., isomorphism in $D(\mathcal{T}^{\circ})$ to an object h_z) Denote the representing object by $z = \mathrm{cone}(f)$

3. The shifts and cone above makes $[\mathcal{T}]$ into an idempotent complete triangulated category.

It seems to be a folklore that the converse is also true, but the author knows of no statement explicitly written down.

Definition 2.5.7. *A DG functor $F : \mathcal{T} \to \mathcal{S}$ induces (by $\otimes_{\mathcal{T}} \mathcal{S}$) an extension functor of modules. These fit into a commutative diagram*

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{F} & \mathcal{S} \\ \downarrow & & \downarrow \\ \widehat{\mathcal{T}}_{\mathrm{pf}} & \xrightarrow{\widehat{F}_{\mathrm{pf}}} & \widehat{\mathcal{S}}_{pe} \end{array}$$

A DG functor F is called a derived Morita equivalence *if the induced DG-functor*

$$\widehat{F}_{\mathrm{pf}} : \widehat{\mathcal{T}}_{\mathrm{pf}} \to \widehat{\mathcal{S}}_{\mathrm{pf}}$$

is a quasi-equivalence.

Note that since $\widehat{\mathcal{T}}_{pe}$ and $\widehat{\mathcal{S}}_{pe}$ have shift functors, we have:

$$\begin{aligned} & \mathrm{H}^0(\underline{\mathrm{Hom}}_{\widehat{\mathcal{T}}_{\mathrm{pf}}}(x, y)) \xrightarrow{\sim} \mathrm{H}^0(\underline{\mathrm{Hom}}_{\widehat{\mathcal{S}}_{\mathrm{pf}}}(\widehat{F}_{\mathrm{pf}}(x), \widehat{F}_{\mathrm{pf}}(y))) \qquad \forall x, y \in \widehat{\mathcal{T}}_{\mathrm{pf}} \\ \iff \quad & \mathrm{H}^\bullet(\underline{\mathrm{Hom}}_{\widehat{\mathcal{T}}_{\mathrm{pf}}}(x, y)) \xrightarrow{\sim} \mathrm{H}^\bullet(\underline{\mathrm{Hom}}_{\widehat{\mathcal{S}}_{\mathrm{pf}}}(\widehat{F}_{\mathrm{pf}}(x), \widehat{F}_{\mathrm{pf}}(y))) \qquad \forall x, y \in \widehat{\mathcal{T}}_{\mathrm{pf}} \end{aligned}$$

Therefore, F is a derived Morita equivalence if and only if it induces an isomorphism

$$\begin{array}{ccc} \mathrm{H}^0(\widehat{\mathcal{T}}_{\mathrm{pf}}) & \xrightarrow{\sim} & \mathrm{H}^0(\widehat{\mathcal{S}}_{\mathrm{pf}}) \\ \| & & \| \\ \mathsf{D}_{\mathrm{perf}}(\mathsf{mod}(\mathcal{T}^\circ)) & \longrightarrow & \mathsf{D}_{\mathrm{perf}}(\mathsf{mod}(\mathcal{S}^\circ)) \end{array}$$

hence the term "derived Morita equivalence."

2.6 Derived Morita Equivalence

In the last subsection, we introduced several notions related to DG categories in general. In this subsection, we will state a general result in the homotopy theory of DG categories, which is geared towards the application to the matrix factorization category towards the end of this subsection.

Theorem 2.6.1. *Let $\mathcal{T}$ be a triangulated 2-periodic DG category which admits coproducts. Let $\mathcal{S}$ be a full DG subcategory of $\mathcal{T}$ whose objects are compact in $[\mathcal{T}]$. Assume that the smallest triangulated subcategory of $[\mathcal{T}]$ which contains the objects of $[\mathcal{S}]$ and is closed under coproducts is $[\mathcal{T}]$ itself. Then the map*

$$f : \mathcal{T} \to \mathcal{S}^{op} - mod \ , \ x \mapsto \mathcal{T}(-, x)|_{\mathcal{S}}$$

actually restricts to

$$f : \mathcal{T} \to Int(\mathcal{S}^{op} - mod) \ , \ x \mapsto \mathcal{T}(-, x)|_{\mathcal{S}}$$

and this restricted functor is a quasi-equivalence.

Furthermore, f induces a quasi-equivalence

$$\mathcal{T}_{pe} \cong Int(\mathcal{S}^{op} - mod)_{pe} = \widehat{\mathcal{S}}_{pe}$$

Proof. We proceed in several steps:

1. The first claim that each $h_x|_{\mathcal{S}}$ is fibrant cofibrant in $\mathcal{S}^{op}mod$ is straight-forward.

2. Check that f commute with coproducts.

3. Show that f induces isomorphism on $H^0(-)$ of the complexes. i.e., Show by the following steps that:

$$Hom_{[\mathcal{T}]}(x, y) \xrightarrow{\sim} Hom_{D(\mathcal{S}^{op})}(f(x), f(y) \qquad (*)$$

   (a) True for all $x, y \in \mathcal{S}$, by the Yoneda lemma.

   (b) Fix $x \in \mathcal{S}$, let $\mathcal{A}$ be the full subcategory of those $y \in Ob([\mathcal{T}])$ such that (*) holds. Then by (2), (3a), and 5-lemma, we have $\mathcal{A} = [\mathcal{T}]$.

   (c) Fix $y \in \mathcal{T}$, and repeat the argument analogous to (b)

4. Since $\mathcal{T}$ is triangulated, it is quasi-equivalence to $\widehat{\mathcal{T}}_{pe}$ which has a shift functor. Hence, the fact that f induces isomorphism on $H^0(-)$ of Hom complexes implies that f induces isomorphism on $H^i(-)$ of Hom complexes. (i,e., f is a quasi-fully-faithful)

5. To show that $[f]$ is essentially surjective, we simply note that its essential image contain every representable module $h_x, x \in Ob(\mathcal{T})$. Then, an application

6. The last statement is obvious.

□

Note that $[MF(R, w)]$ has arbitrary coproducts, and hence is idempotent complete (cf [Kra07]). Therefore, by the folklore mentioned above, we see that $MF^{infty}(R, w)$, defined in Section 3.2, is a triangulated DG category,

Hence, by Theorem 2.4.7, we may apply Theorem 2.6.1 to the cases:

1. $\mathcal{T} = MF^{\infty}(R, w)$ and $\mathcal{S} = MF(R, w)$

2. $\mathcal{T} = MF^{\infty}(R, w)$ and $\mathcal{S} = \{k^{stab}\}$

and we have:

Theorem 2.6.2. *Let (R, w) has isolated singulated at a rational point. Denote A as the full subcateogry $\{k^{stab}\} \subset MF(R, w)$. (Alternatively, A can be viewed as the $\mathbb{Z}/2$-graded endomorphism DG algebra $\underline{Hom}_{MF(R,w)}(k^{stab}, k^{stab})$.) Then there exists quasi-equivalences:*

$$\begin{array}{lll} 1. & MF^{\infty}(R, w) & \xrightarrow{\sim} \widehat{MF(R, w)} \\ 2. & MF^{\infty}(R, w) & \xrightarrow{\sim} \widehat{A} \\ 3. & \widehat{MF(R, w)}_{pe} & \xrightarrow{\sim} \widehat{A}_{pe} \end{array}$$

Thus, in particular, (3) says that $MF(R, w)$ is derived Morita equivalent to the $\mathbb{Z}/2$-graded DG algebra A.

2.7 Hochschild cohomology

Warning: The author is not completely familiar with the material in this subsection. Therefore, the material presented might be inaccurate. But the author has tried to make sure that what is written here is at least "morally right".

Recall that, for two $\mathbb{Z}/2$-graded DG categories $\mathcal{T}$,$\mathcal{S}$, there is a $\mathbb{Z}/2$-graded DG categories $\underline{Hom}(\mathcal{T}, \mathcal{S})$.

This gives a functor:

$$\underline{Hom} : dgcat_{k[u,u^{-1}]}^{op} \times dgcat_{k[u,u^{-1}]} \to dgcat_{k[u,u^{-1}]}$$

This functor can be derived to:

$$R\underline{Hom} : Ho(dgcat_{k[u,u^{-1}]})^{op} \times Ho(dgcat_{k[u,u^{-1}]}) \to Ho(dgcat_{k[u,u^{-1}]})$$

Definition 2.7.1. *Given $\mathbb{Z}/2$-graded DG categories $\mathcal{T}$, the image under*

$$\underline{Hom}(\mathcal{T}, \mathcal{T}) \to R\underline{Hom}(\mathcal{T}, \mathcal{T})$$

of the identity natural transformation $id_{\mathcal{T}}$ is an object of $R\underline{Hom}(\mathcal{T}, \mathcal{T})$, which we still denote by $id_{\mathcal{T}}$. Then the Hochschild complex of $\mathcal{T}$ is defined as the complex

$$\underline{Hom}_{(R\underline{Hom}(\mathcal{T},\mathcal{T}))}(id_{\mathcal{T}}, id_{\mathcal{T}})$$

Hochschild cohomology of $\mathcal{T}$ is defined to be the cohomology of the Hochschild complex:

$$HH^*(\mathcal{T}) := H^*(\underline{Hom}_{(R\underline{Hom}(\mathcal{T},\mathcal{T}))}(id_{\mathcal{T}}, id_{\mathcal{T}}))$$

which is a $\mathbb{Z}/2$-graded algebra.

It turns out that Hochschild cohomology is invariant under derived Morita equivalence (cf. [Toë11]). Thus, to compute the Hochschild cohomology of $MF(R, w)$, it suffices to compute that of $MF^{\infty}(R, w)$. To do that, we need to do two things:

1. Find a simple description of $R\underline{Hom}(MF^{\infty}(R,w), MF^{\infty}(R,w))$.

2. Determine which object in that description correspond to the identity functor.

Let (R,w) and (R',w') be as in 2.2.1. Given an object of $T \in MF^{\infty}(R \otimes R', -w \otimes 1 + 1 \otimes w')$, and an object $X \in MF^{\infty}(R,w)$, it is straightforward to check that the $X \otimes_R T$ is an object in $MF^{\infty}(R',w')$. Indeed, keeping the Koszul sign rule in mind, we can calculate:

$$\begin{aligned}
d^2(x \otimes t) &= d(dx \otimes t + (-1)^{|x|} x \otimes dt) \\
&= d^2 x \otimes t + (-1)^{|x|+1} dx \otimes dt + (-1)^{|x|} dx \otimes dt + (-1)^{|x|+|x|} x \otimes d^2 t) \\
&= wx \otimes t + x \otimes (-wt + tw') \\
&= w' x \otimes t
\end{aligned}$$

Thus, tensoring with a given object $T \in MF^{\infty}(R \otimes R', -w \otimes 1 + 1 \otimes w')$ gives a natural transformation from $MF^{\infty}(R,w)$ to $MF^{\infty}(R',w')$. This determines a DG functor

$$MF^{\infty}(R \otimes R', -w \otimes 1 + 1 \otimes w') \to R\underline{Hom}(MF^{\infty}(R,w), MF^{\infty}(R',w')) \qquad (7.1)$$

In order for this functor to be a quasi-equivalence, we have to impose conditions on the pairs (R,w) and (R',w').

Definition 2.7.2. *Let R be essentially of finite type over k, equidimensional of dimension n, and the sheaf of Kahler differentials $\Omega_{R/k}$ is locally free of rank n. We say that (R,w) has isolated critical locus at a rational point if $Spec(R/(\partial_1 w, ..., \partial_n w))$ is supported at a closed point $\mathfrak{m} \in Spec(R)$ with residue field $R/\mathfrak{m} = k$.*

Being an isolated critical locus is strictly more stringent than being an isolated singularity (as defined in Defintion 2.4.6) because we require $Spec(R/(\partial_1 w, ..., \partial_n w))$ rather than $Spec(S/(\partial_1 w, ..., \partial_n w))$ to be supported at a rational point.

Proposition 2.7.3. *If (R,w) and (R',w') has isolated critical locus at a rational point, then the functor*

$$MF^{\infty}(R \otimes R', -w \otimes 1 + 1 \otimes w') \to R\underline{Hom}(MF^{\infty}(R,w), MF^{\infty}(R',w'))$$

defined in 7.1 is a quasi-equivalence.

Remark 2.7.4. We shall skip the proof of this theorem, but we merely note that the condition of isolated critical locus at a rational point is used because we have the following fact:

If (R,w) and (R',w') have isolated critical locus,
then so does $(R \otimes R', -w \otimes 1 + 1 \otimes w')$

Thus, we can use apply the results of Section 3 to study $MF^{\infty}(R \otimes R', -w \otimes 1 + 1 \otimes w')$

Assume from now on that (R, w) has an isolated critical locus at a rational point. Denote by $\tilde{w} = -w \otimes 1 + 1 \otimes w \in R \otimes R$.

Let $I \subset R \otimes R$ be the ideal $I = (x \otimes 1 - 1 \otimes x)$, hence the module $\Delta := R \otimes R/I$ is the diagonal module. (i.e., I is the ideal of the diagonal of $Spec(R) \times Spec(R)$.)

By an analogy to Fourier-Mukai theory, the following proposition, whose proof we refer the reader to [Dyc], should not be a surprise.

Proposition 2.7.5. *The stabilization $\Delta^{stab} \in MF(R \otimes R, \tilde{w})$ of the diagonal module corresponds to the identity functor under the quasi-equivalence 2.7.3.*

Before we proceed to the explicit calculation of the Hochschild cohomology, we quote the following result which allows us to pass to the completion of R.

Proposition 2.7.6. *If (R, w) has an isolated singularity at a rational point, then $\otimes_R \widehat{R}$ induces a quasi-equivalence*

$$MF^{\infty}(R, w) \xrightarrow{\sim} MF^{\infty}(\widehat{R}, w)$$

Applying this proposition to $(R \otimes R, \tilde{w})$, we have:

$$\begin{aligned} &\underline{Hom}_{(R\underline{Hom}(\mathcal{T},\mathcal{T}))}(id_{\mathcal{T}}, id_{\mathcal{T}}) \\ \cong &\underline{Hom}_{MF(R\otimes R,\tilde{w})}(\Delta^{stab}, \Delta^{stab}) \\ \cong &\underline{Hom}_{MF(\widehat{R\otimes R},\tilde{w})}(\widehat{\Delta^{stab}}, \widehat{\Delta^{stab}}) \end{aligned}$$

where $\widehat{\Delta^{stab}} = \Delta^{stab} \otimes_R \widehat{R}$ is also the stabilization of the $\widehat{R \otimes R}$-module $\Delta_{\widehat{R}}$

By the Cohen structure theorem, the regular local complete ring $\widehat{R \otimes R}$ must be isomorphic to $k[[x_1, ..., x_n, y_1, ..., y_n]]$. Moreover, the diagonal module is given by $\Delta_{\widehat{R}} = k[[x_1, ..., x_n, y_1, ..., y_n]]/I$ where I is generated by the regular sequence $(\Delta_1, ..., \Delta_n)$, $\Delta_i = x_i - y_i$.

We denote $\tilde{w} = -w \otimes 1 + 1 \otimes w$ as $\tilde{w}(x, y) = w(y) - w(x)$. Since $\tilde{w} \in I$, we have $\tilde{w}(x, y) = \Sigma \tilde{w}_i \Delta_i$ for some $\tilde{w}_i = \tilde{w}_i(x, y)$

Hence, the matrix factorization $\Delta^{stab}_{\widehat{R}}$ has the form

$$\Delta^{stab}_{\widehat{R}} := \left[\bigwedge^{odd} V \underset{s}{\overset{s}{\rightleftarrows}} \bigwedge^{even} V \right]$$

where:

$$\begin{aligned} &V = \widehat{R \otimes R} e_1 \oplus ... \oplus \widehat{R \otimes R} e_n \\ &s_0 = \text{DG algebra differential on } \bigwedge\nolimits^{*} V \text{ such that } s_0(e_i) = \Delta_i \\ &s_1 = (\Sigma \tilde{w}_i e_i) \wedge (-), \text{ where } \tilde{w}_i \text{ are defined by } \tilde{w} = \Sigma \tilde{w}_i \Delta_i \end{aligned}$$

To compute the cohomology of endomorphism complex of this matrix factorization, we use the following Lemma:

Lemma 2.7.7. *Let X and Y be objects in $MF^{\infty}(R,w)$ and assume tat X has finite rank. Let A be the endomorphism DG algebra of X and that Y is the stabilization of the S-module L. Then there exists a natural isomorphism*

$$\underline{Hom}_{MF(R,w)}(X,Y) \cong \underline{Hom}_{R}^{\mathbb{Z}/2}(X,L)$$

in the category $D(A^{op})$

Thus, we have

$$\begin{aligned} &\underline{Hom}_{MF(\widehat{R\otimes R},\tilde{w})}\Delta^{stab},\Delta^{stab}) \\ \cong&\underline{Hom}^{\mathbb{Z}/2}_{\widehat{R\otimes R}}(\Delta^{stab},\widehat{R\otimes R}/I) \\ =&\,[\textstyle\bigwedge^{odd}\overline{V} \underset{s}{\overset{s}{\rightleftarrows}} \bigwedge^{even}\overline{V}] \end{aligned}$$

where $\overline{(-)}$ denotes $\otimes_{\widehat{R\otimes R}}(\widehat{R\otimes R}/I)$

This last complex is just the Koszul complex with respect to the elements $(\tilde{w_1},...,\tilde{w_n})$ in $\widehat{R\otimes R}/I$.

To compute this complex as an R-complex, it suffices to determine what $\tilde{w_n}) \in \widehat{R\otimes R}/I$ corresponds to under the isomorphism

$$\widehat{R} \xrightarrow{x\mapsto x\otimes 1} \widehat{R\otimes R} \longrightarrow \widehat{R\otimes R}/I$$

Thus, we calculate:

$$\begin{aligned} -\partial_i w(x) &= \lim_{\Delta_i\to 0} \frac{w(x-\Delta_i)-w(x)}{\Delta_i} \\ &= \lim_{\Delta_i\to 0} \frac{w(x_1,...,x_{i-1},y_i,...,x_n)-w(x)}{\Delta_i} \\ &= \lim_{\Delta_i\to 0} \frac{w(y_1,...,y_n)-w(x) \qquad (\text{mod } \Delta_1,...,\tilde{\Delta_i},...,\Delta_n)}{\Delta_i} \\ &= \lim_{\Delta_i\to 0} \frac{\tilde{w}(x,y)}{\Delta_i} \qquad (\text{mod } \Delta_1,...,\tilde{\Delta_i},...,\Delta_n) \\ &= \tilde{w_i}(x,y) \end{aligned}$$

Here, the computation is taken inside the power series ring $k[[x_1,...,x_n,y_1,...,y_n]]$. Therefore, the limit operation makes sense.

Theorem 2.7.8. *The Hochschild cochain complex of the $\mathbb{Z}/2$-graded DG category $MF^{infty}(R,w)$ is quasi-isomorphic to the $\mathbb{Z}/2$-graded Koszul complex of the regular sequence $\partial_1 w,...,\partial_n w$ in R. In particular, the Hochschild cohomology is isomorphic, as an algebra, to the Jacobian algebra*

$$HH^*(MF^{\infty}(R,w)) \cong R/(\partial_1 w,...,\partial_n w)$$

concentrated in even degree.

Since Hochschild cohomology is invariant under derived Morita equivalence, we have the following:

Corollary 2.7.9. *The Hochschild cohomology of the* $\mathbb{Z}/2$*-graded DG category* $MF(R, w)$ *is isomorphic to the Jacobian algebra*

$$\mathrm{HH}^\bullet(\mathsf{MF}(R, w)) \simeq R/(\partial_1 w, \ldots, \partial_n w)$$

concentrated in even degree.

3 Some physics

We'll begin with a standpoint that emphasizes physical intuition, at the expense of mathematical rigor. This will be centered on a specific quantum field theory, due to Jones and Witten.

3.1 Physical motivation

We'll approach *gauge invariants* via an example: the group $U(1)$. The *Maxwell equations* are

$$\nabla \cdot B = 0$$
$$\nabla \times E = -\frac{\partial}{\partial t} B$$

From these equations we know that $B = \nabla \times A$ for some potential A, but A is not unique – it can be replaced by any $A + \nabla A'$.

Let $f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where $\partial_\mu = \frac{\partial}{\partial \mu}$, and $dA = \frac{1}{2} f_{\mu\nu} dx^\mu \wedge dx^\nu$. Here, the Maxwell equations are

$$df = 0$$
$$d \times f = 0$$

These equations are invariant under the operation $A_\mu \mapsto A_\mu - \partial_\mu \Lambda$.

In quantum mechanics, one starts with a "state vector," i.e. a function $\psi(x, t)$. A "property" E is identified with the functional $\langle E \rangle = \int d^3 x \psi^* E \psi$. If we replace ψ by $e^{i\theta}\psi$, nothing is changed. On the other hand, if we replace ψ by $e^{i\theta(x)}\psi$, then the operator will not send ψ to the same value. To account for this, we replace ∂_μ with the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$. This plays the role of a connection.

We know have $D_\mu e^{i\theta(x)}\psi(x) = e^{i\theta(x)} D_\mu \psi$. This is a type of parallel transport. It can be defined as

$$R(c; A) = e^{i \int_c A \, dX}$$

where c is some curve. This integral is not gauge-invariant, in the sense that if $A \mapsto A' = A_\mu - \partial_\mu \Lambda$, then $R(c; A') = e^{i\Lambda(P_2)} R(c; A) e^{-i\Lambda(P_2)}$, where P_1 and P_2 are the endpoints of c. This construction is motivated by the Bohm-Aharonov effect.

Let's move to a gauge theory over a general Lie group. In the Lie algebra, we have $[T_a, T_b] = if_{abc}T_c$, and put $U = e^{-\Lambda_a T^a}$, where the $\Lambda^a \in \mathbb{R}$. Define $A_\mu(x) = A^a_\mu T_a$, where x lives in spacetime, and the A^a_μ are some functions on spacetime.

Define a "parallel transport operator"

$$R(x + dx, x; A) = 1 - idx^\mu A_\mu(x).$$

Partition the curve, and get

$$R(c; A) = \lim_{N\to\infty} \prod_{\ell=1}^{N} (1 - i\Delta x^\mu_\ell A^\mu(x_\ell)) = Pe^{-\int_\alpha^\beta A(x)\cdot dx}$$

where α is the start and β is the end of the curve. This is not gauge invariant – it transforms by $U(\beta)R(c; A)U^{-1}(\alpha)$. The Taylor expansion becomes

$$A'_\mu = UA_\mu U^{-1} + i(\partial_\mu U)U^{-1}.$$

We want to compute $R(\Box; A)$, where $\Box$ is a small rectangle at x^μ with side-lengths dx^μ and δx^μ. We have

$$R(\Box; A) = R(x, x + dx)R(x + dx, x + dx + \delta x)R(\delta x + x)R(x),$$

which is not gauge invariant. Its trace is

$$\operatorname{tr} R(\Box; A) = W_c(A) = \operatorname{tr}(Pe^{i\int A(x)\cdot dx}).$$

The operator W_c is "Wilson loop." The result of this holonomy calculation is $e^{iF_{\mu\nu}dx^\mu \delta x^\nu}$. Working out the details, we get a formula for this "curvature," which is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu].$$

In the language of differential forms,

$$dA - A\Lambda A = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu.$$

Let's switch to some Feynman integrals. Start with the Lagrangian $L = T - V$, and let $S = \int dtL$. The Lagrangian L depends only on x and $\dot{x}$. We have the equation

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0.$$

Over a three-dimensional manifold whose metric has signature $-, +, +$, we put

$$L = \frac{1}{2}\int_M d^3x |\partial_\mu \varphi|^2.$$

We also put

$$L_{\text{guage}} = \frac{1}{2}\int_M d^3x|F(A)|^2,$$

where F is the "curvature tensor."

If we have $Z = \int \nabla\varphi e^{iL}$, then $\langle W\rangle = \int_{\varphi_0(x,t)}^{\varphi_T(x,T)} \nabla\varphi e^{iL}W$, where we "average W over all functions interpolating the beginning and end points."

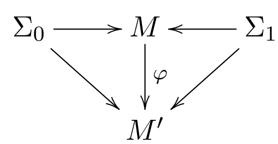
Let $\mathcal{G}$ be the space of maps $M \to G$, and let $\mathcal{A}$ be the space of connections. Essentially,

$$W_K = \int_{\mathcal{A}/\mathcal{G}} \nabla A e^{iK/4\pi L_{\text{cs}}} \prod_\lambda \operatorname{tr}(Pe^{i\oint_{K_\lambda} A}).$$

3.2 Cobordism

Definition 3.2.1. *A* cobordism *is a smooth n-manifold whose boundary is the disjoint union of two $(n-1)$-manifolds.*

We often write $\partial M = \Sigma_0 \sqcup \Sigma_1$ if M is a cobordism from Σ_0 to Σ_1. We do not require either M or ∂M to be connected. We *do* require both M and the Σ_i to be oriented. The choice of an orientation on some Σ_i determines one on M in a natural way. So we could have redefined a cobordism to be two closed oriented $(n-1)$-manifolds, a smooth n-manifold, and orientation-preserving diffeomorphisms $f_0 \sqcup f_1 : \partial M \xrightarrow{\sim} \Sigma_0 \sqcup \Sigma_1$. There is an obvious notion of equivalence of cobordisms, requiring a commutative diagram:



Cobordisms $M_0 : \Sigma_0 \to \Sigma_1$ and $M_1 : \Sigma_1 \to \Sigma_2$ can be composed: $M_1 \circ M_0 = M_1 \coprod_{\Sigma_1} M_0$. It is not immediately obvious that the composite has a smooth structure, but this is in fact the case. One uses Morse functions $f_i : M_i \to [i, i+1]$, such that f_0 is regular on $[1-\epsilon, 1]$ and f_1 is regular on $[1, 1+\epsilon]$ for some $\epsilon > 0$.

The composite of cobordisms to be independent of the isomorphism classes of the individual cobordisms. We can define the *n-cobordism category* $\mathsf{cob}(n)$, whose objects are closed oriented $(n-1)$-manifolds, and whose morphisms are differmorphism classes of cobordisms. It is an easy exercise to check that the $\mathsf{cob}(n)$ are in fact categories. The category $\mathsf{cob}(n)$ has a monoidal structure coming from disjoint union. The "identity element" for this monoidal structure is the empty $(n-1)$-manifold. In $\mathsf{cob}(n)$, there are natural isomorphisms $\lambda_\Sigma : 1_C \sqcup \Sigma \xrightarrow{\sim} \Sigma$, etc. Even better, $\mathsf{cob}(n)$ is a *symmetric monoidal category.* In other words, there is a natural transformation $g_{1,2} : \Sigma_1 \sqcup \Sigma_2 \xrightarrow{\sim} \Sigma_2 \sqcup \Sigma_1$ satisfying $g_{1,2} \circ g_{2,1} = 1$.

Definition 3.2.2. *Let $\mathcal{C}$, $\mathcal{D}$ be monoidal categories. We call a functor $F : \mathcal{C} \to \mathcal{D}$* monoidal *if for all $X, Y \in \mathcal{C}$, there are natural isomorphisms $FX \otimes FY \xrightarrow{\sim} F(X \otimes Y)$ and $F1_{\mathcal{C}} \xrightarrow{\sim} 1_{\mathcal{D}}$. Moreover, we require that F commute with the associativity constraints of $\mathcal{C}$ and $\mathcal{D}$.*

3.3 Topological quantum field theories

We think of a TQFT as assigning to each manifold M a vector space $F(M)$ of "fields," together with an "action" $S : F(M) \to \mathbb{R}$ and some functional ∇_φ. One usually writes

$$S = \int_M L\left(\varphi, \frac{\partial \varphi}{\partial x^\mu}\right).$$

Definition 3.3.1. *A* topological quantum field theory *is a monoidal functor* $\mathsf{cob}(n) \to \mathsf{Vect}_k$ *for some field* k*.*

If we are thinking of physical intuition, one should set $n = 4$. To each Σ, associate to Σ the space $\mathrm{Fun}(\mathscr{O}(\Sigma))$, i.e. the space of *all* (not necessarily linear) functions $\mathscr{O}(\Sigma) \to \mathbb{R}$. To each cobordism $M : \Sigma_0 \to \Sigma_1$, associate an integral operator $ZM : \mathscr{O}(\Sigma_0)^\vee \to \mathrm{Fun}(\mathscr{O}(\Sigma_1))$ defined via the kernel

$$K_M(\varphi_1, \varphi_2) = \int_{\substack{\varphi \in \mathscr{O}(M) \\ \varphi|_{\Sigma_0} = \varphi_1 \\ \varphi|_{\Sigma_1} = \varphi_2}} e^{iS(\varphi)} \nabla \varphi.$$

(better: introduce space of distributions)

For example, consider

$$K = \int \nabla x e^{\frac{1}{2} m \dot{x}^2 + \frac{1}{2} m u^2 x^2}.$$

In physics, all the fundamental parameters $\hbar$, mass, coupling, etc. are contained in the Lagrangian, and in S.

If M is a closed n-manifold, we can think of M as a cobordism from the empty space to itself. By convention, $\mathscr{O}(\varnothing) = 0$, so $\mathrm{Fun}(\mathscr{O}(\varnothing)) = \mathbb{R}$. So $ZM : \mathbb{R} \to \mathbb{R}$ is just a number, which turns out to be

$$K = \int_{\varphi \in \mathscr{O}(M)} e^{iS(\varphi)} \nabla \varphi.$$

Example 3.3.2 (Witten, Dijkgraaf)**.** Let G be a finite group. Let M be an n-cobordism, and let $\mathcal{F}(M)$ be isomorphism classes of principal G-bundles on M. In this case, for $\partial M = \Sigma_0 \sqcup \Sigma_1$, the kernel is

$$K_M(\pi_0, \pi_1) = \sum_{\substack{\pi|_{\Sigma_0} = \pi_0 \\ \pi|_{\Sigma_1} = \pi_1 \\ \pi \in \mathcal{F}(M)}} \frac{1}{\# \mathrm{Aut}(\pi)}.$$

Moreover,

$$Z_M(f) = \sum_{\pi_0 \in \mathcal{F}(\Sigma_0)} K_M(\pi_0, \pi_1) f(\pi_0).$$

3.4 Chern-Simons

Let M be a three-dimensional "spacetime" manifold, G a Lie group, and let

$$CS = A \wedge dA + \frac{2}{3} A \wedge A \wedge A$$

where A is some connection on the trivial G-bundle on M. Our functional is

$$S(A) = \frac{1}{i\pi} \int_M \operatorname{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$

We have $\frac{\delta S}{\delta A} = F_A = 0$, so the space of solutions consists of flat connections.

3.5 Atiyah-Segal axioms

Recall that a TQFT is a functor $Z : \mathsf{cob}(n) \to \mathsf{Vect}_k$ for some field k.

4 Perfectoid rings, almost mathematics, and the cotangent complex

Scholze introduced perfectoid fields (and more generally, perfectoid spaces) in his paper [Sch12], in which he proved a wide range of special cases of Deligne's Weight Mondromy Conjecture for p-adic fields.

4.1 Perfectoid fields

Recall that a *valued field* is a field k together with a homomorphism $|\cdot| : k^\times \to \Gamma$ for some totally ordered abelian group Γ (whose operation we will write multiplicatively). One requires $|\cdot|$ to satisfy the *ultrametric inequality*:

$$|x + y| \leqslant \max\{|x|, |y|\}.$$

As is standard, write 1 for the unit in Γ, and introduce an extra element 0, stipulating that $0 < \gamma$ for all $\gamma \in \Gamma$. We define $|0| = 0$, thus extending $|\cdot|$ to a function $k \to \Gamma \cup \{0\}$. One calls the *rank* of $|\cdot|$ the dimension $\dim_{\mathbb{Q}}(|k^\times| \otimes \mathbb{Q})$. We will say that the valuation $|\cdot|$ is *non-discrete* if $|k^\times| \not\simeq \mathbb{Z}$.

Put

$$k^\circ = \{x \in k : |x| \leqslant 1\}.$$

This is called the *ring of integers* of k. It is a valuation ring with (unique) maximal ideal

$$k^+ = \{x \in k : |x| < 1\}.$$

We call k°/k^+ the *residue field* of k.

Definition 4.1.1. *A* perfectoid field *is a complete valued field k with respect to a non-discrete rank-one valuation, with residue characteristic $p > 0$, such that the Frobenius* $\mathrm{Fr} : k^\circ/p \to k^\circ/p$ *is surjective.*

A typical example of a perfectoid field is $\mathbb{Q}_p(\zeta_{p^\infty})^\wedge$, the completion of $\mathbb{Q}_p(\zeta_{p^n} : n \geqslant 1)$ with respect to the p-adic topology. Similarly, $\mathbb{Q}_p(p^{1/p^\infty})^\wedge$ and $\mathbb{C}_p = \left(\overline{\mathbb{Q}_p}\right)^\wedge$ are perfectoid. An example in characteristic p is the t-adic completion of $\mathbb{F}_p(t^{1/p^\infty})$. For a perfectoid field k of residue characteristic p, choose $\pi \in k^\times$ with $|p| \leqslant |\pi| < 1$. Note that the Frobenius map $a \mapsto a^p$ is defined on A/π for any k°-algebra A.

Definition 4.1.2. *Let k be a perfectoid field. A* perfectoid k-algebra *is a Banach k-algebra A such that $A^\circ = \{x \in A : |x| \leqslant 1\}$ is open and for which* $\mathrm{Fr} : A^\circ/\pi \to A^\circ/\pi$ *is surjective.*

If k is a perfectoid field, let $\mathsf{Perf}(k)$ denote the category of perfectoid k-algebras, with continous k-maps as morphisms. We will construct, for any perfectoid field k with residue characteristic p, a perfectoid field $k^\flat$ of characteristic p. Start by defining

$$k^{\flat\circ} = \varprojlim_{\mathrm{Fr}} (k^\circ/\pi) = \left\{ (x_i) \in \prod_{i \geqslant 0} k^\circ/\pi : x_{i+1}^p = x_i \right\}.$$

It is not too difficult to check directly that $k^{\flat\circ}$ is a valuation ring, and we put $k^\flat = \mathrm{Frac}(k^{\flat\circ})$. There is a canonical map $(-)^\sharp : k^{\flat\circ} \to k^\circ$ defined by

$$(x_0, x_1, \dots)^\sharp = \lim_{n\to\infty} \widetilde{x_n}^{p^n},$$

where $\widetilde{x_n}$ is an arbitrary lift of $x_n \in k^\circ/\pi$ to k°. The map $(-)^\sharp$ is *not* additive unless k already has characteristic p, in which case $k = k^\flat$. In general, $(-)^\sharp$ extends to an isomorphism of multiplicative groups $k^{\flat\times} \to k^\times$, and we can use this to define a valuation on $k^\flat$ by $|x|_{k^\flat} = |x^\sharp|_k$. See Lemma 3.4 of [Sch12] for a proof that $(-)^\sharp$ has the claimed properties, and that $k^\flat$ is a perfectoid field with the same value group as k.

Example 4.1.3. Let N be a lattice (i.e. a finite free $\mathbb{Z}$-module) and let $\sigma \subset N_\mathbb{R}$ be a strongly convex polyhedral cone. Let $\sigma^\wedge \subset N_\mathbb{R}^\vee$ be its dual. If we put $M = N^\vee$, then the spectra of algebras of the form $k[\sigma] = k[\sigma^\vee \cap M]$ form affine charts for toric varieties over k. There is a "perfectoid version" of this. Write $k\langle\sigma\rangle = k\langle\sigma^\vee \cap M[\frac{1}{p}]\rangle$ for the ring

$$\left(k^\circ[\sigma^\vee \cap M \otimes \mathbb{Z}[\frac{1}{p}]]^\wedge\right) \otimes k.$$

Then $k\langle\sigma\rangle$ is a perfectoid algebra over k. (*Note*: $k\langle\sigma\rangle$ is *not*, in this context, a non-commutative polynomial algebra over k.

If A is a perfectoid k-algebra, define $A^\flat$ in much the same way, via

$$A^\flat = \left(\varprojlim_{\mathrm{Fr}} A^\circ/\pi\right) \otimes_{k^{\flat\circ}} k^\flat.$$

It turns out that if A is perfectoid, then $A^\flat$ is also perfectoid, and we have the following deep theorem:

Theorem 4.1.4 (Scholze)**.** *The functor* $(-)^\flat : \mathsf{Perf}(k) \to \mathsf{Perf}(k^\flat)$ *is an equivalence of categories.*

In fact, much more can be shown, e.g. $(-)^\flat$ induces equivalences of categories between $\mathsf{FEt}(A)$ and $\mathsf{FEt}(A^\flat)$ for all A, where $\mathsf{FEt}(A)$ is the category of finite étale algebras over A (it turns out that such algebras are perfectoid).

Example 4.1.5. If $A = k\langle\sigma\rangle = k\langle\sigma^\vee \cap M[\frac{1}{p}]\rangle$ as in Example 4.1.3, then $A^\flat = k^\flat\langle\sigma\rangle = k^\flat\langle\sigma^\vee \cap M[\frac{1}{p}]\rangle$.

Theorem 4.1.4 is proved without introducing much heavy machinery in Section 3.6 of [KL13]. The basic idea is that the map $(-)^\sharp : k^{\flat\circ} \to k^\circ$ induces a ring homomorphism $\theta : W(k^{\flat\circ}) \to k^\circ$, where $W(-)$ is the ring of p-typical Witt vectors. The inverse to the functor $(-)^\flat$ is $A^\sharp = W(A^\circ) \otimes_{W(k^{\flat\circ})} k$. Scholze's proof is more conceptual, and passes through a diagram

$$\begin{array}{ccccc}
\mathsf{Perf}(k) & \xrightarrow{\sim} & \mathsf{Perf}(k^{\circ a}) & \xrightarrow{\sim} & \mathsf{Perf}(k^{\circ a}/\pi) \\
\Big\downarrow{\scriptstyle (-)^\flat} & & & & \Big\| \\
\mathsf{Perf}(k^\flat) & \xrightarrow{\sim} & \mathsf{Perf}(k^{\flat\circ a}) & \xrightarrow{\sim} & \mathsf{Perf}(k^{\flat\circ a}/\pi^\flat)
\end{array}$$

in which most of the categories have yet to be defined. The superscript $(-)^{\mathrm{a}}$ should be read "almost," following the "almost mathematics" initially created by Faltings, and developed systematically in [GR03].

4.2 Almost mathematics

Almost mathematics was first introduced by Faltings in [Fal88], where he proved a deep conjecture of Fontaine on the étale cohomology of varieties over p-adic fields. We follow the treatment in Section 2.2 of [GR03].

Let V be a valuation ring with maximal ideal $\mathfrak{m}$. Throughout this section, we assume that *the value group of V is non-discrete*. This implies $\mathfrak{m}^2 = \mathfrak{m}$. In fact, much of the theory works for any unital ring V with idempotent two-sided ideal $\mathfrak{m}$, but we have no need to work at this level of generality. The reader should keep in mind the example $V = k^\circ$ for a perfectoid field k.

Let $\mathsf{Mod}(V)$ be the category of all V-modules, and let $\mathsf{Ann}(\mathfrak{m})$ be the full subcategory consisting of those modules killed by $\mathfrak{m}$.

Lemma 4.2.1. $\mathsf{Ann}(\mathfrak{m})$ *is a Serre subcategory of* $\mathsf{Mod}(V)$*.*

Proof. We need to show that if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of V-modules for which M' and M'' are killed by $\mathfrak{m}$, then M is also killed by $\mathfrak{m}$. We trivially have $\mathfrak{m}^2 M = 0$, but $\mathfrak{m}^2 = \mathfrak{m}$, whence the result. $\square$

Definition 4.2.2. *The category of* V^{a}-modules *is the Serre quotient* $\mathsf{Mod}(V^{\mathrm{a}}) = \mathsf{Mod}(V)/\mathsf{Ann}(\mathfrak{m})$.

Write $(-)^{\mathrm{a}} : \mathsf{Mod}(V) \to \mathsf{Mod}(V^{\mathrm{a}})$ for the quotient functor. Even though "V^{a}" does not exist as a ring we will write $\mathrm{Hom}_{V^{\mathrm{a}}}(-,-)$ for hom-sets in $\mathsf{Mod}(V^{\mathrm{a}})$. While in general, hom-sets in quotient categories can be difficult to describe, the category $\mathsf{Mod}(V^{\mathrm{a}})$ is relatively easy to understand via the following theorem.

Theorem 4.2.3. *There is a natural isomorphism* $\mathrm{Hom}_{V^{\mathrm{a}}}(M^{\mathrm{a}}, N^{\mathrm{a}}) = \mathrm{Hom}_V(\mathfrak{m} \otimes M, N)$.

It follows that $\mathsf{Mod}(V^{\mathrm{a}})$ is naturally a $\mathsf{Mod}(V)$-enriched category. We define two functors $(-)_!, (-)_* : \mathsf{Mod}(V^{\mathrm{a}}) \to \mathsf{Mod}(V)$:

$$M_* = \mathrm{Hom}_{V^{\mathrm{a}}}(V^{\mathrm{a}}, M)$$
$$M_! = \mathfrak{m} \otimes M_*.$$

Theorem 4.2.4. *The triple* $((-)_!, (-)^{\mathrm{a}}, (-)_*)$ *is adjoint. Moreover, these adjunctions induce natural isomorphisms* $(M_*)^{\mathrm{a}} = M = (M_!)^{\mathrm{a}}$.

This is suggestive of the situation in which $j : U \hookrightarrow X$ is an open embedding of topological spaces. The restriction functor $j^* : \mathsf{Sh}(X) \to \mathsf{Sh}(U)$ fits into an adjoint triple $(j_!, j^*, j_*)$ in which $j^* j_* = 1 = j^* j_!$. So we should think of $\mathsf{Mod}(V^{\mathrm{a}})$ as the category of quasi-coherent sheaves on some subscheme of $\operatorname{Spec} V$, even though there is no such subscheme. In fact, since $\mathsf{Mod}(V^{\mathrm{a}})$ does not contain enough projectives, it should be thought of as some kind of "non-affine" object.

The category $\mathsf{Mod}(V^{\mathrm{a}})$ inherits the structure of a tensor category from $\mathsf{Mod}(V)$. In fact, we have internal tensor and hom defined by

$$M^{\mathrm{a}} \otimes N^{\mathrm{a}} = (M \otimes N)^{\mathrm{a}}$$
$$\mathrm{Hom}^{\mathrm{a}}(M^{\mathrm{a}}, N^{\mathrm{a}}) = \mathrm{Hom}(M, N)^{\mathrm{a}}$$

There is a tensor-hom adjunction $\mathrm{Hom}(L \otimes N, M) = \mathrm{Hom}(L, \mathrm{Hom}^{\mathrm{a}}(M, N))$. This allows us to speak of algebra objects in $\mathsf{Mod}(V^{\mathrm{a}})$ as commutative unital monoid objects in the tensor category $(\mathsf{Mod}(V^{\mathrm{a}}), \otimes)$. We call such objects V^{a}*-algebras.* It is easy to check that they are all of the form A^{a}, for A some V-algebra.

If A is a V^{a}-algebra, we can form the category $\mathsf{Mod}(A)$ of A-modules in the obvious way. This is also an abelian tensor category, so it makes sense to speak of "flat objects" in the usual way, i.e. an A-module M is *flat* if the functor $M \otimes_A -$ is exact.

Definition 4.2.5. *Let k be a perfectoid field. A* perfectoid $k^{\circ\mathrm{a}}$-algebra *is a flat, π-adically complete $k^{\circ\mathrm{a}}$-algebra A for which* $\mathrm{Fr} : A/\pi^{1/p} \to A/\pi$ *is an isomorphism.*

Note that even though $\pi^{1/p}$ may not exist as an actual element of k°, the ideal $(\pi^{1/p})$ is well-defined, so it makes sense to write $M/\pi^{1/p}$ if M is any k°- (or $k^{\circ\mathrm{a}}$)-module. Write $\mathsf{Perf}(k^{\circ\mathrm{a}})$ for the category of perfectoid $k^{\circ\mathrm{a}}$-algebras.

Theorem 4.2.6. *The functor $A \mapsto A^{\circ\mathrm{a}}$ induces an equivalence of categories* $\mathsf{Perf}(k) \xrightarrow{\sim} \mathsf{Perf}(k^{\circ\mathrm{a}})$.

Idea of proof. See the first part of Section 5 in [Sch12]. Besides some technicalities, one has the existence of an inverse functor $A \mapsto A_* \otimes_{k^\circ} k$. □

4.3 The cotangent complex

We have seen in the last section that localization induces an equivalence between the category of perfectoid k-algebras and the category of perfectoid $k^{\circ a}$-algebras. Since $k^\times$ and $k^{\flat\times}$ are canonically isomorphic via $(-)^\sharp$, also write π for the element of $k^\flat$ corresponding to $\pi \in k$. It is easy to check that $k^\circ/\pi = k^{\flat\circ}/\pi$. It easily follows that $\mathsf{Mod}(k^{\circ a}/\pi) = \mathsf{Mod}(k^{\flat\circ a}/\pi)$. The idea of this section is to pass from $\mathsf{Perf}(k^{\circ a})$ to a suitable category of "perfectoid $k^{\circ a}/\pi$-algebras."

Definition 4.3.1. *Let k be a perfectoid field. A $k^{\circ a}/\pi$-algebra A is* perfectoid *if it is flat and* $\mathrm{Fr} : A/\pi^{1/p} \to A$ *is an isomorphism.*

Let $\mathsf{Perf}(k^{\circ a}/\pi)$ denote the category of perfectoid $k^{\circ a}/\pi$-algebras. We will show that the functor $A \mapsto A/\pi$ from $\mathsf{Perf}(k^{\circ a})$ to $\mathsf{Perf}(k^{\circ a}/\pi)$ is an equivalence of categories by introducing an "almost version" of the cotangent complex. Let's start by recalling the classical theory:

Theorem 4.3.2. *There is a functorial way of assigning to a flat map $A \to B$ of (commutative, unital) rings an object (the* cotangent complex*) $\boldsymbol{L}_{B/A}$ of $\mathsf{D}^{\leqslant 0}(B)$. This complex satisfies the following properties:*

1. *There is a functorial way of assigning to a square-zero extension $0 \to I \to \widetilde{A} \twoheadrightarrow A$ is a square-zero extension an* obstruction class
$$\boldsymbol{o}(I) \in ext^2(\boldsymbol{L}_{B/A}, I_B)$$
such that $\boldsymbol{o}(I) = 0$ if and only if there is a flat $\widetilde{A}$-algebra $\widetilde{B}$ such that $\widetilde{B} \otimes_{\widetilde{A}} A = B$. (One calls such a $\widetilde{B}$ a deformation *of B to $\widetilde{A}$.)*

2. *If a deformation of B to $\widetilde{A}$ exists, then the set $\mathrm{Def}_A^{\widetilde{A}}(B)$ of deformations of B to $\widetilde{A}$ is a*
$ext^1(\boldsymbol{L}_{B/A}, I_B)$-torsor.

3. *If $\widetilde{B}$ and $\widetilde{B}'$ are deformations of A-algebras B, B' to $\widetilde{A}$, and if $f : B \to B'$ is an A-algebra map, then there is a functorial way of assigning to f an* obstruction class
$$\boldsymbol{o}(f) \in ext^1(\boldsymbol{L}_{B/A}, I_{B'})$$
such that the set $\mathrm{Def}_A^{\widetilde{A}}(f)$ of isomorphism classes of lifts of f to $\widetilde{f} : \widetilde{B} \to \widetilde{B}'$ is nonempty if and only if $\boldsymbol{o}(f) \neq 0$.

4. *If a lift of f to $\widetilde{A}$ exists, then $\mathrm{Def}_A^{\widetilde{A}}(f)$ is a $\mathrm{Hom}(\boldsymbol{L}_{B/A}, I_{B'})$-torsor.*

Proof. See Proposition 2.1.2.3 in Chapter III of [Ill71] for parts 1 and 2. □

Gabber and Ramero were able to generalize the "classical" theory of the cotangent complex to an almost setting. To be precise, Theorem 4.3.2 remains true if we work V^{a}-algebras, for any non-discrete valuation ring V. In other words, there is a canonical object $\boldsymbol{L}^{\mathrm{a}}_{B/A} \in \mathsf{D}^{\leqslant 0}(B)$ such that the theore still works.

Theorem 4.3.3 (Scholze)**.** *Let k be a perfectoid field. If A is a perfectoid $k^{\circ a}/\pi$-algebra, then $\boldsymbol{L}^{\mathrm{a}}_{A/(k^{\circ a}/\pi)} = 0$ as an object of $\mathsf{D}(A)$.*

Idea of proof. This is a deep result, but at the heart of its proof is the fact that if R is a smooth perfect $\mathbb{F}_p$-algebra, then $\boldsymbol{L}_{R/\mathbb{F}_p} = 0$. Smoothness yields $\boldsymbol{L}_{R/\mathbb{F}_p} = \Omega^1_{R/\mathbb{F}_p}[0]$, and from $d(r^p) = pdr^{p-1} = 0$ we see that $\Omega^1_{R/\mathbb{F}_p} = 0$. □

Corollary 4.3.4 (Scholze)**.** *The functor $A \mapsto A/\pi$ induces an equivalence of categories $\mathsf{Perf}(k^{\circ a}) \xrightarrow{\sim} \mathsf{Perf}(k^{\circ a}/\pi)$.*

Proof sketch. Let $A_n = k^{\circ a}/\pi^{n+1}$. We content ourselves with showing that objects and morphisms in $\mathsf{Alg}(A_0)$ lift uniquely to each $\mathsf{Alg}(A_n)$. Let B_0 be a perfectoid A_0-algebra. Theorem 4.3.3 tells us that $\boldsymbol{L}^{\mathrm{a}}_{B_0/A_0} = 0$, so B_0 and any morphisms from B_0 to other perfectoid A_0-algebras lift uniquely to A_1 by the almost version of Theorem 4.3.2. All that remains for the induction to work is to show that $\boldsymbol{L}^{\mathrm{a}}_{B_n/A_n} = 0$ implies $\boldsymbol{L}^{\mathrm{a}}_{B_{n+1}/A_{n+1}} = 0$. There is an exact sequence

$$0 \longrightarrow B_0 \xrightarrow{\pi^n} B_{n+1} \longrightarrow B_n \longrightarrow 0,$$

and general theory gives us an exact triangle in $\mathsf{D}(A_{n+1})$:

$$\boldsymbol{L}^{\mathrm{a}}_{B_0/A_0} \longrightarrow \boldsymbol{L}^{\mathrm{a}}_{B_{n+1}/A_{n+1}} \longrightarrow \boldsymbol{L}^{\mathrm{a}}_{B_n/A_n} \longrightarrow .$$

Since $\boldsymbol{L}^{\mathrm{a}}_{B_0/A_0} = 0$ by Theorem 4.3.3 and $\boldsymbol{L}^{\mathrm{a}}_{B_n/A_n} = 0$ by assumption, we get that $\boldsymbol{L}^{\mathrm{a}}_{B_{n+1}/A_{n+1}} = 0$. □

5 Nearby and Vanishing Cycle Functors

We work in the following setting: let X be a complex-analytic manifold, $f : X \to \mathbb{C}$ a non-constant analytic map, for which we suppose that $Z = f^{-1}(0)$ is non-empty. Put $U = X \smallsetminus Z$. Throughout, we work with the middle perversity $p_{1/2}(S) = -\frac{1}{2}\dim_{\mathbb{R}} S$.

Denote by $\mathsf{D}(X)$ the bounded derived category $\mathsf{D}^b_c(X, \mathbb{C})$ of constructible sheaves of complex vector spaces. Let $\mathsf{Perv}(X) = \mathsf{Perv}^{p_{1/2}}(X, \mathbb{C})$ be the corresponding category of perverse sheaves. Recall that $\mathsf{Perv}(X)$ is an abelian category in which every object has finite length.

5.1 Construction

The following is originally from Deligne's section *La formalisme des cycles évanescents* in [DK73].

Consider the diagram

$$\begin{array}{ccccccc} Z & \overset{i}{\hookrightarrow} & X & \overset{j}{\hookleftarrow} & U & \leftarrow & \widetilde{X}^* \\ \downarrow & & \downarrow f & & \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & \mathbb{C} & \overset{j}{\hookleftarrow} & \mathbb{C}^\times & \overset{p}{\leftarrow} & \widetilde{\mathbb{C}^\times} \end{array}$$

where $p : \widetilde{\mathbb{C}^\times} \to \mathbb{C}^\times$ is the universal covering; and $\widetilde{X^*} = X \times_{\mathbb{C}} \widetilde{\mathbb{C}^\times}$. We denote the projection $\widetilde{X^*} \twoheadrightarrow X$ by π.

Recall that we have the adjunction $\pi^* : \mathsf{Sh}(X, \mathbb{C}) \rightleftarrows \mathsf{Sh}(\widetilde{X^*}, \mathbb{C}) : \pi_*$ The functor π^* is exact, and π_* is left-exact, so we have an adjunction at the level of derived categories:

$$\pi^* : \mathsf{D}(X) \rightleftarrows \mathsf{D}(\widetilde{X^*}) : \mathsf{R}\pi_*.$$

Definition 5.1.1. *The* nearby cycles functor *is the functor* $\Psi_f : \mathsf{D}(X) \to \mathsf{D}(Z)$ *defined by*

$$\Psi_f(\mathscr{F}^\bullet) = i^* \mathsf{R}\pi_* \pi^*(\mathscr{F}^\bullet).$$

Note that since $\pi(\widetilde{X^*}) = U \subset X$, the functor Ψ_f factors through $\mathsf{D}(U)$. In other words, there exists $\psi_f : \mathsf{D}(U) \to \mathsf{D}(Z)$ such that $\Psi_f = \psi_f \circ j^*$. In other words, $\Psi_f(\mathscr{F}^\bullet)$ only depends on $\mathscr{F}^\bullet|_U$. Moreover, Ψ_f comes with a natural transformation

$$\Theta : i^* \to \Psi_f$$

defined by applying i^* to the adjunction transformation $1 \to \mathsf{R}\pi_*\pi^* = \Psi_f$

Definition 5.1.2. *The* vanishing cycles functor *is the functor* $\phi_f : \mathsf{D}(X) \to \mathsf{D}(Z)$ *given by*

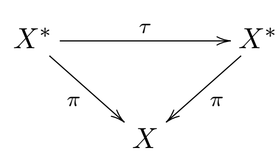
$$\phi_f(\mathscr{F}^\bullet) = \text{cone}(\Theta_{\mathscr{F}^\bullet} : i^*(\mathscr{F}^\bullet) \to \Phi_f(\mathscr{F}^\bullet)).$$

Proposition 5.1.3. *Both ψ_f and ϕ_f restricts to functors on the corresponding subcategories of perverse sheaves:*

$$\psi_f : \mathsf{Perv}(U) \to \mathsf{Perv}(Z)$$
$$\phi_f : \mathsf{Perv}(X) \to \mathsf{Perv}(Z).$$

5.2 Monodromy

As in the previous section, let $p : \widetilde{\mathbb{C}^\times} \to \mathbb{C}^\times$ be the universal cover. The fundamental group $\pi_1(\mathbb{C}^\times) = \mathbb{Z}$ acts on $\widetilde{\mathbb{C}^\times}$ by deck transformations. Denote by $t : \widetilde{\mathbb{C}^\times} \to \widetilde{\mathbb{C}^\times}$ the deck transformation corresponding to the generator 1 of $\pi_1(\mathbb{C}^\times)$. The transformation t satisfies $pt = p$. If we base-change to X, we have a commutative diagram



It follows that $\pi^* = \tau^* \circ \pi^*$, whence we have a natural isomorphism at the level of derived functors:

$$\mathsf{R}\pi_* \circ \mathsf{R}\tau_* = \mathsf{R}\pi_*.$$

The standard adjunction gives us a morphism of functors $\lambda : \mathrm{id} \to \mathsf{R}\tau_*\tau^*$. Thus we have a commutative diagram

$$\begin{array}{ccc} \mathsf{R}\pi_*\pi^* & \xrightarrow{\mathsf{R}\pi_*\lambda\pi^*} & \mathsf{R}\pi_*\mathsf{R}\tau_*\tau^*\pi^* \\ & \searrow^{T} & \| \\ & & \mathsf{R}\pi_*\pi^* \end{array}$$

where $T : \Psi_f \to \Psi_f$ is the endomorphism coming from τ. One calls T the *monodromy operator*.

Moreover, T restricts to identity under $\Theta : i^* \to \Psi_f$, i.e. it extends to a morphism of triangles:

$$\begin{array}{ccccccc} i^* & \xrightarrow{\Theta} & \Psi_f & \longrightarrow & \phi_f & \longrightarrow & i^*[1] \\ \| & & \downarrow T & & \downarrow T & & \downarrow \\ i^* & \xrightarrow{\Theta} & \Psi_f & = & \phi_f & \longrightarrow & i^*[1]. \end{array}$$

For any $\mathscr{F}^\bullet \in \mathsf{D}(X)$, this gives us a morphism of triangles in D_Z:

$$\begin{array}{ccccccc} i^*(\mathscr{F}^\bullet) & \xrightarrow{\Theta_{\mathscr{F}^\bullet}} & \Psi_f(\mathscr{F}^\bullet) & \xrightarrow{\mathrm{can}_{\mathscr{F}^\bullet}} & \phi_f(\mathscr{F}^\bullet) & \longrightarrow & i^*(\mathscr{F}^\bullet)[1] \\ \downarrow & & \downarrow T-\mathrm{id} & & \downarrow \mathrm{var}_{\mathscr{F}^\bullet} & & \downarrow \\ 0 & \longrightarrow & \Psi_f(\mathscr{F}^\bullet) & = & \Psi_f(\mathscr{F}^\bullet) & \longrightarrow & 0 \end{array}$$

It follows that

$$\mathrm{var} \circ \mathrm{can} = T - \mathrm{id}.$$

5.3 Gluing problems

Given $i : Z \hookrightarrow X \hookleftarrow U : j$ as above, we want to "glue" $\mathsf{Perv}(U)$ and $\mathsf{Perv}(Z)$.

Definition 5.3.1. *Define a category* $\mathsf{Glue}(Z, U)$ *whose objects are quadruples* $(\mathscr{G}^\bullet, \mathscr{H}^\bullet, a, b)$*, where:*

1. $\mathscr{G}^\bullet \in \mathsf{Perv}(U)$
2. $\mathscr{H}^\bullet \in \mathsf{Perv}(Z)$
3. $a : \psi_f(\mathscr{G}^\bullet) \to \mathscr{H}^\bullet$ *is a morphism in* $\mathsf{Perv}(Z)$
4. $b : \mathscr{H}^\bullet \to \psi_f(\mathscr{G}^\bullet)$ *is a morphism in* $\mathsf{Perv}(Z)$
5. $b \circ a = T_{\mathscr{G}^\bullet} - \mathrm{id}$

Theorem 5.3.2. *The functor* $\mathsf{Perv}(X) \to \mathsf{Glue}(Z, U)$ *given by*

$$\mathscr{F}^\bullet \mapsto (j^*\mathscr{F}^\bullet, \phi_f(\mathscr{F}^\bullet), \mathrm{can}_{\mathscr{F}^\bullet}, \mathrm{var}_{\mathscr{F}^\bullet})$$

is an equivalence of categories.

Example 5.3.3. Consider $Z = \{0\} \hookrightarrow X = \mathbb{C} \hookleftarrow U = \mathbb{C}^\times$, via $f(z) = z$. One has that $\mathsf{Perv}(Z) = \mathsf{Vect}_{\mathbb{C}}^{\text{fin}} \subset \mathsf{D}(\mathsf{Vect}_{\mathbb{C}})$. The category $\mathsf{Perv}(U)$ consists of locally constant sheaves on $\mathbb{C}^\times$, i.e. $\mathsf{Perv}(U) \simeq \mathsf{Rep}_{\mathbb{C}}(\pi_1(\mathbb{C}^\times))$, which is just $\mathsf{Mod}^{\text{fin}}(\mathbb{C}[t^{\pm}])$. To be explicit,

$$\mathsf{Perv}(U) = \left\{(V, A) : V \in \mathsf{Vect}_{\mathbb{C}}^{\text{fin}} \text{ and } V : A \xrightarrow{\sim} V\right\}.$$

The functor ψ_f acts via $(V, A) \mapsto V$ and $T_{(V,A)} : \psi_f(V, A) \to \psi_f(V, A)$ is the operator A. Theorem 5.3.2 now says that

$$\mathsf{Perv}(X) = \{(W, V, E : V \to W, F : W \to V) : FE + \text{id is invertible}\}.$$

There is a general way of assigning an algebra $\Lambda^{q,h}(Q)$ (called the multiplicative preprojective algebra) to a quiver Q such that

$$\mathsf{Perv}(X) = \Lambda^{q,h}(\bullet \to \bullet).$$

Example 5.3.4. Let $n \geqslant 1$, let $X = \mathbb{C}^n$, and let $\mathcal{S} = \{X_I\}_{I \subset [n]}$, where

$$X_I = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : x_i = 0 \text{ if and only if } i \in I\}.$$

Let $\mathsf{Perv}(X)$ be the category of perverse sheaves on X with respect to $\mathcal{S}$. Combinatorialists are interested in perverse sheaves because of the following theorem. We first define a category $\mathsf{A}_{(n)}$ whose objects are tuples

$$\left(\{V_I\}_{I \subset [n]}, E_{I,i} : V_I \to V_{I \cup \{i\}} \text{ for } i \notin I, F_{I,i} : V_{I \cup \{i\}} \to V_I \text{ for } i \notin I\right)$$

such that

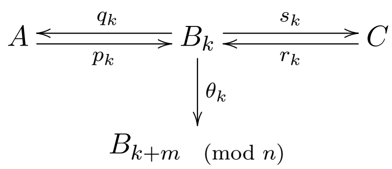
1. $E_{I \cup \{j\},i} \circ E_{I,j} = E_{I \cup \{i\},j} \circ E_{I,i}$ for $i \notin I$
2. $F_{I,j} F_{I \cup \{i\},i} = F_{I,i} F_{I \cup \{i\},j}$ for $i \neq j$ both not in I
3. $F_{I \setminus \{j\},i} I F_{I \setminus \{j\},j} = F_{I \cup \{i\} \setminus \{j\}} F_{I,i}$ for $i \notin I$ and $j \in I$
4. $F_{I,i} E_{I,i} + \text{id}$ is invertible for all i

Morphisms are $\{f_I : V_I \to V_I'\}$ commuting with the Es and Fs.

Theorem 5.3.5. *With X as above, there is an equivalence of categories* $\mathsf{Perv}(X) \simeq \mathsf{A}_{(n)}$.

Proof. See [GMV96]. ☐

Example 5.3.6. Let $X = \mathbb{C}^2$ and $\{0\} \subset \{x^n = ym\} \subset \mathbb{C}^2$ be our stratification (we assume $n \leqslant m$). Then there is an equivalence of categories $\mathsf{Perv}(X) \simeq \mathsf{A}_{(n,m)}$, where $\mathsf{A}_{(n,m)}$ consists of tuples $(A, B_1, \ldots, B_n, C)$ with arrows



satisfying certain requirements, including that $1+q_kp_k$ and θ_k are all invertible.

As a special case, if we put $(n,m)=(2,3)$, define

$$H=\mathbb{C}\langle A,B\rangle/(ABA+A^2+A, BAB+B^2+B).$$

Define $\mathscr{H}$ by the algebra homomorphism $H\to\mathscr{H}$, which is universal among algebra homomorphisms $\varphi: H\to R$ such that $\varphi(A+1)$ and $\varphi(1+B)$ are units. Alternatively,

$$\mathscr{H}=H\langle X,Y\rangle/((1+A)X=X(1+A)=1,(1+B)Y=Y(1+B)=1).$$

We have $\mathsf{A}_{(2,3)}\simeq\mathsf{Mod}^{\mathrm{fg}}(\mathscr{H})$.

5.4 Monadology

Recall that an *exact category* (in the sense of Quillen) is an additive category $\mathcal{A}$ together with a special class $\mathcal{E}$ of "exact pairs," consisting of diagrams of the form $A\xrightarrow{i}B\xrightarrow{p}C$ such that $(A,i)=\mathrm{Ker}(p)$ and $(C,p)=\mathrm{Coker}(i)$. These exact pairs are required to satisfy certain axioms. [**reference earlier section for these**]

Example 5.4.1. If $\mathcal{A}$ is an abelian category, then we can let $\mathcal{E}$ be the class of all short exact sequences. More interestingly, we could restrict to the class of all split exact sequences.

Example 5.4.2. The category of finitely-generated projective modules over a unital ring is exact (when given the obvious class of exact pairs).

Example 5.4.3. Let $\mathcal{A}=\mathsf{Com}(\mathcal{B})$ for some abelian category $\mathcal{B}$. We can set $\mathcal{E}$ to be the class of termwise-split short exact sequences in $\mathcal{A}$.

In an exact category, there is are notions of *admissible monic* (a morphism which fits into the first half of an exact pair) and *admissible epic* (a morphism which fits into the second half of an exact pair). We write $\hookrightarrow$ for admissible monics and $\twoheadrightarrow$ for admissible epics.

Fix an exact category $\mathcal{A}$.

Definition 5.4.4. *A* monad *in $\mathcal{A}$ is a complex of the form $\mathcal{P}=[P_-\xrightarrow{\alpha_-}P\xrightarrow{\alpha_+}P_+]$ with α_- admissible monic and α_+ admissible epic.*

By "complex" we require $\alpha_+\circ\alpha_-=0$. Denote by $\mathrm{H}(\mathcal{P})$ the quotient $\mathrm{Ker}(\alpha_+)/\mathrm{Im}(\alpha_-)$ [**prove that this quotient exists**]. We call this the *cohomology* of the monad $\mathcal{P}$. Let $\widetilde{\mathcal{A}}$ be the category of monads in $\mathcal{A}$. This is naturally an exact category with $\mathrm{H}:\widetilde{\mathcal{A}}\to\mathcal{A}$ an exact functor.

Let $\widetilde{\mathcal{A}}_1$ be the category whose objects are $\mathcal{P}_1=[P_{-1}\overset{\gamma_{-1}}{\hookrightarrow}P_0\overset{\gamma_0}{\hookrightarrow}P_1]$. We call $\widetilde{\mathcal{A}}_1$ the category of 3-*step filtations.* Finally, let $\widetilde{\mathcal{A}}_2$ be the category of $[L_-\xrightarrow{(\delta_-,\varepsilon_-)}A\oplus B\xrightarrow{(\delta_+,\varepsilon_+)^t}L_+]$ with δ_- monic and δ_+ epic.

Proposition 5.4.5. *There are canonical exact equivalences of categories $\widetilde{\mathcal{A}}\simeq\widetilde{\mathcal{A}}_1\simeq\widetilde{\mathcal{A}}_2$.*

Proof. We consider the equivalence $\widetilde{\mathcal{A}} \simeq \widetilde{\mathcal{A}}_1$. To a sequence $P_- \xrightarrow{\alpha_-} P \xrightarrow{\alpha_+} P_+$ associate the sequence $P_- \xrightarrow{\alpha_-} \mathrm{Ker}(\alpha_+) \hookrightarrow P$, and in reverse send $P_{-1} \overset{\gamma_-}{\hookrightarrow} P_0 \overset{\gamma_0}{\hookrightarrow} P_1$ to the sequence $P_{-1} \xrightarrow{\gamma_0\gamma_{-1}} P_1 \twoheadrightarrow P_1/P_0$.

For $\widetilde{\mathcal{A}}_1 \simeq \widetilde{\mathcal{A}}_2$, consider

$$[P_{-1} \xrightarrow{\gamma_-} P_0 \xrightarrow{\gamma_0} P_1] \mapsto [P_0 \xrightarrow{(\gamma_0, can)} P_1 \oplus P_0/P_- \xrightarrow{P}_1 /P_{-1}]$$

$$\dots$$

□

The motivation for all of this is quite classical. Recall the following theorem in [Bar77]. Let $\mathcal{M}(n,r)$ be the moduli space of framed torsion-free coherent sheaves on $\mathbb{P}^2_{\mathbb{C}}$ of rank r and Chern class $c_2 = n$. Let $i : \ell_\infty \hookrightarrow \mathbb{P}^2$ be the inclusion of the line at infinity. "Elements" of $\mathcal{M}(n,r)$ are pairs $(\mathscr{F}, \phi)$, where $\mathscr{F}$ is a rank-r torsion free sheaf on $\mathbb{P}^2_{\mathbb{C}}$ with $c_2(\mathscr{F}) = n$, and $\phi : i^*\mathscr{F} \xrightarrow{\sim} \mathscr{O}_{\mathbb{P}^1}^{\oplus r}$ is a "framing at infinity." There is a natural way to give $\mathcal{M}(n,r)$ the structure of a scheme.

Theorem 5.4.6 (Barth)**.** *There is an isomorphism between $\mathcal{M}(n,r)$ and the moduli space of isomorphism classes of triples (B_1, B_2, i, j), where $B_1, B_2 \in M_n(\mathbb{C})$, $i : \mathbb{C}^r \to \mathbb{C}^n$ and $j : \mathbb{C}^n \to \mathbb{C}^r$, satisfying $[B_1, B_2] + ij = 0$. Also, the triples are* stable *in the sense that there are no $S \subset \mathbb{C}^n$ such that $B_\alpha(S) \subset S$ for $\alpha \in \{1,2\}$ and* $\mathrm{Im}(i) \subset S$*.*

The linear-algebraic data in this theorem can be encoded in terms of the quiver **[couldn't live-TEXthe quiver: it has $i : W \to V$, $j : V \to W$ and $B_i : V \to V$]**

Sketch of proof. We will construct a functor in one direction. To a tuple (B_1, B_2, i, j), assign

$$\mathscr{F} = \mathrm{H}\left(V \otimes \mathscr{O}_{\mathbb{P}^2}(-1) \xrightarrow{a} (V \otimes \mathscr{O}_{\mathbb{P}^2})^2 \oplus W \otimes \mathscr{O}_{\mathbb{P}^2} \xrightarrow{b} V \otimes \mathscr{O}_{\mathbb{P}^2}(1)\right).$$

We can identify $\mathbb{P}^2_{\mathbb{C}}$ with the "proj" of $\mathbb{C}[z_0, z_1, z_2]$, where we can regard the z_i as global sections of $\mathscr{O}_{\mathbb{P}^2}(1)$. Even better, we think of the z_i as maps $\mathscr{O}_{\mathbb{P}^2}(i) \to \mathscr{O}_{\mathbb{P}^2}(i+1)$ for all i. The maps a and b are given by

$$a = \begin{pmatrix} z_0 B - z_1 \\ z_0 B_2 - z_2 \\ z_0 j \end{pmatrix}$$

$$b = \begin{pmatrix} -(z_0 B_2 - z_2) & z_0 B_1 - z_1 & z_0 i \end{pmatrix}$$

One can check that $ba = 0$ if and only if $[b_1, b_2] + ij = 0$.

Now we define the operation $\mathscr{F} \mapsto (B_1, B_2, i, j)$. Define a rank-two vector bundle $\mathscr{E}$ on $\mathbb{P}^2$ by the "Euler exact sequence"

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-1) \to \mathscr{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathscr{E} \to 0.$$

We put

$$\mathscr{F} \mapsto \left(\mathrm{H}^1(\mathbb{P}^2, \mathscr{F}(-2)) \otimes \mathscr{O}_{\mathbb{P}^2}(-1) \to \mathrm{H}^1(\mathbb{P}^2, \mathscr{F}(-1) \otimes \mathscr{E}^\vee) \otimes \mathscr{O}_{\mathbb{P}^2} \to \mathrm{H}^1(\mathbb{P}^2, \mathscr{F}) \otimes \mathscr{O}_{\mathbb{P}^2}(1)\right).$$

□

There is an example from [BW02]. Let $r = 1$ and n be any integer. Let C_n be the space of isomorphism classes of tuples (X, Y, i, j) such that $X, Y \in M_n(\mathbb{C})$, $i \in \mathbb{C}^n$ and $j \in (\mathbb{C}^n)^\times$ such that $[X, Y] + 1 = ij$. This is generally known as the Calosero-Moser space.

There is a bijection between $\coprod_{n \geqslant 0} C_n$ and the set of isomorphism classes of right ideals in $A = A_1(\mathbb{C}) = \mathbb{C}\langle X, Y\rangle/([X, Y] - 1)$.

5.5 Reflection functors

Let $\mathcal{A}$ be an exact category.

Definition 5.5.1. *The category $\mathcal{A}^\sharp$ of* dyads *consists of commutative diagrams*

$$\begin{array}{ccc} C_- & \xhookrightarrow{\alpha_-} & A \\ \big\downarrow{\scriptstyle \beta_-} & & \big\downarrow{\scriptstyle \alpha_+} \\ B & \xrightarrow{\beta_+} & C_+ \end{array}$$

where we do not *require that $\alpha_+ \circ \alpha_- = 0$.*

Let $\mathcal{A}_1^\sharp$ be the category of *special monads*, i.e. those of the form

$$C_+ \xhookrightarrow{\alpha_-} A \oplus B \xrightarrow{\alpha_+} C_+$$

where $\alpha_+\alpha_- = 0$. Finally, let $\mathcal{A}_2^\sharp$ be the category of *special exact sequences*, i.e. those of the form

$$D_- \xhookrightarrow{(\gamma_-, \delta_-^1, \delta_-^2)} A \oplus B^1 \oplus B^2 \xrightarrow{\begin{pmatrix}\gamma_+ \\ \delta_+^2 \\ \delta_+^2\end{pmatrix}} D_+$$

such that (γ_-, δ_-^i) is mmonic for each i, and such that (γ_+, δ_+^i) is epic for each i.

Proposition 5.5.2. *There are canonical exact equivalences $\mathcal{A}^\sharp \simeq \mathcal{A}_1^\sharp \simeq \mathcal{A}_2^\sharp$.*

Proof. We first construct a functor $\mathcal{A}^\sharp \to \mathcal{A}_1^\sharp$. Send a dyad to the monad

$$C_- \xhookrightarrow{(\alpha_-, -\beta_-)} A \oplus B \xrightarrow{\begin{pmatrix}\alpha_+ \\ \beta_+\end{pmatrix}} C_+.$$

For the functor $\mathcal{A}_1^\sharp \to \mathcal{A}_2^\sharp$, send a monad Q_1 to

$$\operatorname{Ker}(\alpha_+) \hookrightarrow A \oplus B \oplus \mathrm{H}(Q_1) \twoheadrightarrow \operatorname{Coker}(\alpha_-).$$

□

A key observation is that on $\mathcal{A}_2^\sharp$, here is a natural auto-equivalence $r : \mathcal{A}_2^\sharp \to \mathcal{A}_2^\sharp$, sending Q_2 to $r(Q_2)$, which is

$$D_- \to A \oplus B^2 \oplus B^1 \to D_+.$$

Our equivalences "transport" this to an auto-equivalence of $\mathcal{A}^\sharp$, which we call *reflection.*A dyad

$$\begin{array}{ccc} C_- & \xhookrightarrow{\alpha_-} & A \\ \downarrow{\scriptstyle \beta_-} & & \vdots\twoheaddownarrow{\scriptstyle \alpha_+} \\ B & \xrightarrow{\beta_+} & C_+ \end{array}$$

is sent by r to

$$\begin{array}{ccc} \operatorname{Ker}(\alpha_+) & \hookrightarrow & A \\ \downarrow{\scriptstyle \beta'_-} & & \twoheaddownarrow \\ \mathrm{H}(Q_1) & \xrightarrow{\beta'_+} & \operatorname{Coker}(\alpha_-) \end{array}$$

where

$$\beta'_- = \operatorname{Ker}(\alpha_+) \hookrightarrow \operatorname{Ker}(\alpha_+, \beta_+) \twoheadrightarrow \mathrm{H}(Q_1)$$
$$\beta'_+ = \operatorname{Ker}(\alpha_-, -\beta_-) \twoheadrightarrow B \to \operatorname{Coker}(\alpha_-).$$

Exercise 1. Check that any Barth or C-M monad is naturally special. Compute the reflection functor on the corresponding dyads.

Exercise 2. Recall the moduli space $\mathcal{M}(n, 1)$ of torsion-free coherent sheaves on $\mathbb{P}^2_{\mathbb{C}}$ of rank one, framed over ℓ_∞. In other words, we specify $\phi : i^*\mathscr{F} \xrightarrow{\sim} \mathscr{O}_{\mathbb{P}^1}$. We claim that $\mathcal{M}(n, 1) \simeq \operatorname{Hilb}_n(\mathbb{C}^2)$. To see this, first identify $\operatorname{Hilb}_n(\mathbb{C}^2)$ with the space of isomorphism classes of triples (X, Y, i) where $X, Y \in M_n(\mathbb{C})$, $i \in \mathbb{C}^n$, $[X, Y] = 0$, and i generates $\mathbb{C}^2$ as a $\mathbb{C}[X, Y]$-module. This is clearly just the space of isomorphism classes of (V, i), where V is an n-dimensional $\mathbb{C}[X, Y]$-module and $i \in V$ is a cyclic vector. This in turn is isomorphism classes of surjections $\mathbb{C}[X, Y] \twoheadrightarrow V$. Recall Barth's description of $\mathscr{F} \in \mathcal{M}(n, 1)$ in terms of monads. Every such sheaf is the cohomology of a monad of the form

$$\mathscr{O}_{\mathbb{P}^2}(-1) \otimes V \xrightarrow{m} (\mathscr{O}_{\mathbb{P}^2} \otimes V)^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^2} \xrightarrow{m} \mathscr{O}_{\mathbb{P}^2}(1) \otimes V.$$

Via the restriction functor $j^* : \mathsf{coh}(\mathbb{P}^2_{\mathbb{C}}) \to \mathsf{coh}(\mathbb{C}^2) \simeq \mathsf{mod}(\mathbb{C}[X, Y])$, this corresponds to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_0 \otimes V & \xrightarrow{m} & (A_0 \otimes V)^{\oplus 2} \oplus A_0 & \longrightarrow & A_0 \otimes V & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \widetilde{\pi}} & & \downarrow{\scriptstyle \pi} & & \\ 0 & \longrightarrow & 0 & \longrightarrow & A_0 & \longrightarrow & V & \longrightarrow & 0 \end{array}$$

where we write $A_0 = \mathbb{C}[X,Y]$. Everything comes down to showing that $(\widetilde{\pi}, \pi)$ is a quasi-isomorphism of A_0-modules.

Exercise 3. Replace $A_0 = \mathbb{C}[x,y] = \mathbb{C}\langle x,y\rangle/[x,y]$ by the Weyl algebra $A_\hbar = \mathbb{C}\langle x,y\rangle/([x,y] = \hbar)$. In fact $A_\hbar \simeq A_1$ for $\hbar \neq 0$. We replace $\mathrm{Hilb}_n(\mathbb{C}^2)$ with the C-(?)-Moeser space C_n of quadruples (X,Y,i,j) satisfying $[X,Y] + \hbar = ij$. We have an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A\otimes V & \longrightarrow & (A\otimes V)^{\oplus 2}\oplus A & \longrightarrow & A\otimes V & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle\widetilde{\pi}} & & \downarrow{\scriptstyle\pi} & & \\ 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & V & \longrightarrow & 0. \end{array}$$

The claim is that there is such a quasi-isomorphism in the category $\mathsf{mod}_\infty(A)$ of "A_∞-modules over A."

This is worked out in the paper [BC07]. It reflects the fact (from commutative algbra) that any ideal in $\mathbb{C}[x,y]$ is isomorphic to a *unique* ideal of finite codimension. So we can think of $\coprod_{n\geqslant 0} \mathrm{Hilb}_n(\mathbb{C}^2)$ as the moduli space of *all* ideals in $\mathbb{C}[x,y]$. Similarly, we could interpret $\coprod_n C_n$ as the space of "isomorphism classes of right ideals in $\bigoplus 2^{A_\hbar}$."

6 Abstract Nearby Cycles functor

6.1 The setup

Consider a general recollement of triangulated categories:

$$\mathsf{D}_Z \rightleftarrows \mathsf{D}_X \rightleftarrows \mathsf{D}_U$$

We recall the following

Theorem 6.1.1 ([BBD82])**.** *For any two t-structures* $(\mathsf{D}_Z^{\leq 0}, \mathsf{D}_Z^{\geq 0})$ *and* $(\mathsf{D}_U^{\leq 0}, \mathsf{D}_U^{\geq 0})$ *There exists a unique t-structure* $(\mathsf{D}_X^{\leq 0}, \mathsf{D}_X^{\geq 0})$ *compatible with both.*

We denote by $\mathsf{Perv}_Z = \mathsf{D}_Z^{\leq 0} \cap \mathsf{D}_Z^{\geq 0}$, $\mathsf{Perv}_X = \mathsf{D}_X^{\leq 0} \cap \mathsf{D}_X^{\geq 0}$, $\mathsf{Perv}_U = \mathsf{D}_U^{\leq 0} \cap \mathsf{D}_U^{\geq 0}$ and call them the abelian categories of perverse sheaves.

Note that for any $M \in \mathsf{Perv}_X$, we have:

$$i^! M \in \mathsf{D}_Z^{\geq 0}$$

$$i^* M \in \mathsf{D}_Z^{\leq 0}$$

Definition 6.1.2. $M \in \mathsf{Perv}_X$ *is called a* tilting perverse sheaf *if*

$$i^* M \in \mathsf{Perv}_Z \quad \textit{and} \quad i^! M \in \mathsf{Perv}_Z$$

Lemma 6.1.3. *Given any* $M \in \mathsf{Perv}_X$*, then* M *is tilting perverse if and only if both of the followings hold:*

1. j_*j^*M *and* $j^!j_!M$ *are both in* Perv_X

2. *The adjunction maps*
$$M \to j_*j^*M \quad \textit{is surjective}$$
$$j_!j^*M \to M \quad \textit{is injective}$$

Proposition 6.1.4. *Let* $M_U \in \mathsf{Perv}_U$ *be such that* $j_*(M_U) \in \mathsf{Perv}_X$ *and* $j_!(M_U) \in \mathsf{Perv}_X$. *Then there exists a tilting perverse sheaf* $M \in \mathsf{Perv}_X$ *such that* $j^*M \cong M_U$.

Sketch of proof. Define $A, B \in \mathsf{Perv}_X$ by the following exact sequence:

$$0 \to A \to j_!M_U \xrightarrow{\phi_{M_U}} j_*M_U \to B \to 0$$

Case 1. $\mathrm{Ext}^2(B, A) = 0$

Then we have a three-step filtration, and this allows us to construct a minimal tilting extension
$$M = \Xi(M_U)$$

Case 2. $\mathrm{Ext}^2(B, A) \neq 0$

This case requires more work. □

Denote by $\mathsf{Perv}^{\mathsf{mte}}(U)$ the full subcategory of $\mathsf{Perv}(U)$ consisting of objects M_U which have a minimal tilting extension $\Xi(M_U)$.

Then $\Xi(M_U)$ is functorial in $M_U \in \mathsf{Perv}^{\mathsf{mte}}(U)$.

Denote by $\mathsf{Perv}^{\mathsf{mte}}(X)$ the full subcategory of $\mathsf{Perv}(X)$ consisting of those objects M such that $j^*M \in \mathsf{Perv}^{\mathsf{mte}}(U)$.

Definition 6.1.5. *Define a category* $\mathsf{Glue}(Z, U)$ *whose objects are quadruples* $(M_U, \Phi, \alpha, \beta)$, *where:*

1. $M_U \in \mathsf{Perv}^{\mathsf{mte}}(U)$

2. $\Phi \in \mathsf{Perv}(Z)$

3. $\alpha : i^!\Xi(M_U) \to \Phi$ *is a morphism in* $\mathsf{Perv}(Z)$

4. $\beta : \Phi \to i^*\Xi(M_U)$ *is a morphism in* $\mathsf{Perv}(Z)$

5. $\beta \circ \alpha = \tau : i^!\Xi(M_U) \to i^*\Xi(M_U)$ *is the canonical map*

Theorem 6.1.6. *There is a canonical commutative diagram of equivalences of categories:*

$$\begin{array}{ccc} {\mathsf{Perv}^{\mathsf{mte}}}_X & \longrightarrow & \mathsf{Glue}(Z, U) \\ {\scriptstyle Q}\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\downarrow{\scriptstyle R} \\ \mathsf{diads} & \underset{\sim}{\xrightarrow{r}} & \mathsf{diads} \end{array}$$

where:

1. $r : \mathsf{diads} \to \mathsf{diads}$ *is the reflection functor.*

2. *Q is defined by the following:*

   Given $M \in \mathsf{Perv}^{\mathsf{mte}}(X)$, define the diad

$$Q(M) := \left[\begin{array}{ccccc} & & \Xi(j^*M) & & \\ & \overset{can}{\hookrightarrow\!\nearrow} & & \overset{can}{\searrow} & \\ j_!j^*M & & & & j_*j^*M \\ & \underset{can}{\searrow} & & \underset{can}{\nearrow} & \\ & & M & & \end{array}\right]$$

3. *R is defined by the following:*

   Given $(M_U, \Phi, \alpha, \beta) \in \mathsf{Glue}(Z, U)$, define the diad

$$R(M_U, \Phi, \alpha, \beta) := \left[\begin{array}{ccccc} & & \Xi(M_U) & & \\ & \overset{can}{\hookrightarrow\!\nearrow} & & \overset{can}{\searrow} & \\ i_*i^!(\Xi(M_U)) & & & & i_*i^*\Xi(M_U) \\ & \underset{i_*\alpha}{\searrow} & & \underset{i_*\beta}{\nearrow} & \\ & & i_*\Phi & & \end{array}\right]$$

6.2 Multiplicative Preprojective Algebra Associated to a Quiver

The main references are [CBS06] and [BRT13].

Let $Q = (I, Q)$ be a finite quiver, where I denotes the set of vertices and Q denotes the set of arrows.

Denote $\overline{Q}$ as the double quiver of Q. i.e.,

1. Vertices are the same : $I(Q) = I(\overline{Q})$

2. Arrows are doubled in the opposite direction.

Any construction on a quiver that does not depend on orientation should be a construction on the double quiver

For any arrow $a \in Q$, we its double (in the opposite direction) by $a^* \in \overline{Q}$. By defining $a^{**} = a$, we have an involution

$$\begin{aligned} (-)^* : k(\overline{Q}) &\to k(\overline{Q}) \\ a &\mapsto a^* \end{aligned}$$

Also, define $\epsilon : \overline{Q} \to \{\pm 1\}$ by

$$\epsilon(a) = \begin{cases} 1 & : a \in Q \\ -1 & : a^* \in Q \end{cases}$$

Given two sets of parameters:

$$\{q_v\}_{v \in I}, \qquad \{\hbar_a\}_{a \in Q}$$

Extend this set of parameters to $\overline{Q}$ by setting $\hbar_{a^*} = \hbar_a$

Definition 6.2.1. *The multiplicative preprojective algebra $\Lambda^{q,\hbar}(Q)$ associated to the quiver Q and the sets of parameters $q = \{q_v\}_{v \in I}, \hbar = \{\hbar_a\}_{a \in Q}$ is defined as the algebra homomorphism that is initial among all algebra homomorphism $\phi : k(\overline{Q}) \to R$ satisfying:*

1. *$\phi(aa^* + \hbar_a)$ is a unit in R*
2. $\phi\left(\prod_{a \in \overline{Q}} (aa^* + \hbar_a)^{\epsilon(a)} - \sum_{v \in I} q_v e_v\right) = 0$

Lemma 6.2.2. *Such $\Lambda^{q,\hbar}(Q)$ always exists for any quiver Q and any sets of parameters $q = \{q_v\}_{v \in I}, \hbar = \{\hbar_a\}_{a \in Q}$.*

In [CBS06], they defined the preprojective algebra only for the case $\hbar_a = 1$ for all $a \in Q$.

Example 6.2.3. Consider the one-loop quiver. (i.e., the quiver $Q = Q_{loop}$ with one vertex v and one arrow a).

Write $q_v = q$ and $\hbar_a = \hbar$.

Then the multiplicative preprojective algebra is:

$$k(\overline{Q}) \to \Lambda^{q,\hbar}(Q) = \frac{k\langle x, y, (\hbar + xy)^{-1}\rangle}{(xy - qyx - \hbar)}$$

Note that when $\hbar = 0$, we have:

$$\Lambda^{q,0}(Q) = \frac{k\langle x^{\pm 1}, y^{\pm 1}\rangle}{(xy - qyx)}$$

while for $\hbar = 1$, we have:

$$\Lambda^{q,1}(Q) = \frac{k\langle x, y, (1 + xy)^{-1}\rangle}{(xy - qyx - 1)}$$

Example 6.2.4. Consider the one-arrow quiver.

$$Q = \left[\bullet_\infty \xrightarrow{a} \bullet_0\right]$$

Then the multiplicative preprojective algebra is:

$$\Lambda^{q,\hbar}(Q) = \frac{k\langle e_0, e_\infty, a, a^*, (aa^*+\hbar)^{-1}, (a^*a+\hbar)^{-1}\rangle}{(aa^*+\hbar = q_0 e_0, \quad (a^*a+\hbar)^{-1} = q_\infty e_\infty, \quad e_0 e_\infty = e_\infty e_0 = 0, \quad e_i^2 = e_i)}$$

while for $\hbar = 1$, we have:

$$\Lambda^{q,1}(Q) = \frac{k(\overline{Q})}{(e_\infty + a^*a = q_\infty e_\infty, \quad e_0 + aa^* = q_0 e_0)}$$

Theorem 6.2.5. *We have a canonical bijection:*

$$\prod_{(q_0, q_\infty \in \mathbb{C}\times\mathbb{C}^*)} \mathsf{Rep}_{\mathsf{f.d.}}\left(\Lambda^{q,1}(\left[\bullet_\infty \xrightarrow{a} \bullet_0\right])\right)$$
$$\cong \{\mathcal{F}^\bullet \in \mathsf{Perv}_{\mathbb{C}} : j^*(\mathcal{F}^\bullet) \textit{ has scalar monodromy}\}$$

Example 6.2.6. Let Q_∞ be the quiver with two vertices 0 and ∞, one loop x from 0 to 0, and one arrow i from ∞ to 0.

Denote their opposite arrows by $y = x^*$ and $j = i^*$.

Set $\hbar_x = \hbar_y = 0$, $\hbar_i = \hbar_j = 1$, $(q_0, q_\infty) = (q, q^{-n})$ for some $q^n \neq 1$.

Now we have:

$$\begin{array}{ccccc} k(\overline{Q_\infty}) & \xrightarrow{e_\infty \mapsto 0} & k(\bar{(}Q_{loop})) & & \\ \downarrow & & \downarrow & & \\ \Lambda^{q,\hbar}(Q_\infty) & \xrightarrow{i} & \Lambda^{q,\hbar}(Q_{loop}) & = & \frac{k\langle x^{\pm 1}, y^{\pm 1}\rangle}{(xy-qyx)} \end{array}$$

Theorem 6.2.7. *There is a natural recollement set-up:*

$$\mathsf{D}^b(\mathsf{Mod}{-}\Lambda^{q,\hbar}(Q_{loop})) \longrightarrow \mathsf{D}^b(\mathsf{Mod}{-}\Lambda^{q,\hbar}(Q_\infty)) \longrightarrow \mathsf{D}^b(\mathsf{Mod}{-}\mathbb{U}^{q,\hbar}(Q_{loop}))$$

Moreover, the map

$$i^* : \coprod_{n=0}^{\infty} \mathsf{Rep}\left(\Lambda^{q,\hbar}(Q_\infty), \underline{n} = (1,n)\right) //GL(\underline{n})$$
$$\xrightarrow{\sim} \{\textit{Isomorphism class of ideals of } A_q\}$$

is the Calogero-Moser map.

Appendix A

Miscellaneous topics

1 Characteristic classes of representations (after Quillen)

Let A be an associative unital ring. Consider the (*discrete*) group $\mathrm{GL}_n(A) = M_n(A)^\times$ of invertible $n \times n$ matrices with coefficients in A. We call $\mathrm{GL}_n(A)$ the *general linear group over A*. We have embeddings $\mathrm{GL}_n(A) \to \mathrm{GL}_{n+1}(A)$ given by

$$\theta \mapsto \begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix}.$$

We define

$$\mathrm{GL}(A) = \mathrm{GL}_\infty(A) = \varinjlim \mathrm{GL}_n(A).$$

Put

$$\mathrm{H}_\bullet\left(\mathrm{GL}(A), k\right) = \bigoplus_{i \geqslant 0} \mathrm{H}_i\left(\mathrm{GL}(A), k\right) = \mathrm{H}_\bullet(\mathrm{BGL}(A), k),$$

where $\mathrm{BGL}(A)$ is the classifying space of $\mathrm{GL}(A)$ (see 2.5.3). The diagonal map $\mathrm{GL}_n(A) \to \mathrm{GL}_n(A) \times \mathrm{GL}_n(A)$ gives $\mathrm{H}_\bullet(\mathrm{GL}(A), k)$ the structure of a coassociative cocommutative coalgebra over k.

Note that $\mathrm{H}_\bullet(\mathrm{GL}(A), k)$ has also the structure of a graded-commutative k-algebra. There is a natural map "taking direct sum" $\oplus : \mathrm{GL}_n(A) \times \mathrm{GL}_m(A) \to \mathrm{GL}_{n+m}(A)$, defined by

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},\ A \in \mathrm{GL}_n(A),\ B \in \mathrm{GL}_m(A)$$

It induces a map $\oplus : \mathrm{GL}(A) \times \mathrm{GL}(A) \to \mathrm{GL}(A)$, which in turn induces a map $\mathrm{BGL}(A) \times \mathrm{BGL}(A) \to \mathrm{BGL}(A)$. The latter map turns out to be associative and commutative up to homotopy. It follows that $\mathrm{H}_\bullet(\mathrm{GL}(A), k)$ naturally has the structure of a graded k-algebra.

We want to define a "functor of points" $\mathsf{GrComAlg}_k \to \mathsf{Set}$ for which the representing object is $\mathrm{H}_\bullet\left(\mathrm{GL}(A), k\right)$. First we need to extend the notion of representation. Let G be a (discrete) group. We can think of a group G as a category (actually, groupoid) with one object.

Let $\mathcal{A}$ be any (small) additive category. Let $\mathsf{Isom}(\mathcal{A})$ be the groupoid of isomorphisms in $\mathcal{A}$. That is, the only morphisms in $\mathcal{A}$ which we keep in $\mathsf{Isom}(\mathcal{A})$ are isomorphisms. Let $S = \pi_0(\mathsf{Isom}(\mathcal{A}))$ be the set of isomorphism classes of objects in $\mathcal{A}$.

Definition 1.0.8. *A* representation *of G in $\mathcal{A}$ is a functor $\rho : G \to \mathsf{Isom}(\mathcal{A})$.*

We denote by $\mathsf{Isom}(G, \mathcal{A})$ the groupoid $\mathsf{Isom}(\mathsf{Fun}(G, \mathcal{A}))$ of isomorphism classes of representations of G in $\mathcal{A}$. That is,

$$\mathsf{Isom}(G, \mathcal{A}) = \coprod_{s \in S} \mathrm{Hom}(G, \mathrm{Out}(P_s)),$$

where P_s is a representative of the class of $s \in S$.

Exercise Check that if G and G' are groups then $\mathsf{Isom}(\mathsf{Fun}(G, G'))$ is the quotient of $\mathrm{Hom}(G, G')$ by inner automorphisms in G', i.e.

$$\mathsf{Isom}(\mathsf{Fun}(G, G')) = \mathrm{Hom}(G, G')/\,\mathrm{Inn}(G').$$

Definition 1.0.9. *We say that representation $\rho : G \to \mathsf{Isom}(\mathcal{A})$ is* trivial *if $\forall f \in G$, $\rho(f) = \mathrm{id} \in \mathrm{Hom}_{\mathsf{Isom}(\mathcal{A})}(\rho(*), \rho(*))$, where $* \in \mathrm{Ob}(G)$ is the unique object of G.*

Definition 1.0.10. *Two representations $E, E' \colon G \to \mathsf{Isom}(\mathcal{A})$ are* stably isomorphic *if $E \oplus \varepsilon \simeq E' \oplus \varepsilon'$, where ε and ε' are trivial representations.*

Define $\mathsf{St}(G, \mathcal{A}) = \mathsf{Isom}(G, \mathcal{A})/\sim$, where $E \sim E'$ if E and E' are stably isomorphic.

Recall that $S = \pi_0(\mathsf{Isom}(\mathcal{A}))$ is the set of isomorphism classes of objects in $\mathcal{A}$. The set S is naturally a commutative monoid, via $[P] + [Q] = [P \oplus Q]$. We denote by $S \ltimes S$ the category with $\mathrm{Ob}(S \ltimes S) = S$, and

$$\mathrm{Hom}_{S \ltimes S}(s, s') = \{t \in S : s' = s + t\}.$$

There is a natural functor $[G, \mathrm{Aut}(-)] : S \ltimes S \to \mathsf{Set}$, which sends an object s to $[G, \mathrm{Aut}(P_s)]$. Given a morphism t from s to s' (i.e. $s' = s + t$), there is a natural map $u \mapsto i \oplus \mathrm{id}$ from $\mathrm{Aut}(P_s)$ to $\mathrm{Aut}(P_s \oplus P_{s'}) \simeq \mathrm{Aut}(P_{s'})$. Denote by t_* the induced morphism $[G, \mathrm{Aut}(P_s)] \to [G, \mathrm{Aut}(P_{s'})]$.

Lemma 1.0.11. *There is a natural isomorphism $\mathsf{St}(G, \mathcal{A}) \simeq \varinjlim_{S \ltimes S}[G, \mathrm{Aut}(-)]$.*

We should think of $\mathrm{Hom}(G, \mathrm{Out}(P))$ as a functor from (S, S) to Set, where $s \mapsto [G, \mathrm{Aut}(P_s)] = \mathrm{Hom}(G, \mathrm{Out}(P_s))$. An arrow $(s, t) \colon s \to s'$ is send to $u \mapsto u \oplus \mathrm{id}$ as a function $[G, \mathrm{Aut}(P_s)] \to [G, \mathrm{Aut}(P_s \oplus P_t)]$.

Example 1.0.12. Let A be an associative unital ring. Let $\mathcal{A}$ be the category of all finitely-generated projective (right) A-modules. A general representation of G in $\mathcal{A}$ will be a homomorphism $G \to \mathrm{Aut}_A(P)$ for some P. Certainly the category $\mathcal{A}$ contains the objects $A^{\oplus n}$. An automorphism of $A^{\oplus n}$ is just an element of $\mathrm{GL}_n(A)$, so a representation of G in $A^{\oplus n}$ is just a homomorphism $G \to \mathrm{GL}_n(A)$.

There is a monoid homomorphism $(\mathbb{N},+) \to (S,\oplus)$ given by $n \mapsto [A^{\oplus n}]$. This induces a functor between groupoids $F : \mathbb{N} \ltimes \mathbb{N} \to S \ltimes S$. It turns out that this functor is cofinal. In other words, for every $s \in S \ltimes S$, there exists $n \geqslant 1$ and a morphism $f : s \to F(n)$ in $S \ltimes S$. Indeed, if $s = [P]$ for some projective P, then since P is projective there exists another projective Q such that $P \oplus Q \simeq A^{\oplus n}$ for some n. Setting $f = [Q]$, we have $f : s \to F(n)$.

It follows that colimits over the category $S \ltimes S$ are the same as colimits over the subcategory $\mathbb{N} \ltimes \mathbb{N}$. This gives us the much more manageable description of $\mathsf{St}(G,\mathcal{A})$:

$$\mathsf{St}(G,\mathcal{A}) = \varinjlim_n [G, \mathrm{GL}_n(A)].$$

There is a canonical map $\rho : \varinjlim_n [G, \mathrm{GL}_n(A)] \to [G, \mathrm{GL}(A)]$. Concretely, let $E : G \to \mathrm{Aut}(P)$ represent an element of $\mathsf{St}(G,\mathcal{A})$. Choose Q with $P \oplus Q \simeq A^{\oplus n}$. We let ρ_E be the composite

$$G \xrightarrow{E} \mathrm{GL}_A(P) \to \mathrm{GL}_A(P \oplus Q) = \mathrm{Aut}(A^{\oplus n}) = \mathrm{GL}_n(A) \hookrightarrow \mathrm{GL}(A).$$

Remark 1.0.13. Our construction is parallel to the topological situation. Namely, let X be a paracompact topological space. We replace "representations of G in $\mathcal{A}$" by complex vector bundles on X. Let $\mathsf{VB}(X)$ be the set of isomorphism classes of vector bundles on X. There is an obvious decomposition $\mathsf{VB}(X) = \coprod_n \mathsf{VB}_n(X)$, where $\mathsf{VB}_n(X)$ consists of isomorphism classes of n-dimensional vector bundles on X. In fact, $\mathsf{VB}_n(X) = [X, BU_n]$, where BU_n is the classifying space of the group U_n. Instead of $\mathsf{St}(G,\mathcal{A})$, we think of $\widetilde{\mathrm{K}}_0(X) := [X, BU_\infty]$, the (reduced) topological K-theory. The analogue of our map $\rho : \mathsf{St}(G,\mathcal{A}) \to [G, \mathrm{GL}(A)]$ is the map $[X, \coprod_n \mathsf{VB}_n(X)] \to \widetilde{\mathrm{K}}_0(X)$, which is an isomorphism if X is compact or a finite-dimensional CW complex.

Let $M = \bigoplus_{i\geqslant 0} M_i$ be a graded abelian group. For an arbitrary group G, define

$$\underline{\mathrm{H}}^0(G,M) = \prod_{i\geqslant 0} \mathrm{H}^i(G, M_i),$$

where we view each M_i as a trivial G-module.

Definition 1.0.14. *A* characteristic class (of representations) *in $\mathcal{A}$ with coefficients in M is a natural transformation of contravariant functors*

$$\theta : \mathsf{Isom}(-,\mathcal{A}) \to \underline{\mathrm{H}}^0(-,M)$$

between functors $\mathsf{Grp}^\circ \to \mathsf{Set}$. *We call θ* stable *if $\theta(E \oplus \varepsilon) = \theta(E)$ for any trivial representation ε.*

Now suppose M is a graded-commutative ring. Recall that graded commutativity means $ab = (-1)^{|a|\cdot|b|} ba$ for any homogeneous $a, b \in M$. Then $\underline{\mathrm{H}}^0(G,M)$ has a natural ring structure given by the cup product $\smile : \mathrm{H}^i(G,M_i) \times \mathrm{H}^j(G,M_j) \to \mathrm{H}^{i+j}(G,M_{i+j})$. Given cocycles $f_1 : G^i \to M_i$ and $f_2 : G^j \to M_j$, the cup-product $f_1 \smile f_2$ is given by

$$[f_1 \smile f_2](g_1, \ldots, g_{i+j}) = f_1(g_1, \ldots, g_i) f_2(g_{i+1}, \ldots, g_{i+j}).$$

Definition 1.0.15. *A characteristic class θ is called* exponential *if $\theta(0) = 1$ and $\theta(E_1 \oplus E_2) = \theta(E_1) \smile \theta(E_2)$.*

Let $E : G \to \mathsf{Isom}(\mathcal{A})$ be a representation. We apply to E the homology functor

$$\mathrm{H}_\bullet(E) : \mathrm{H}_\bullet(G) \to \mathrm{H}_\bullet(\mathsf{Isom}(\mathcal{A})) = \mathrm{H}_\bullet(\mathsf{Nerve}(\mathsf{Isom}(\mathcal{A})), \mathbb{Z}).$$

Observe that $\underline{\mathrm{H}}^0(G, M) = \underline{\mathrm{Hom}}^0(\mathrm{H}_\bullet(G), M)$.

Let k be a field. Then we have constructed a functor $\mathsf{GrComAlg_k} \to \mathsf{Set}$ which assigns to M the set of stable exponential characteristic classes with coefficients in G.

Theorem 1.0.16. *Assume that X is "nice." Then $\mathrm{H}_\bullet(\mathsf{Isom}(\mathcal{A}))$ represents this functor.*

It turns out that $\mathrm{H}_\bullet(\mathsf{Isom}(\mathcal{A})) = \mathrm{H}_\bullet(\mathrm{GL}_\infty(A), k)$. So, at least morally, $\mathrm{H}_\bullet(\mathrm{GL}(A), k)$ is the "universal algebra of stable exponential classes." More concretely, given such a characteristic class $\theta : \mathsf{Isom}(-, M) \to \underline{\mathrm{H}}^0(-, M)$, then given any $f : M \to M'$, we have $f_*\theta : \mathsf{Isom}(-, M') \to \underline{\mathrm{H}}^0(-, M) \xrightarrow{f} \underline{\mathrm{H}}^0(-, M')$. One says that $f_*\theta$ is induced from θ by f. The theorem asserts that there is a natural bijection between stable exponential characteristic classes in $\mathcal{A}$ with coefficients in M and algebra maps $\mathrm{H}_\bullet(\mathsf{Isom}(\mathcal{A})) \to M$.

2 Generalized manifolds

The ideas in this example come from Freed and Hopkins' paper [FH13].

Let Man be the category of smooth (finite-dimensional) manifolds and smooth maps. A *generalized manifold* is a sheaf on Man. That is, a generalized manifold is a functor $\mathscr{F} : \mathsf{Man}^\circ \to \mathsf{Set}$ such that whenever $M \in \mathrm{Ob}\,\mathsf{Man}$ has an open cover $\{U_\alpha\}_{\alpha \in I}$, the following diagram is an equalizer:

$$\mathscr{F}(M) \longrightarrow \prod_{\alpha \in I} \mathscr{F}(U_\alpha) \rightrightarrows \prod_{\alpha, \beta \in I} \mathscr{F}(U_\alpha \cap U_\beta)$$

The main idea is to try to extend differential geometry to the category of generalized smooth manifolds. One example is differential forms. We have the functor $\Omega^\bullet : \mathsf{Man}^\circ \to \mathsf{Set}$, which assigns to a manifold M the de Rham complex $\Omega^\bullet(M) = \bigwedge \Omega^1(M)$.

The Yoneda embedding $\mathsf{Man} \hookrightarrow \widetilde{\mathsf{Man}}$ allows us to treat any manifold as a generalized manifold. Recall that the Yoneda lemma shows that there is a natural bijection

$$\mathrm{Hom}_{\widetilde{\mathsf{Man}}}(h_X, \mathscr{F}) \simeq \mathscr{F}(X).$$

Definition 2.0.17. *For any generalized manifold $\mathscr{F}$, we can define the de Rham complex of $\mathscr{F}$ by*

$$\Omega^\bullet(\mathscr{F}) = \mathrm{Hom}_{\widetilde{\mathsf{Man}}}(\mathscr{F}, \Omega^\bullet).$$

The Yoneda lemma tells us that this definition agrees with the usual one if $\mathscr{F}$ is representable. Can we compute the de Rham cohomology of $\mathscr{F}$? We have a direct-sum decomposition of functors on generalized manifolds, $\Omega^\bullet = \bigoplus_q \Omega^q$. Let $\mathscr{F} = \Omega^1 : M \mapsto \Omega^1(M)$. It is a highly nontrivial theorem that $\Omega^\bullet(\Omega^1)$ is isomorphic to

$$\mathbb{R} \xrightarrow{0} \mathbb{R} \xrightarrow{1} \mathbb{R} \xrightarrow{0} \cdots$$

Thus $\mathrm{H}^\bullet_{\mathrm{dR}}(\Omega^1) = \mathbb{R}$, concentrated in degree zero. (This is Theorem 7.19 in [FH13].)

Recall that (by definition) $\Omega^q(\Omega^1) = \mathrm{Hom}_{\widetilde{\mathsf{Man}}}(\Omega^1, \Omega^q)$. So $\tau \in \Omega^q(\Omega^1)$ should be thought of as a natural way to associate q-forms to 1-forms. For M fixed, $\omega \in \Omega^1(M)$, we have $\tau_M(\omega) \in \Omega^q(M)$, such that for any $f : M' \to M$, $\tau(f^*\omega) = f^*\tau(M)$.

In his paper [Che77] Chen generalized notion of differential forms to loop spaces. Namely, take X to be a smooth manifold. Consider the path space $P(X)$, which is a set of all smooth paths $\gamma\colon I \to X$ endowed with compact-open topology. Then $P(X)$ is not strictly speaking a smooth manifold, but we can still define what is a smooth map $N \to P(X)$ and what is differential form on $P(X)$.

First, any map $f\colon N \to P(X)$ defines unique map $\tilde{f}\colon N \times I \to X$ given by $\tilde{f}(n,t) = f(n)(t)$. We call a map f *smooth* if he correspondent $\tilde{f}$ is smooth.

We can define differential q-form ω on $P(X)$ by assigning to each smooth map $f_N\colon N \to P(X)$ a q-form on N, denoted by $f_N^*\omega$, in such a way that for any morphism of manifolds $g\colon M \to N$ the compatibility condition $g^*\left(f_N^*\omega\right) = f_M^*\omega$ is satisfied. This notion allows us to define in the similar fashion de Rham algebra of differential forms on $P(X)$.

It turns out that the Chen's construction is coherent with the notion of generalized manifold given in [FH13]. Indeed, the path space $P(X)$ is just an exponent object X^I in $\widetilde{\mathsf{Man}}$, i.e. it is defined by

$$\mathrm{Hom}_{\widetilde{\mathsf{Man}}}(N \times I, X) \simeq \mathrm{Hom}_{\widetilde{\mathsf{Man}}}(N, X^I),$$

where N, X and I are viewed as elements of $\widetilde{\mathsf{Man}}$ via Yoneda embedding. By definition 2.0.17, $\Omega^\bullet(X^I) = \mathrm{Hom}_{\widetilde{\mathsf{Man}}}(X^I, \Omega^\bullet)$. Having a morphism of functors $\alpha \in \Omega^\bullet(X^I)$ is the same as having a compatible family of maps $\alpha_N\colon X^I(N) \to \Omega^\bullet(N)$. But this is exactly how Chen defined differential forms on X^I. So the two constructions are coherent.

Now take forms $\omega_1, \ldots, \omega_k$ on X of degrees $s_1, \ldots, s_k$ respectively. For any smooth map $f\colon N \to P(X)$ these forms define pull-back forms $\tilde{f}^*(\omega_1), \ldots, \tilde{f}^*(\omega_k)$ on $N \times I$. Each of these pull-backs can be written as $\tilde{f}^*(\omega_i) = \omega_i'(n) + \omega_i''(n)dt_i$. Then we can integrate the form $\omega_1''(n_1)\ldots\omega_k''(n_k)dt_1\ldots dt_k$ over the simplex $\Delta_{k-1} = \{(t_1, \ldots, t_k) \in I^k \mid t_1 \leq \cdots \leq t_k \text{ and } \sum\limits_i t_i = 1\}$ to obtain a $(s_1-1)+\cdots+(s_k-1)$-form ω on $N \times \cdots \times N = N^k$. Then using the diagonal embedding $i\colon N \hookrightarrow N^k$, we get a $(s_1-1)+\cdots+(s_k-1)$-form $i^*(\omega)$ on N. The map that associates to the set of forms $\omega_1, \ldots, \omega_k$ the form ω is called *iterated path integral* $\iint$.

One of the main results in Chen's paper is that $\iint$ induces a morphism of Hopf algebras $B^c(\Omega^\bullet(X)) \to \Omega^\bullet(P(X))$ which turns out to be a quasi-isomorphism. Here $B^c(\Omega^\bullet(X))$ denotes the cyclic bar construction of the algebra $\Omega^\bullet(X)$.

3 Quivers and path algebras

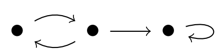
Path algebras of quivers provide a nice class of examples of non-commutative algebras. We will use the machinery of classical derived functors to see why quivers can be viewed as "noncommutative spaces".

3.1 Basic definitions

First we recall (see 1.1.6) basic definitions about quivers. Also, it will fix the notations.

Definition 3.1.1. *A* quiver *is just a directed graph with finitely many vertices. More formally, a quiver is a quadruple $Q = \{Q_0, Q_1, s, t\}$, where Q_0 is the (finite) set of vertices, Q_1 is the set of arrows, and $s, t : Q_1 \to Q_0$ are the "incidence maps" assigning to an arrow its source (resp. target).*

For example, a quiver may look like



Definition 3.1.2. *A (nontrivial)* path *in Q, $\boldsymbol{\rho} = \rho_1\rho_2\cdots\rho_m$, is a sequence $\{\rho_1, \ldots, \rho_m\}$ of edges such that $s(\rho_i) = t(\rho_{i+1})$. A trivial path is just a vertex.*

Remark 3.1.3. We can naturally regard any quiver Q as a category $\underline{Q}$, where $\mathrm{Ob}(\underline{Q}) = Q_0$, and $\mathrm{Hom}_{\underline{Q}}(i,j)$ is the set of paths from i to j. Composition in $\underline{Q}$ is concatenation of paths. There is another category that can be naturally associated to quiver, see example 1.1.6 in the Chapter 3.

Remark 3.1.4. One can also think of a quiver as a "finite noncommutative space." In other words, vertices are the "points," and arrows represent some kind of "higher homological link" between points. If A is a finitely generated reduced commutative algebra over an algebraically closed field k of characteristic zero, then we can consider its maximal spectrum $X = \mathrm{mSpec}(A)$, whose points are maximal ideals $\mathfrak{m} \subset A$, or equivalently k-algebra homomorphisms $u : A \to k$. Given points $p, q \in X$, we can consider the corresponding skyscraper sheaves $\mathscr{O}_p, \mathscr{O}_q$. It turns out that $\mathrm{Ext}^i(\mathscr{O}_p, \mathscr{O}_q) = 0$ if $p \neq q$. In contrast, for a quiver Q, the modules corresponding to different vertices can have nontrivial Ext-groups, whose dimensions encode the number of arrows between points. We will make this precise below, see 3.4.3.

Remark 3.1.5. We will usually denote vertices of a quiver simply by natural numbers: $1, 2, \ldots, n$, sometimes ∞. But when we are thinking of vertices as trivial paths, or as elements of $\mathrm{Path}(Q)$ then we will denote them by $e_1, e_2, \ldots, e_n$, or e_∞.

Let $\mathrm{Path}(Q)$ be the set of paths in Q. We can naturally extend s and t to maps $s, t : \mathrm{Path}(Q) \to Q_0$ by

$$s(\boldsymbol{\rho}) = s(\rho_m)$$
$$t(\boldsymbol{\rho}) = t(\rho_1).$$

Definition 3.1.6. *The* length function $\ell : \mathrm{Path}(Q) \to \mathbb{Z}_+$ *is a map where* $\ell(\boldsymbol{\rho})$ *is the number of arrows in* $\boldsymbol{\rho}$*. We put* $\ell(e) = 0$ *if* e *is a vertex.*

3.2 Path algebra of a quiver

Definition 3.2.1. *Let* Q *be a quiver. The* path algebra *of* Q *over* k*, denoted* kQ*, is the* k*-vector space with basis* $\mathrm{Path}(Q)$*. The product on* kQ *is given by*

$$\boldsymbol{\rho} \cdot \boldsymbol{\sigma} = \begin{cases} \boldsymbol{\rho\sigma} & \text{if } t(\boldsymbol{\sigma}) = s(\boldsymbol{\rho}) \\ 0 & \text{otherwise} \end{cases}$$

Example 3.2.2. Let the vertices of Q be $\{1, 2, 3\}$, with arrows $\rho : 1 \to 2$ and $\sigma : 2 \to 3$. Then $\mathrm{Path}(Q) = \{e_1, e_2, e_3, \rho, \sigma, \sigma\rho\}$, so kQ is the 5-dimensional k-algebra spanned by the above elements. Some basic computation yields the following presentation of kQ:

$$kQ = k\langle e_1, e_2, e_3, \rho, \sigma, \sigma\rho : e_{ij} = \delta_{ij} e_i, \rho e_1 = e_2 \rho = \rho, e_1 \rho = \rho e_2 = e_3 \rho = \rho e_3 = \sigma\rho = 0\rangle$$

It is easy to see that $e_1 + e_2 + e_3 = 1$ in kQ.

Example 3.2.3. Let Q be the quiver with a unique edge and vertex. Then $kQ \simeq k[x]$, where the unique vertex corresponds to q, and the edge corresponds to x. Similarly, let Q_n be the unique quiver with a single vertex and n edges. One has $kQ_n = k\langle x_1, \ldots, x_n\rangle$.

Example 3.2.4. In general, if A and B are k-algebras and M is an (A, B)-bimodule, then we write $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ for the algebra, which as a vector space is isomorphic to $A \oplus M \oplus B$, and with multiplication given by

$$\begin{pmatrix} a_1 & m_1 \\ 0 & b_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & m_2 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 & a_1 m_2 + m_1 b_2 \\ 0 & b_1 b_2 \end{pmatrix}.$$

Similarly we define for (B, A)-bimodule N an algebra $\begin{pmatrix} A & 0 \\ N & B \end{pmatrix}$.

If now Q is the *Kronecker quiver* $1 \rightrightarrows 2$, then $kQ \simeq \begin{pmatrix} k & 0 \\ k^{\oplus 2} & k \end{pmatrix}$. The isomorphism is given by

$$e_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_1 \mapsto \begin{pmatrix} 0 & 0 \\ (1,0) & 0 \end{pmatrix}, \quad \rho_2 \mapsto \begin{pmatrix} 0 & 0 \\ (0,1) & 0 \end{pmatrix},$$

Example 3.2.5. Let Q be a quiver with at most one path between any two vertices. Write $n = \#Q_0$. Then $kQ \simeq \{A \in M_n(k) : A_{ij} = 0 \text{ if there is no path } j \to i\}$. For example, if $Q = 1 \to 2 \to \cdots \to n$, then kQ is the algebra of $n \times n$ lower-triangular matrices.

Example 3.2.6. Finally, let Q be the quiver

$$\infty \longrightarrow 0 \circlearrowleft$$

Then $kQ \simeq \begin{pmatrix} k[x] & k[x] \\ 0 & k \end{pmatrix}$. This is called a *framed* 1*-loop quiver*. Path algebra Path(Q) gives an example of the ring which is right Noetherian, but not left Noetherian.

Let Q be a quiver. We write $kQ_0 = \bigoplus_{e \in Q_0} ke \subset kQ$. This is a semisimple k-algebra. Let $kQ_1 = \bigoplus_{\rho : i \to j} k\rho$ be the span of all arrows in Q.

Lemma 3.2.7. *Let Q be a quiver. Then kQ_1 is naturally a kQ_0-bimodule, and $kQ \simeq T_{kQ_0}(kQ_1)$.*

Proof. Recall that if S is a k-algebra and M is an S-bimodule, then the tensor algebra $T_S M$ satisfies the following universal property: given any k-algebra map $f_0 : S \to A$ and an S-bimodule map $f_1 : M \to A$, there is a unique $f : T_S M \to A$ such that $f|_S = f_0$ and $f|_M = f_1$. We apply this to the case $S = kQ_0$, $M = kQ_1$. These both embed into kQ, giving a map $T_S M \to kQ$. Surjectivity follows from the definition of kQ, and injectivity follows from induction on the grading. □

3.3 The structure of the path algebra

First of all, note that $\{e_i : i \in Q_0\}$ is a complete set of orthogonal idempotents in kQ, i.e.

$$\sum_i e_i = 1, \qquad e_i e_j = \begin{cases} e_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Moreover, if $A = kQ$, then Ae_i is the span of all paths starting at i, while $e_j A$ is the span of all paths ending at j. Thus $e_j A e_i$ is the span of all paths starting at i and ending at j. As a left A-module, $A = \bigoplus_i Ae_i$, and as right A-modules, $A = \bigoplus_i e_i A$. This implies that the Ae_i are *projective* left ideals and the $e_i A$ are *projective* right ideals.

Lemma 3.3.1. *Let Q be a quiver and put $A = kQ$.*

(a) For any left A-module M and right A-module N,

$$\mathrm{Hom}_A(Ae_i, M) \simeq e_i M \text{ and } \mathrm{Hom}_A(e_j A, N) \simeq N e_j.$$

(b) If $0 \neq a \in Ae_i$ and $0 \neq b \in e_i A$, then $ab \neq 0$.

(c) If $e_j \in Ae_i A$, then $i = j$.

Proof. (a) Any $f : Ae_i \to M$ is determined by $f(e_i) \in e_iM$, etc.

(b) Write $a = a_0 \cdot (\text{longest path in } a) + \cdots$ and $b = b_0 \cdot (\text{longest path in } b) + \cdots$, where $a_0b_0 \neq 0$. Then $ab = a_0b_0 \cdot (\text{longest path}) + \cdots \neq 0$.

(c) The module Ae_iA has a basis given by all paths going through i, so $e_j \in Ae_iA$ implies $i = j$. □

Lemma 3.3.2. *Each $e_i \in kQ$ is a primitive idempotent, i.e. Ae_i is an indecomposable A-module. Similarly, each e_jA is an indecomposable right projective.*

Proof. Recall, if A is a k-algebra and M is a left A-module, we say that M is *decomposable* if $M \simeq M_0 \oplus M_1$ for nonzero M_0, M_1. Note that M is decomposable if and only if $\mathrm{End}_A(M)$ has a nontrivial idempotent (projector onto one of the factors). Consider $\mathrm{End}_A(Ae_i) = \mathrm{Hom}_A(Ae_i, Ae_i) \simeq e_iAe_i$. The last isomorphism is an isomorphism of k-algebras if we consider e_i as the unit in e_iAe_i. If Ae_i were decomposable, then e_iAe_i would have a nontrivial idempotent f. Then $f^2 = f = fe_i$, so $f(f - e_i) = 0$. Part (b) of Lemma 3.3.1 tells us that this cannot be the case unless $f = 0$. □

Lemma 3.3.3. *Let $A = kQ$. Then $Ae_i \not\simeq Ae_j$ (as A-modules) unless $i = j$. Therefore $\{e_i\}$ is a complete set of primitive idempotents in A.*

Proof. Let $f \in \mathrm{Hom}_A(Ae_i, Ae_j) = e_jAe_i$ and $g \in \mathrm{Hom}_A(Ae_j, Ae_i) = e_iAe_j$. Then $f \circ g \in e_iAe_j^2Ae_i \subset Ae_jA$. If f were an isomorphism with inverse g, we would have $e_i \in Ae_jA$, but part (c) of Lemma 3.3.1 implies $i = j$. □

Remark 3.3.4. In general, it is not easy to distinguish projectives up to isomorphism. In general, a k-algebra A can have many non-equivalent projectives, even if A has no idempotents.

3.4 Representations of quivers

Definition 3.4.1. *For any quiver Q and a field k, define the category $\mathsf{Rep}_k(Q)$ to be just the category $\mathsf{Fun}(\underline{Q}, \mathsf{Vect})$ of functors from the quiver Q viewed as a category to the category of vector spaces.*

Explicitly, objects of $\mathsf{Rep}_k(Q)$ are assignments $i \mapsto X_i \in \mathsf{Vect}_k$, together with k-linear maps $X_\rho : X_{s(\rho)} \to X_{t(\rho)}$ for any edge $\rho \in Q_1$. A morphism $\Theta : X \to X'$ is given by a collection of k-linear maps $\{\Theta_i : X_i \to X'_i\}_{i \in Q_0}$ such that for all edges $\rho \in Q_1$, the following diagram commutes.

$$\begin{array}{ccc} X_{s(\rho)} & \xrightarrow{X_\rho} & X_{t(\rho)} \\ \downarrow{\scriptstyle \Theta_{s(\rho)}} & & \downarrow{\scriptstyle \Theta_{t(\rho)}} \\ X'_{s(\rho)} & \longrightarrow & X'_{t(\rho)} \end{array}$$

Proposition 3.4.2. *There is a natural equivalence of categories kQ-$\mathsf{Mod} \simeq \mathsf{Rep}_k(Q)$. Moreover, $\mathsf{Mod}(kQ) \simeq \mathsf{Rep}_k(Q^\circ)$, where Q° is the* opposite *quiver of Q, with the same vertices and edges as Q, but $s^\circ = t$, $t^\circ = s$.*

Proof. There is an obvious functor $F : kQ\text{-}\mathsf{Mod} \to \mathsf{Rep}_k(Q)$, which assigns to a kQ-module X, the Q-representation $F(X)$ with $F(X)_i = e_iX$. Given a morphism $f : X \to X'$ of kQ-modules, we let $F(f)_\rho : e_iX \to e_iX'$ be the restriction to e_iX of $f : X \to X'$.

Conversely, given $\{X_i\} \in \operatorname{Ob}\mathsf{Rep}_k(Q)$, define $X = \bigoplus_{i\in Q_0} X_i$. There are canonical projection and injection maps $\varepsilon_i : X_i \hookrightarrow X$ and $\pi_i : X \twoheadrightarrow X_i$. Give X a kQ-module structure by

$$(\rho_1 \cdots \rho_m) \cdot x = \varepsilon_{t(\rho_1)} \circ X_{\rho_1} \circ \cdots \circ X_{\rho_m} \circ \pi_{s(\rho_m)} x$$

The similar proof works for the category of right kQ-modules $\mathsf{Mod}(kQ)$. □

If A is a k-algebra, write $\operatorname{Irr}(A)$ for the set of isomorphism classes of irreducible A-modules. Let $\operatorname{Ind}(A)$ be the set of isomorphism classes of indecomposable *projectives* over A.

Theorem 3.4.3. *Assume Q has no oriented cycles. (Equivalently, $A = kQ$ has finite dimension over k.)*

1. *The following assignments are bijections:*

$$Q_0 \to \operatorname{Irr}(A) \qquad i \mapsto S(i)$$
$$Q_0 \to \operatorname{Ind}(A) \qquad i \mapsto Ae_i,$$

where $S(i)_j = k^{\oplus \delta_{ij}}$.

2. *There is a natural linear isomorphism for every $i, j \in Q_0$:*

$$\operatorname{Ext}^1_A(S(i), S(j)) \simeq \operatorname{Span} Q(i,j)$$

where $Q(i,j)$ is the set of all arrows from i to j.

3. $\operatorname{Ext}^k_A(M,N) = 0$ *for all $k \geqslant 2$ and all A-modules M, N.*

Remark 3.4.4. Part 1 of the theorem obviously fails if Q has oriented cycles. For example, let Q consist of a single loop. Then $kQ = k[x]$, and $\operatorname{Irr}(A) = \operatorname{Spec}(A) \supset k$, which is much larger than Q_0.

Before proving Theorem 3.4.3 we will need some other lemmas and facts.

Proposition 3.4.5. *For any Q and any left A-module X, the following sequence is exact.*

$$0 \longrightarrow \bigoplus_{\rho\in Q_1} Ae_{t(\rho)} \otimes e_{s(\rho)}X \xrightarrow{f} \bigoplus_{i\in Q_0} Ae_i \otimes e_iX \xrightarrow{g} X \longrightarrow 0 \qquad (*)$$

Here, $g(a \otimes x) = a \cdot x$ for all $a \in Ae_i$, $x \in e_iX$. The map f is defined by

$$\begin{aligned} f(ae_{t(\rho)} \otimes e_{s(\rho)}x) &= a\rho \otimes x - a \otimes \rho x \\ &= a\rho e_{s(\rho)} \otimes e_{s(\rho)}x - ae_{t(\rho)} \otimes e_{t(\rho)}x \\ &\in (Ae_{s(\rho)} \otimes e_{s(\rho)}X) \oplus (Ae_{t(\rho)} \otimes e_{t(\rho)}X), \end{aligned}$$

where we used that $\rho = \rho \cdot e_{s(\rho)} = e_{t(\rho)} \cdot \rho$.

Remark 3.4.6. Note that $(*)$ is a *projective* resolution because each $Ae_i \otimes_k V$ is a direct summand of the free module $A \otimes_k V$, where $V = kQ_1$. Moreover, if we put $X = A$, then we get a projective bimodule resolution of A. Indeed, $Ae_i \otimes e_j A$ is a direct summand of $A \otimes A = \bigoplus_{i,j \in Q_0} Ae_i \otimes e_j A$, which is a rank one A-bimodule.

Proof. (of Prop.3.4.5) First we show that g is surjective. This can be easily seen from the fact that any element of X can be written as

$$x = 1 \cdot x = \left(\sum_{i \in Q_0} e_i \right) x = \sum_{i \in Q_0} e_i x = g \left(\sum_{i \in Q_0} e_i \otimes e_i x \right).$$

The fact that $\mathrm{Ker}(g) \supseteq \mathrm{Im}(f)$ is just a direct computation. Indeed,

$$g \circ f(a \otimes e_{t(\rho)} \otimes e_{s(\rho)} x) = g(a\rho \otimes x - a \otimes \rho x) = a\rho x - a\rho x = 0.$$

To show that $\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$, we first note that any $\xi \in \bigoplus_{i=1}^n Ae_i \otimes e_i X$ can be written uniquely as

$$\xi = \sum_{i=1}^n \sum_{\substack{\text{paths } a \\ s(a)=i}} a \otimes x_a$$

where all but finitely many of $x_a \in e_{s(a)} X$ are zero. Let $\deg(\xi)$ be the length of the longest path a such that $x_a \neq 0$. If a is a nontrivial path, we can factor it as $a = a'\rho$, with $s(a') = t(\rho)$ and a' a single edge. We have $a' \otimes x_a = a' e_{s(a)} \otimes e_{s(a)} x = a' e_{t(\rho)} \otimes e_{s(a)}$. By definition,

$$f(a' \otimes x_a) = a'\rho \otimes x_a - a' \otimes \rho x_a = a \otimes x_a - a' \otimes \rho x_a$$

We claim that for any ξ, the set $\xi + \mathrm{Im}(f)$ contains elements of degree zero. For, if $\deg(\xi) = d$, then

$$\xi - f \left(\sum_{i=1}^n \sum_{\substack{s(a)=i \\ \ell(a)=d}} a' \otimes x_a \right)$$

has degree strictly less than d. The claim follows by induction on $\ell(a) = d$.

Let now $\xi \in \mathrm{Ker}(g)$, and take $\xi' \in \xi + \mathrm{Im}(f)$ an element of degree zero. In other words, $\xi' = \sum_{i=1}^n e_i \otimes x_{e_i}$. If $g(\xi) = 0$, then because $g \circ f = 0$, we get $g(\xi) = g(\xi') = \sum x e_i$, an element of $\bigoplus_{i=1}^n e_i X$, which can be zero if and only each $xe_i = 0$. But each $xe_i = 0$ implies $\xi' = 0$, i.e. $\xi \in \mathrm{Im}(f)$. This finishes the proof that $\mathrm{Ker}(g) = \mathrm{Im}(f)$.

Finally, we prove that $\mathrm{Ker}(f) = 0$. Suppose $f(\xi) = 0$. Then we can write

$$\xi = \sum_{\rho \in Q_1} \sum_{\substack{\text{path } a \\ s(a)=t(\rho)}} a \otimes x_{\rho,a} = a \otimes x_{\rho,x} + \cdots,$$

where a is a path of maximal length. We get

$$f(\xi) = \sum_\rho \sum_a a\rho \otimes x_{\rho,a} - \sum_\rho \sum_a a \otimes \rho x_{\rho,a} = a\rho \otimes x_{\rho,a} + \text{lower terms}.$$

This contradicts our choice of a.

□

Definition 3.4.7. *If X is a finite-dimensional representation of Q, define $\underline{\dim}(X) = (\dim_k X_1, \ldots, \dim_k X_n) \in \mathbb{Z}^n$, the* dimension vector *of X.*

Definition 3.4.8. *The* Euler form *of Q is a bilinear form $\langle \cdot, \cdot \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$, defined by*

$$\langle \alpha, \beta \rangle = \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{\rho \in Q_1} \alpha_{t(\rho)} \beta_{s(\rho)}.$$

Associated quadratic form is called the Tit's form *of Q, $q : \mathbb{Z}^n \to \mathbb{Z}$, and is defined by*

$$q(\alpha) = \langle \alpha, \alpha \rangle.$$

Corollary 3.4.9. *For any two finite-dimensional representations X and Y of Q, we have*

$$\langle \underline{\dim} X, \underline{\dim} Y \rangle = \dim_k \operatorname{Hom}_A(X, Y) - \dim_k \operatorname{Ext}^1_A(X, Y).$$

Proof. This comes from applying $\operatorname{Hom}_A(-, Y)$ to the standard resolution $(*)$:

$$\begin{array}{ccccccccc} 0 \to \operatorname{Hom}_A(X,Y) \to & \bigoplus_{i \in Q_0} \operatorname{Hom}_A(Ae_i \otimes e_i X, Y) & \to & \bigoplus_{\rho \in Q_0} \operatorname{Hom}_A(Ae_{t(\rho)} \otimes e_{s(\rho)} X, Y) & \to \operatorname{Ext}^1_A(X,Y) \to 0 \\ & \downarrow \wr & & \downarrow \wr & \\ & \bigoplus_{i \in Q_0} \operatorname{Hom}_k(e_i X, e_i Y) & & \bigoplus_{\rho \in Q_1} \operatorname{Hom}_k(e_{s(\rho)} X, e_{t(\rho)} Y). & \end{array}$$

The standard Euler characteristic yields the formula. □

Remark 3.4.10. Taking $X = Y$, the Corollary 3.4.9 tells us that

$$\dim \operatorname{End}_A(X) = \dim \operatorname{Ext}^1(X, X) - q(\underline{\dim} X).$$

Lemma 3.4.11. *Let X, Y be (nonzero) simple A-modules. Then an extension of X by Y*

$$0 \longrightarrow Y \xrightarrow{\alpha} Z \xrightarrow{\beta} X \longrightarrow 0 \qquad (*)$$

is nonsplit if and only if $\alpha(Y)$ is the only proper submodule of Z.

Proof. Suppose $(*)$ splits. Then there exists $\gamma : X \to Z$ such that $\beta\gamma = \mathrm{id}_X$. Then $\gamma(X) \subset Z$ is a submodule $\neq \alpha(Y)$ because $\beta\gamma(X) = X \neq \beta\alpha(Y) = 0$.

Conversely, suppose K is a proper submodule of Z and $K \neq \alpha(Y)$. Define $\varphi = \beta|_K : K \to X$. Since X is simple, φ is either 0 or surjective. If $\varphi = 0$, then $K \subset \operatorname{Ker}\beta = \operatorname{Im}\alpha = \alpha(Y)$, a simple module. This implies $K = \alpha(Y)$, a contradiction. Thus φ is surjective. If $k \in K$ is killed by φ, then $K \cap \alpha(Y) \neq 0$, so $K = \alpha(Y)$, a contradiction. We have shown that $\varphi : K \to X$ is an isomorphism. We define the splitting γ by $\gamma = \varphi^{-1}$. □

Lemma 3.4.12. *Let X, Y be two non-isomorphic, simple A-modules. Then*

a) If A is commutative, then every extension of X by Y splits.

b) If $z \in Z(A)$ such that $z \in \operatorname{Ann}_A(Y) \setminus \operatorname{Ann}_A(X)$, then every extension of X by Y splits.

Proof. a) Note that if A is commutative and $0 \neq x \in X$, then the map $A \to X$ given by $a \mapsto ax$ is surjective, with kernel a maximal ideal $\mathfrak{m}_x$. After choosing nonzero $y \in Y$, then $X \simeq Y$ iff $\mathfrak{m}_x = \mathfrak{m}_y$. Let Z be an extension of X by Y. Choose $a \in \mathfrak{m}_y \setminus \mathfrak{m}_x$, and define $\widehat{a} : Z \to Z$ by $z \mapsto az$. Then $\widehat{a}(Y) = 0$ and $X \simeq Z/\operatorname{Ker}(\widehat{a}) \simeq X$. This lets us create a splitting of this exact sequence.

b) Define $\widehat{a} : Z \to Z$ as in part (a). Everything else works. □

We return to the proof of Theorem 3.4.3.

Proof. The only non-trivial part of the Theorem 3.4.3 is about Ext's.

Lemma 3.4.13. *Let Q be a quiver, and let i, j be vertices in Q. Then there exists an arrow $\rho : i \to j$ if and only if there is a nonsplit extension*

$$0 \longrightarrow S(j) \longrightarrow X \longrightarrow S(i) \longrightarrow 0.$$

Proof. Assume $i \neq j$ and there is $\rho : i \to j$. We can construct an extension X by

$$X_n = \begin{cases} k & \text{if } n = i = j \\ 0 & \text{otherwise} \end{cases}$$

and $X_{\rho'} = \mathrm{id}$ if $\rho' = \rho$, with $X_{\rho'} = 0$ otherwise. There is an obvious non-split exact sequence

$$0 \longrightarrow S(j) \longrightarrow X \longrightarrow S(i) \longrightarrow 0. \tag{A.1}$$

If $i = j$ and there is $\rho : i \to i$, let $X_n = k^{\oplus 2}$ if $n = i$, $X_n = 0$ otherwise, with

$$X_{\rho'} = \begin{cases} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \text{if } \rho' = \rho \\ 0 & \text{otherwise} \end{cases}$$

The opposite direction is obvious.

□

Lemma 3.4.14. *If i and j are two vertices in a quiver Q, then*

$$\dim_k \operatorname{Ext}^1_{kQ}(S(i), S(j)) = \#\{\textit{arrows } i \to j\}.$$

Proof. It follows from the Euler formula

$$\langle \underline{\dim} X, \underline{\dim} Y \rangle = \dim_k \operatorname{Hom}_A(X, Y) - \dim_k \operatorname{Ext}^1_A(X, Y)\operatorname{Ext}.$$

Apply this when $X = S(i)$ and $Y = S(j)$. We know that $\operatorname{Hom}_A(S(i), S(j)) = 0$ if $i \neq j$, and has dimension 1 otherwise. Moreover, $\underline{\dim} S(i) = (\delta_{ik})_k$ and $\underline{\dim} S(j) = (\delta_{jk})_k$. It follows that

$$\langle \underline{\dim} S(i), \underline{\dim} S(j) \rangle = \begin{cases} -\sum_\rho \delta_{i,s(\rho)} \delta_{j,t(\rho)} & \text{if } i \neq j \\ 1 - \sum_\rho \delta_{i,s(\rho)} \delta_{j,t(\rho)} & \text{if } i = j \end{cases}$$

which yields the desired fact. □

These two lemmas prove the part (b) of the theorem. Now we only need to show that for $n \geq 2$ all functors Ext^n vanish.

Lemma 3.4.15. *For any k-algebra A, the following three conditions are equivalent:*

1. *every submodule of a (left) projective A-module is projective*
2. *every (left) A-module has a projective resolution of length $\leqslant 1$*
3. $\operatorname{Ext}^n_A(-, -) = 0$ *for all* $n \geqslant 2$

Proof. Directions $1 \Rightarrow 2 \Rightarrow 3$ are obvious.

To see $3 \Rightarrow 1$ consider the sequence $0 \to X \to P \to P/X \to 0$, where P is any projective and $X \subset P$. Apply $\operatorname{Hom}_A(-, Y)$ for any Y. As above, the long exact sequence yields $\operatorname{Ext}^n(X, Y) \simeq \operatorname{Ext}^{n+1}(P/X, Y) = 0$. Hence $\operatorname{Ext}^1(X, Y) = 0$ for all Y, and so X is projective. □

Now the statement (c) about vanishing of Ext^n for $n \geq 2$ follows from the lemma above, knowing that for a path algebra $A = kQ$ every (left) A-module has a projective resolution of length $\leqslant 1$, see Proposition 3.4.5. □

Definition 3.4.16. *Algebras, satisfying one of the three equivalent conditions of Lemma 3.4.15 are called* (left) hereditary.

Theorem 3.4.17. *Let k be a perfect field. Then any finite-dimensional hereditary k-algebra is Morita equivalent to the path algebra of a quiver.*

Definition 3.4.18. *A k-algebra A is called* formally smooth *if it satisfies any of the following equivalent conditions:*

1. *A "behaves like a free algebra" with respect to nilpotent extensions: whenever* $I \subset R$ *has* $I^n = 0$*, maps lift in the following diagram*

$$\begin{array}{ccc} & \overset{\widetilde{\varphi}}{\nearrow} & R \\ & & \downarrow \\ A & \xrightarrow[\varphi]{} & R/I \end{array}$$

2. *A has cohomological dimension* $\leqslant 1$ *with respect to Hochschild cohomology, i.e.*

$$\mathrm{HH}^n(A, M) = 0, \quad \forall A\text{-}bimodule\ M, \quad \forall n \geqslant 2$$

3. *the kernel of the multiplication map* $A \otimes A \to A$ *is a projective A-bimodule*

Lemma 3.4.19. *If A is formally smooth, then A is (left and right) hereditary.*

Proof. Let $\Omega^1(A) = \mathrm{Ker}(A \otimes A \xrightarrow{m} A)$. Then there is a short exact sequence of bimodules:

$$0 \longrightarrow \Omega^1(A) \longrightarrow A \otimes A \longrightarrow A \longrightarrow 0.$$

The fact that A is formally smooth tells us that $\Omega^1(A)$ is a projective bimodule. Tensor by M for some left A-module M, to get

$$0 \longrightarrow \Omega^1(A) \otimes_A M \longrightarrow A \otimes M \longrightarrow M \longrightarrow 0.$$

(we have exactness on the left because $\mathrm{Tor}_1^A(A, M) = 0$ since A is projective.) Since $\Omega^1(A)$ is projective, so is $\Omega^1(A) \otimes M$, whence the result. □

Theorem 3.4.20 (Cuntz–Quillen)**.** *Every hereditary finite-dimensional k-algebra is formally smooth.*

Proof. This is Proposition 6.2 in [CQ95]. □

Example 3.4.21. A commutative k-algebra A is formally smooth if and only if Spec(A) is smooth with Krull dimension $\leqslant 1$. For example, $A = k[x, y]$ is not formally smooth (in the category of all associative algebras) though it is smooth in the category of *commutative* k-algebras.

Example 3.4.22. Let $A = k\langle x, y\rangle/([x, y] = 1)$. It is hereditary but not formally smooth.

Example 3.4.23. Let G be an abstract group. Then $A = k[G]$ is formally smooth if and only if G is *virtually free*, i.e. it contains a free group of a finite index.

Appendix B

Exercises

1 Standard complexes in Algebra and Geometry

Exercise 1. For a morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ show that the presheaf $\mathrm{Im}(\varphi)$ defined in section 3.3 is actually a sheaf.

Exercise 2. Show that if X is an irreducible topological space, then any constant sheaf on X is flabby.

Exercise 3. Suppose G is a group and A is G-module. Check that the set $Z^1(G,A) = \{f \in C^1(G,A) : d^1 f = 0\}$ of 1-cocycles is exactly the set of derivations $d : G \to A$, i.e. $Z^1(G,A) = \mathrm{Der}(G,A)$. Moreover, the set $B^1(G,A) = \mathrm{Im}(d^0)$ of 1-coboundaries is exactly the set of inner derivations.

Exercise 4. Let A be a k-algebra for a commutative ring k, and $(M_\bullet, d_M)$, $(N_\bullet, d_N)$ be two complexes of left A-modules. In 2.5 we defined graded Hom-space $\underline{\mathrm{Hom}}_A(M,N)$. Prove that if M is a finitely generated (as A-module) then $\underline{\mathrm{Hom}}_A(M,N) = \mathrm{Hom}_A(M,N)$.

Exercise 5. Show that any associative star product (see 3.2) on $A_t = A \otimes_k k[\![t]\!]$ is unital, and that for any $*$, there exists $*' \sim *$ such that $1_{*'} = 1_A$.

2 Classical homological algebra

Exercise 1. Let A be a ring, M an A-module. Prove the dual basis theorem 2.3.6. Moreover, show that M is a generator for the category $\mathsf{Mod}(A)$ in the sense of definition 2.3.5 if and only if $M^*M = A$.

Exercise 2. Show that if A and B are commutative rings, then A and B are Morita equivalent if and only if they are isomorphic.

Exercise 3. Prove the properties 2.4 of functors $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$ from the "yoga" of six functors.

Exercise 4. Suppose $F\colon \Delta \to \mathsf{Top}$ is the functor defined by $F([n]) = \Delta_n$, where by Δ_n we denote the standard topological n-simplex. Denote by $Y\colon \Delta \hookrightarrow \mathsf{sSet}$ the Yoneda embedding. Prove that the left Kan extension $\mathrm{Lan}_Y F$ of F along Y exists and is isomorphic to the geometric realization functor, i.e. $\mathrm{Lan}_Y F = |-|$.

Exercise 5. Let $\mathcal{A}$ be an additive category. Show that finite products and finite coproducts exist, and coincide. On the other hand, show that *infinite* products and coproducts need not be the same.

Exercise 6. For an additive category $\mathcal{A}$ and any $X, Y \in \mathcal{A}$, we can define the *diagonal* $\Delta_X : X \to X \times X$ by $\Delta_X = \mathrm{id}_X \times \mathrm{id}_X$, and the *folding map* $\nabla_Y : Y \sqcup Y \to Y$ by $\nabla_Y = \mathrm{id}_Y \sqcup \mathrm{id}_Y$. Show that the abelian group structure on $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is given by

$$f + g = \nabla_Y \circ (f \times g) \circ \Delta_X.$$

Exercise 7. For additive $\mathcal{A}$ find a categorical definition of $-f$ for any $f : X \to Y$, $X, Y \in \mathcal{A}$.

Exercise 8. Show that if $\mathcal{A}$ is additive, then $\mathcal{A}^\circ$ is additive, and that $\mathcal{A} \times \mathcal{B}$ is additive whenever $\mathcal{A}$ and $\mathcal{B}$ are.

Exercise 9. If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are categories, show that there is an equivalence of categories

$$\mathsf{Fun}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \simeq \mathsf{Fun}(\mathcal{A}, \mathsf{Fun}(\mathcal{B}, \mathcal{C})).$$

If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are additive, show this equivalence restricts to an equivalence

$$\mathsf{Fun}_{\mathrm{add}}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \simeq \mathsf{Fun}_{\mathrm{add}}(\mathcal{A}, \mathsf{Fun}_{\mathrm{add}}(\mathcal{B}, \mathcal{C})).$$

Exercise 10. Show that if $F : \mathcal{A} \to \mathcal{B}$ is additive, then the canonical map $F(X \oplus Y) \to F(X) \oplus F(Y)$ is an isomorphism.

Exercise 11. Show that if $\mathcal{A}$ is additive, then the "functors of points" $h_X : \mathcal{A}^\circ \to \mathsf{Set}$ are actually functors $h_X : \mathcal{A}^\circ \to \mathsf{Ab}$, and similarly for $h^X = \mathrm{Hom}(X, -)$. Show that the Yoneda embedding $h : \mathcal{A} \to \mathsf{Fun}(\mathcal{A}^\circ, \mathsf{Ab})$ is additive.

Exercise 12. Let $\mathcal{A}$ be an additive category, $\varphi : X \to Y$ a morphism in $\mathcal{A}$. We defined kernel of φ (if it exists) as the object $\mathrm{Ker}(\varphi)$ representing the functor $\underline{\mathrm{Ker}}(\varphi)\colon \mathcal{A}^\circ \to \mathsf{Ab}$ defined by

$$Z \mapsto \mathrm{Ker}_{\mathsf{Ab}}(\varphi_* : \mathrm{Hom}_{\mathcal{A}}(Z, X) \to \mathrm{Hom}_{\mathcal{A}}(Z, Y)).$$

Prove that this definition is equivalent to letting $\mathrm{Ker}(\varphi)$ be the equalizer of the diagram

$$X \underset{0}{\overset{\varphi}{\rightrightarrows}} Y.$$

Exercise 13. If $\mathcal{A}$ is an abelian category show that $\mathsf{Com}(\mathcal{A})$ is also an abelian category. More generally, show that for any small category I the category $\mathsf{Fun}(I, \mathcal{A})$ is naturally an abelian category.

Exercise 14. Prove the Mittag-Leffler condition 2.1.

Exercise 15. Prove lemma 2.2.4.

Exercise 16. If $\mathcal{A}$ is an abelian category satisfying AB5, then every finitely generated object is compact. This was the lemma 2.4.6.

Exercise 17. For a Grothendieck category $\mathcal{A}$ show that $\mathcal{A}^\circ$ is also Grothendieck if and only if $\mathcal{A}$ is the zero category.

Exercise 18. Show that the category $\mathsf{Com}^b(\mathcal{A})$ of bounded complexes (see subsection 3.2) is "generated" by $\mathcal{A}$, i.e. every $X^\bullet \in \mathsf{Com}^b(\mathcal{A})$ can be obtained by taking iterated suspensions and cones of objects in $\mathcal{A}$.

Exercise 19. Take the complex I to be $I = (0 \to \mathbb{Z}^{\oplus 2} \to \mathbb{Z} \to 0)$. For any morphism of complexes $f^\bullet : X^\bullet \to Y^\bullet$ there is a natural inclusion $X^\bullet \hookrightarrow X^\bullet \otimes I^\bullet$. Show that the mapping cylinder $\mathrm{Cyl}(f)$ (see 3.2) can be equivalently defined as a push-forward

$$\begin{array}{ccc} X & \xrightarrow{i_0} & X \otimes I \\ {\scriptstyle f}\downarrow & & \downarrow \\ Y & \dashrightarrow & \mathrm{Cyl}(f) \end{array}$$

3 Residues and Lie cohomology

Let V be a vector space over a field k. We *do not* assume that V is finite-dimensional.

3.1 Commensurable subspaces

Exercise 1. Let A and B be subspaces of V. Prove that the following conditions are equivalent:

(i) $\dim (A + B)/(A \cap B) < \infty$

(ii) $\dim A/(A \cap B) < \infty$ and $\dim B/(A \cap B) < \infty$

(iii) $\dim (A + B)/A < \infty$ and $\dim (A + B)/B < \infty$.

Call the subspaces A and B *commensurable* (and write $A \sim B$) if they satisfy the above conditions.

Exercise 2. Prove that commensurability is an equivalence relation on the set of subspaces of V. (Hint: prove first that if $A \sim B$ and $B \sim C$ then $\dim (A + B + C)/(A \cap B \cap C) < \infty$.)

Exercise 3. Let A, B, A' and B' be subspaces in V such that $A \sim A'$ and $B \sim B'$. Prove then that

$$A + B \sim A' + B' \quad \text{and} \quad A \cap B \sim A' \cap B'.$$

Exercise 4. Given two commensurable subspaces A and B in V, define the *relative dimension* of A and B by

$$[A|B] := \dim A/(A\cap B) - \dim B/(A\cap B).$$

Prove that if A, B and C are pairwise commensurable, then

$$[A|B] + [B|C] = [A|C].$$

Exercise 5. Let A, B, A' and B' be subspaces in V such that $A \sim A'$ and $B \sim B'$. Prove that

$$[A|A'] + [B|B'] = [A\cap B|A'\cap B'] + [A+B|A'+B'].$$

3.2 Traces

Exercise 6. For a subspace A in V, define

$$\mathrm{End}(V,A) := \{g \in \mathrm{End}\, V \ :\ A + gA \sim A\}.$$

and

$$\mathrm{End}_{\mathrm{fin}}(V,A) := \{g \in \mathrm{End}\ V \ :\ \dim gA < \infty \text{ and } A + gV \sim A\}.$$

Prove that $\mathrm{End}_{\mathrm{fin}}(V,A)$ and $\mathrm{End}(V,A)$ are subalgebras of $\mathrm{End}\, V$ depending only on the commensurability class of A. Moreover,

$$f \in \mathrm{End}(V,A)\,,\ g \in \mathrm{End}_{\mathrm{fin}}(V,A) \quad \Rightarrow \quad fg\,,\ gf \in \mathrm{End}_{\mathrm{fin}}(V,A).$$

Exercise 7. Let $\mathrm{End}_{\mathrm{fin}}(V) = \{g \in \mathrm{End}\, V \ :\ \dim gV < \infty\}$. Prove that

$$g_1,\, g_2 \in \mathrm{End}_{\mathrm{fin}}(V,A) \quad \Rightarrow \quad g_1\,,\, g_2 \in \mathrm{End}_{\mathrm{fin}}(V).$$

Conclude that, for any $g \in \mathrm{End}_{\mathrm{fin}}(V,A)$, $\dim g^2V < \infty$.

Exercise 8. Using the previous exercise 7, we can define a trace map $\mathrm{tr}_V : \mathrm{End}_{\mathrm{fin}}(V,A) \to k$ as follows. Given $g \in \mathrm{End}_{\mathrm{fin}}(V,A)$, choose any finite-dimensional g-invariant subspace U containing g^2V, restrict g to U and define

$$\mathrm{tr}_V(g) := \mathrm{tr}_U(g|_U).$$

where tr_U is the usual trace on U. Prove that this definition is independent of the choice of U.

Exercise 9. Prove that tr_V is a linear map: that is, for any $g_1, g_2 \in \mathrm{End}_{\mathrm{fin}}(V,A)$,

$$\mathrm{tr}_V(\alpha_1 g_1 + \alpha_2 g_2) = \alpha_1\,\mathrm{tr}_V(g_1) + \alpha_2\,\mathrm{tr}_V(g_2)\ , \quad \alpha_1,\, \alpha_2 \in k.$$

(Hint: Use the (finite-dimensional) subspace $U = g_1^2V + g_2^2V + g_1g_2V + g_2g_1V$.)

Exercise 10. For any $f, g \in \mathrm{End}\, V$, define $[f,g] = fg - gf$. Notice that, by part (a), if $f \in \mathrm{End}(V,A)$ and $g \in \mathrm{End}_{\mathrm{fin}}(V,A)$, then $[f,\, g] \in \mathrm{End}_{\mathrm{fin}}(V,A)$, and thus $\mathrm{tr}_V[f,g]$ is well defined. Prove that $\mathrm{tr}_V[f,g] = 0$. (Hint: Use the (finite-dimensional) subspaces $U_1 = gfgV \supset (gf)^2V$ and $U_2 = (fg)^2V$.)

Exercise 11. For any $f, g, h \in \mathrm{End}(V,A)$, with one of them lying in $\mathrm{End}_{\mathrm{fin}}(V,A)$, prove

$$\mathrm{tr}_V\left([h,f]g\right) = \mathrm{tr}_V\left(h[f,g]\right).$$

3.3 Residues

Let A be a subspace of V. Fix a projection $\pi : V \to A$.

Exercise 12. Prove that if $f \in \mathrm{End}(V, A)$, then $[\pi, f] \in \mathrm{End}_{\mathrm{fin}}(V, A)$.

Exercise 13. Using 12, define the function $\psi^V_{(\pi)} : \mathrm{End}(V, A) \times \mathrm{End}(V, A) \to k$ by

$$\psi^V_{(\pi)}(f, g) = \mathrm{tr}_V([\pi, f]g).$$

Prove that $\psi^V_{(\pi)}$ is a skew-symmetric bilinear form on $\mathrm{End}(V, A)$.

Exercise 14. Let $f, g \in \mathrm{End}(V, A)$. Let U be a subspace of V containing A and invariant under f and g. Prove that $\psi^V_{(\pi)}(f, g) = \psi^U_{(\pi)}(f, g)$. (Because of this property we can drop the superscript V in $\psi^V_{(\pi)}$.) Check that, if A is invariant under f and g, then $\psi_{(\pi)}(f, g) = 0$.

Exercise 15. Let A' be another subspace of V, and let $\pi' : V \to A'$ be a projection onto A'. Assume that $A \sim A'$. Then $\pi - \pi' \in \mathrm{End}_{\mathrm{fin}}(V, A)$, and

$$\psi_{(\pi')}(f, g) - \psi_{(\pi)}(f, g) = \mathrm{tr}_V \left((\pi - \pi')[f, g]\right).$$

(Hint: Use property 11 of 3.2.)

Exercise 16. Using 15, prove that, if $f, g \in \mathrm{End}(V, A)$ are such that $[f, g] = 0$, then $\psi_{(\pi)}(f, g)$ is independent of the choice of the projection $\pi : V \to A$ and depends only on the commensurability class of A. (We will write $\psi_A(f, g)$ instead of $\psi_{(\pi)}(f, g)$ when $[f, g] = 0$.)

Exercise 17. Prove that if $f, g \in \mathrm{End}(V, A)$ commute and if $\dim A < \infty$ or $\dim V/A < \infty$, then $\psi_A(f, g) = 0$.

Exercise 18. Let A and B be subspaces of V. Choose projections $\pi_A : V \to A$, $\pi_B : V \to B$ and $\pi_{A\cap B} : V \to A \cap B$ such that $\pi_A + \pi_B - \pi_{A\cap B}$ is a projection π_{A+B} of V onto $A + B$ (this is clearly always possible). Prove that, for any $f, g \in \mathrm{End}\,(V, A) \cap \mathrm{End}\,(V, B)$, the following formula holds

$$\psi_{\pi_A}(f, g) + \psi_{\pi_B}(f, g) = \psi_{\pi_{A+B}}(f, g) + \psi_{\pi_{A\cap B}}(f, g).$$

Conclude that, if $f, g \in \mathrm{End}\,(V, A) \cap \mathrm{End}\,(V, B)$ are such that $[f, g] = 0$, then

$$\psi_A(f, g) + \psi_B(f, g) = \psi_{A+B}(f, g) + \psi_{A\cap B}(f, g).$$

This last formula is called the (abstract) *Tate Residue Formula.*

3.4 Interpretation in terms of Lie algebra cohomology

We denote the Lie algebras $\mathrm{End}(V, A)$, $\mathrm{End}_{\mathrm{fin}}(V, A)$ etc. by $\mathfrak{gl}(V, A)$, $\mathfrak{gl}_{\mathrm{fin}}(V, A)$ etc. respectively, with the usual (commutator) bracket.

Exercise 19. Check that $\psi_{(\pi)}$ is a 2-cocycle on the Lie algebra $\mathfrak{gl}(V, A)$. Hence it defines a canonical central extension

$$0 \longrightarrow k \longrightarrow \widetilde{\mathfrak{gl}}(V, A) \longrightarrow \mathfrak{gl}(V, A) \longrightarrow 0$$

Exercise 20. Show that the cohomology class $c_A := [\psi_{(\pi)}] \in \mathrm{H}^2\left(\mathfrak{gl}(V, A), k\right)$ is independent of the choice of π and depends only on the commensurability class of A. (Hint: Use 15 from 3.3.)

Exercise 21. Show that c_A induces a cohomology class $\bar{c}_A \in \mathrm{H}^2\left(\mathfrak{gl}(V, A)/\mathfrak{gl}_{\mathrm{fin}}, k\right)$.

3.5 Adeles and residues on algebraic curves

The above formalism comes from algebraic geometry. In what follows, we briefly outline a classic construction of residues of differential forms on curves due to Tate [Tat68]. This material requires familiarity with basic algebraic geometry.

Let X be a smooth connected algebraic curve over k, which we now assume to be algebraically closed. Let $K = k(X)$ be the field of rational functions on X. For a (closed) point $x \in X$, let $\mathscr{O}_x$ denote its local ring. Write $A_x = \hat{\mathscr{O}}_x$ for the completion of $\mathscr{O}_x$ and K_x for the field of fractions of A_x. Note that A_x is canonically a subspace of K_x, so we can consider $\mathrm{End}(K_x, A_x)$ defined as in 3.2.

Now, choose a local parameter (coordinate) t at x and identify $A_x \simeq k[\![t]\!]$ and $K_x \simeq k(\!(t)\!)$ in the usual way, where $k[\![t]\!]$ and $k(\!(t)\!)$ are the rings of formal power and Laurent series in t, respectively. In addition, identify the elements of K_x with the corresponding multiplication operators on K_x: this gives us the embedding

$$\iota : K_x \hookrightarrow \mathrm{End}(K_x) \quad f \mapsto [f : g \mapsto fg].$$

Exercise 22. Show that $\mathrm{Im}(\iota) \subseteq \mathrm{End}(K_x, A_x)$.

Exercise 23. Prove that, for all $f, g \in K_x$,

$$-\psi_{A_x}(f, g) = \text{coefficient of } t^{-1} \text{ in } f(t)g'(t)\ ,$$

which is the usual residue of the differential form $\omega = f\,dg$ at x.

Exercise 24. For a set S of closed points of X, denote $\mathscr{O}(S) := \bigcap_{x\in S} \mathscr{O}_x \subset K$. Prove that

$$\sum_{x\in S} \psi^{K_x}_{A_x}(f, g) = \psi^{K}_{\mathscr{O}(S)}(f, g) \quad \forall f, g \in K_x.$$

(Hint: To prove the above formula, consider the spaces

$$A_S = \prod_{x \in S} A_x$$

$$V_S = \left\{ (f_x) \in \prod_{x \in S} K_x : f_x \in A_x \text{ for all but a finite number of } x \right\},$$

(The elements of V_X are called *adeles* on the curve X.) Note that K embeds diagonally in V_S. Using the short exact sequence of sheaves

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{\phi} \mathscr{F}^0 \longrightarrow \mathscr{F}^1 \longrightarrow 0.$$

where $\mathscr{F}^0(U) = K \times A_U$ and $\mathscr{F}^1(U) = V_U$ for an open $U \subseteq X$ and ϕ is the diagonal map, show that

$$V_X/(K + A_X) \simeq \mathrm{H}^1(X, \mathscr{O}_X).$$

Conclude that $\dim_k [V_X/(K + A_X)] < \infty$. Then, use the abstract Residue Formula from exercise 18 in 3.3 for $V = V_S$, $A = A_S$ and $B = K$.)

Exercise 25. Conclude from 24 that the sum of residues of $\omega = f\,dg$ over all points of X is zero.

Bibliography

[AZ94] M. Artin and J. J. Zhang. Noncommutative projective schemes. *Adv. Math.*, 109:228–287, 1994. 134

[Bae40] Reinhold Baer. Abelian groups that are direct summands of every containing abelian group. *Bull. Amer. Math. Soc.*, 46:800–806, 1940. 137

[Bar77] W. Barth. Moduli of vector bundles on the projective plane. *Invent. Math.*, 42:63–91, 1977. 209

[BBD82] A.A. Beilinson, J. Bernstein, and P. Deligne. Faisceaux pervers. *Société Mathématique de France*, 100, 1982. 79, 161, 164, 165, 171, 212

[BC07] Yuri Berest and Oleg Chalykh. A_∞-modules and Calogero-Moser spaces. *J. Reine Angew. Math.*, 607:69–112, 2007. 212

[BCE08] Yuri Berest, Oleg Chalykh, and Farkhod Eshmatov. Recollement of deformed preprojective algebras and the Calogero-Moser correspondence. *Mosc. Math. J.*, 8(1):21–37, 183, 2008. 170

[Bei78] A. A. Beilinson. Coherent sheaves on $\mathbf{P}^n$ and problems in linear algebra. *Funktsional. Anal. i Prilozhen.*, 12(3):68–69, 1978. 126

[Ber74] George M. Bergman. Coproducts and some universal ring constructions. *Trans. Amer. Math. Soc.*, 200:33–88, 1974. 65

[BGG78] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand. Algebraic vector bundles on $\mathbf{P}^n$ and problems of linear algebra. *Funktsional. Anal. i Prilozhen.*, 12(3):66–67, 1978. 147

[BKS03] D. Benson, H. Krause, and S. Schwede. Realizability of modules over Tate cohomology. *Trans. AMS*, 356(9):3621–3668, 2003. 41

[BL04] Andrew Baker and Andrey Lazarev. Topological Hochschild cohomology and generalized Morita equivalence. *Algebr. Geom. Topol.*, 4:623–645 (electronic), 2004. 86

[BO95] Oleg Bondal and Dmitry Orlov. Semiorthogonal decompositions for algebraic varieties. *preprint*, 1995. 166

[BRT13] Yuri Berest, Ajay Ramadoss, and Xiang Tang. The Picard group of a noncommutative algebraic torus. *J. Noncommut. Geom.*, 7(2):335–356, 2013. 214

[Buc86] Ragnar-Olaf Buchweitz. Maximal cohen-macaulay modules and tate-cohomology over gorenstein rings. 1986. 177

[BW02] Yuri Berest and George Wilson. Ideal classes of the Weyl algebra and noncommutative projective geometry. *Int. Math. Res. Not.*, (26):1347–1396, 2002. With an appendix by Michel Van den Bergh. 210

[CB92] William Crawley-Boevey. Lectures on representations of quivers. 1992. available at `http://www1.maths.leeds.ac.uk/~pmtwc/quivlecs.pdf`. 72

[CBS06] William Crawley-Boevey and Peter Shaw. Multiplicative preprojective algebras, middle convolution and the Deligne-Simpson problem. *Adv. Math.*, 201(1):180–208, 2006. 214, 215

[Che77] Kuo-Tsai Chen. Iterated path integrals. *Bulletin of Am. Math. Soc.*, 83, Number 2:831–879, 1977. 221

[CQ95] Joachim Cuntz and Daniel Quillen. Algebra extensions and nonsingularity. *J. Amer. Math. Soc.*, 8(2):251–289, 1995. 32, 231

[DHKS04] William G. Dwyer, Phillip S. Hirschhorn, Daniel M. Kan, and Jeffry H. Smith. *Homotopy limit functors on model categories and homotopical categories*, volume 113 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2004. 128

[DK73] Pierre Deligne and Nicholas Katz. *Groupes de monodromie et géométrie algébrique. II*, volume 340 of *Lecture Notes in Mathematics*. Springer-Verlag, 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II). 204

[DS95] W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995. 186

[Dyc] T. Dyckerhoff. Compact generators in categories of matrix factorizations. `arXiv:0904.4713`. 177, 183, 184, 185, 192

[EH00] David Eisenbud and Joe Harris. *The geometry of schemes*, volume 197 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. 17

[Eis80] David Eisenbud. Homological algebra on a complete intersection, with an application to group representations. *Trans. Amer. Math. Soc.*, 260(1):35–64, 1980. 177, 179

[ES53] B. Eckmann and A. Schopf. Über injektive Moduln. *Arch. Math. (Basel)*, 4:75–78, 1953. 137

[Fal88] Gerd Faltings. p-adic Hodge theory. *J. Amer. Math. Soc.*, 1(1):255–299, 1988. 201

[FH91] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics. 75

[FH13] Daniel S. Freed and Michael J. Hopkins. Chern-Weil forms and abstract homotopy theory. *Bull. Amer. Math. Soc. (N.S.)*, 50(3):431–468, 2013. 220, 221

[FL98] Daniel R. Farkas and Gail Letzter. Ring theory from symplectic geometry. *J. Pure Appl. Algebra*, 125(1-3):155–190, 1998. 43

[Gab62] Pierre Gabriel. Des catégories abéliennes. *Bull. Soc. Math. France*, 90:323–448, 1962. 134

[Gin05] Victor Ginzburg. Lectures on noncommutative geometry. 2005. Available at `http://arxiv.org/abs/math/0506603`. 32

[GM03] Sergei I. Gelfand and Yuri I. Manin. *Methods of homological algebra*. Springer Monographs in Mathematics. Springer-Verlag, second edition, 2003. 41

[GMV96] Sergei Gelfand, Robert MacPherson, and Kari Vilonen. Perverse sheaves and quivers. *Duke Math. J.*, 83(3):621–643, 1996. 171, 207

[GR03] Ofer Gabber and Lorenzo Ramero. *Almost ring theory*, volume 1800 of *Lecture Notes in Mathematics*. Springer-Verlag, 2003. 201

[Hak72] Monique Hakim. *Topos annelés et schémas relatifs*. Springer-Verlag, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 64. 15

[Ill71] Luc Illusie. *Complexe cotangent et déformations. I*, volume 239 of *Lecture Notes in Mathematics*. Springer-Verlag, 1971. 203

[Kela] Bernhard Keller. Derived categories and their uses. Available at `http://school2011.incarne.net/pdf/Literature/Keller-derived.pdf`. 180

[Kelb] Bernhard Keller. On differential graded categories. Available at `http://atlas.mat.ub.edu/grgta/articles/Keller.pdf`. 186, 187

[Kel03] Bernhard Keller. *Notes for an introduction to Kontsevich's quantization theorem*. 2003. availeble at `http://www.math.jussieu.fr/~keller/publ/emalca.pdf`. 41

[KKP08] L. Katzarkov, M. Kontsevich, and T. Pantev. Hodge theoretic aspects of mirror symmetry. In *From Hodge theory to integrability and TQFT tt*-geometry*, volume 78 of *Proc. Sympos. Pure Math.*, pages 87–174. Amer. Math. Soc., Providence, RI, 2008. 177

[KL13] Kiran Kedlaya and Ruochuan Liu. Relative p-adic hodge theory, i: Foundations. 2013. `arXiv:1301.0792`. 200

[Kra07] Henning Krause. Derived categories, resolutions, and Brown representability. In *Interactions between homotopy theory and algebra*, volume 436 of *Contemp. Math.*, pages 101–139. Amer. Math. Soc., Providence, RI, 2007. 190

[LS75] T. Y. Lam and M. K. Siu. K_0 and K_1–an introduction to algebraic K-theory. *Amer. Math. Monthly*, 82:329–364, 1975. 76

[Mat89] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid. 31

[Maz73] Barry Mazur. Notes on étale cohomology of number fields. *Ann. Sci. École Norm. Sup. (4)*, 6:521–552, 1973. 79

[MLM92] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in geometry and logic*. Universitext. Springer-Verlag, New York, 1992. A first introduction to topos theory. 15

[MV88] R. MacPherson and K. Vilonen. Perverse sheaves with singularities along the curve $y^n = x^m$. *Comment. Math. Helv.*, 63(1):89–102, 1988. 171, 172

[Nee] A. Neeman. Triangulated categories. Available at `http://hopf.math.purdue.edu/Neeman/triangulatedcats.pdf`. 185

[Oka50] Kiyoshi Oka. Sur les fonctions analytiques de plusieurs variables. VII. Sur quelques notions arithmétiques. *Bull. Soc. Math. France*, 78:1–27, 1950. 17

[Orl04] D. O. Orlov. Triangulated categories of singularities and D-branes in Landau-Ginzburg models. *Tr. Mat. Inst. Steklova*, 246(Algebr. Geom. Metody, Svyazi i Prilozh.):240–262, 2004. 177, 178, 182

[OSS11] Christian Okonek, Michael Schneider, and Heinz Spindler. *Vector bundles on complex projective spaces*. Modern Birkhäuser Classics. Birkhäuser/Springer, 2011. Corrected reprint of the 1988 edition, with an appendix by S. I. Gelfand. 147

[Qui73] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories*, number 541 in Lecture Notes in Mathematics, pages 85–147. Springer, Berlin, 1973. 156

[Rei10] Ryan Reich. Notes on Beilinson's "How to glue perverse sheaves". *J. Singul.*, 1:94–115, 2010. 171

[Sch12] Peter Scholze. Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.*, 116:245–313, 2012. 199, 200, 202

[Ste75] Bo Stenström. *Rings of quotients*, volume 217 of *Die Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, New York, 1975. 134

[Tab05] Goncalo Tabuada. Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories. *C. R. Math. Acad. Sci. Paris*, 340(1):15–19, 2005. 187

[Tat68] John Tate. Residues of differentials on curves. *Ann. Sci. École Norm. Sup. (4)*, 1:149–159, 1968. 238

[Toë] Bertrand Toën. The homotopy theory of dg-categories and derived morita theory. `arXiv:math/0408337`. 177, 186

[Toë11] Bertrand Toën. Lectures on dg-categories. In *Topics in algebraic and topological K-theory*, volume 2008 of *Lecture Notes in Math.*, pages 243–302. Springer, Berlin, 2011. 129, 188, 191

[Ver96] Jean-Louis Verdier. Des catégories dérivées des catégories abéliennes. *Astérisque*, (239):xii+253 pp. (1997), 1996. With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis. 143