

Mapping class groups of surfaces and quantization

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- ① Mapping class groups.
- ② Quantum representations from skein theory.
- ③ Quantum representations from geometric quantization.
- ④ Character varieties as the tensor product of functors.

Mapping class group of a surface

Let Σ_g be a closed compact oriented surface of genus g .

Definition

The **mapping class group** $\mathcal{M}(\Sigma_g)$ is the group of orientation-preserving diffeomorphisms modulo isotopy:

$$\mathcal{M}_g := \mathcal{M}(\Sigma_g) := \text{Diff}^+(\Sigma_g) / \text{Diff}_0^+(\Sigma_g)$$

Examples:

- 1 $\mathcal{M}(\mathbb{S}^2) = \{1\}$;
- 2 $\mathcal{M}(\mathbb{T}) \simeq \text{SL}_2(\mathbb{Z})$.

Structure of $\mathcal{M}(\Sigma_g)$



Theorem (Dehn)

\mathcal{M}_g is generated by Dehn twists along non-separating circles in Σ_g .

Fact: \mathcal{M}_g is a finitely presented group, there are explicit generators and relations.

Skein modules and algebras

Definition

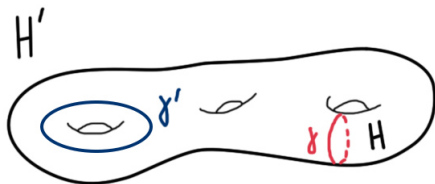
Fix $\xi \in \mathbb{C}^\times$. For a 3-manifold N , the **skein module** $\mathcal{K}_\xi(N)$ is a \mathbb{C} -vector space spanned by the isotopy classes of (framed) links in N modulo the **skein relations**:

$$\begin{aligned} \text{Crossing} &= \xi \text{ (positive crossing)} + \xi^{-1} \text{ (negative crossing)} \\ L \cup \text{circle} &= (-\xi^2 - \xi^{-2})L \end{aligned}$$

Fact: $\mathcal{K}_\xi(\mathbb{S}^3) \simeq \mathbb{C}$.

Skein pairing and the action of $\mathcal{M}(\Sigma)$

$\Sigma \hookrightarrow \mathbb{S}^3$ such that $\mathbb{S}^3 \setminus \Sigma = H \sqcup H'$ is the union of two handlebodies, let $\xi = \sqrt[4k+8]{1}$.



$$\langle -, - \rangle: \mathcal{K}_\xi(H) \times \mathcal{K}_\xi(H') \rightarrow \mathcal{K}_\xi(\mathbb{S}^3) \simeq \mathbb{C}$$
$$\langle -, - \rangle: V_k \times V'_k \rightarrow \mathbb{C}$$

Need: define how Dehn twists acts on V_k .

Fact: if γ bounds a disk in H , the Dehn twist τ_γ acts on $\mathcal{K}_\xi(H)$, inducing an action on V_k . Similarly, for γ' bounding a disk in H' , $\tau_{\gamma'}$ acts on V'_k .

Note: using the pairing, $\tau_{\gamma'}$ also act on V_k .

Fact: This gives a well-defined action of $\mathcal{M}(\Sigma)$ on $\mathbb{P}(V_k)$.

Definition

Call $\mathbb{P}(V_k)$ the **quantum representation of \mathcal{M}_g** of level k .

Theorem (Lickorish)

Spaces V_k are finite dimensional, and their dimension is

$$d_g(k) := \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} \left(\sin \frac{\pi j}{k+2}\right)^{2-2g}$$

Character variety

Let Γ be a group, and G a compact Lie group.

Let $\text{Rep}(\Gamma, G)$ be the variety of representations of Γ into G .

- **Example:** $\text{Rep}(\mathbb{Z}, G) \simeq G$;
- **Example:** $\text{Rep}(\mathbb{Z} \times \mathbb{Z}, G) = \{(A, B) \in G \times G \mid AB = BA\}$.

Note: G acts on $\text{Rep}(\Gamma, G)$ by conjugation.

The quotient $\mathcal{X}(\Gamma, G) = \text{Rep}(\Gamma, G)/G$ is called the **character variety**.

Note: in general $\text{Rep}(\Gamma, G)$ is quite singular, even for “nice” Γ .

Character variety of surface groups

Let $\Gamma = \pi_1(\Sigma, x_0) \simeq \langle a_1, \dots, a_g, b_1, \dots, b_g \mid [a_1, b_1] \dots [a_g, b_g] = 1 \rangle$.

Then $\mathcal{X}(\Gamma, G) \simeq \{(A_1, \dots, B_g) \in G^{2g} \mid [A_1, B_1] \dots [A_g, B_g] = 1\} / G$.

$\mathcal{X}(\Gamma, G)$ is singular, and let $\mathcal{X}^{\text{reg}} \subset \mathcal{X}(\Gamma, G)$ be the regular part.

Theorem (Atiyah–Bott)

For simply-connected G , \mathcal{X}^{reg} has a natural symplectic form ω . The form ω only depends on the choice of a symmetric form on $\mathfrak{g} = \text{Lie}(G)$.

Geometric quantization

Fact: there exists a line bundle \mathcal{L} on \mathcal{X}^{reg} such that

$$c_1(\mathcal{L}) = [\omega]$$

Pick σ – a complex structure on Σ . Then $\sigma \rightsquigarrow$ complex structure on \mathcal{X}^{reg} .
 $\mathcal{X} \rightsquigarrow \mathcal{X}_\sigma^{\text{reg}}$ a complex manifold, and $\mathcal{L} \rightsquigarrow \mathcal{L}_\sigma$ a holomorphic line bundle.

Let $W_{k,\sigma} := H^0(\mathcal{X}^{\text{reg}}, \mathcal{L}_\sigma^{\otimes k})$ the space of holomorphic sections.

Theorem

There is a natural action of the mapping class group $\mathcal{M}(\Sigma)$ on the spaces $\mathbb{P}(W_{k,\sigma})$. Moreover, $W_{k,\sigma}$ are finite dimensional, and for $G = SU(2)$

$$\dim(W_{k,\sigma}) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} \left(\sin \frac{\pi j}{k+2}\right)^{2-2g}$$

Theorem (Andersen–Ueno)

The projective representations $\mathbb{P}(V_k)$ and $\mathbb{P}(W_k)$ of the mapping class group $\mathcal{M}(\sigma)$ are isomorphic.

Remarks:

- 1 Both constructions can be carried for any compact simply-connected Lie group G .
- 2 In skein theory: choice of H, H' , k comes from $\xi = \sqrt[4k+8]{1}$.
In geom.quant.: choice of complex structure σ , k comes from $k\omega$.

Category \mathcal{H}

Let \mathcal{H} be a symmetric monoidal category with

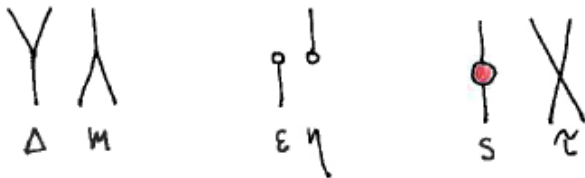
$\text{Ob}(\mathcal{H}) = \mathbb{N} = \{[0], [1], [2], \dots\}$, $[n] \otimes [m] := [n + m]$,

and $\text{Mor}(\mathcal{H})$ generated by

$$m: [2] \rightarrow [1], \quad \eta: [0] \rightarrow [1], \quad S: [1] \rightarrow [1]$$

$$\Delta: [1] \rightarrow [2], \quad \varepsilon: [1] \rightarrow [0], \quad \tau: [2] \rightarrow [2]$$

satisfying the obvious (?) axioms. Graphically,



Relations in \mathcal{H}

$(ab)c = a(bc)$

$\Delta(a)\Delta(b) = \Delta(ab)$

Note: cocommutative Hopf algebras \equiv monoidal functors $F: \mathcal{H} \rightarrow \text{Vect}$.

Representation and character varieties

Any functors $F: \mathcal{H} \rightarrow \text{Vect}_{\mathbb{K}}$ and $E: \mathcal{H}^{\text{op}} \rightarrow \text{Vect}_{\mathbb{K}}$ give $E \otimes_{\mathcal{H}} F \in \text{Vect}_{\mathbb{K}}$.

If F, E are **weakly monoidal**, then $E \otimes_{\mathcal{H}} F$ is an **algebra**.

Let Γ be a discrete group, and G be an affine algebraic group.

Then $\mathbb{K}[\Gamma]$ is a cocommutative Hopf algebra, $\mathbb{K}(G)$ is a commutative Hopf algebra, and so they define functors

$$\begin{aligned} F_{\Gamma}: \mathcal{H} &\rightarrow \text{Vect}, & [n] &\mapsto \mathbb{K}[\Gamma]^{\otimes n} \\ E_G: \mathcal{H}^{\text{op}} &\rightarrow \text{Vect}, & [n] &\mapsto \mathbb{K}(G)^{\otimes n} \simeq \mathbb{K}(G^n) \\ E'_G: \mathcal{H}^{\text{op}} &\rightarrow \text{Vect}, & [n] &\mapsto \mathbb{K}(G^n)^G. \end{aligned}$$

Theorem (Kassabov–P)

There are natural **algebra isomorphisms**

$$E_G \otimes_{\mathcal{H}} F_{\Gamma} \simeq \mathbb{K}(\text{Rep}(\Gamma, G))$$

$$E'_G \otimes_{\mathcal{H}} F_{\Gamma} \simeq \mathbb{K}(\mathcal{X}(\Gamma, G))$$

Character variety: $\mathbb{K}(\mathcal{X}(\Gamma, G)) \simeq E'_G \otimes_{\mathcal{H}} F_{\Gamma} = \bigoplus_n \mathbb{K}(G^n)^G \otimes \mathbb{K}[\Gamma]^{\otimes n} / \sim$

Idea: “quantize” $\mathbb{K}(G)$, $\mathbb{K}[\Gamma]$ and \mathcal{H} .

Assume: $\Gamma = \pi_1(\Sigma)$

Replace: $\mathbb{K}[\Gamma]^{\otimes n} \rightsquigarrow$

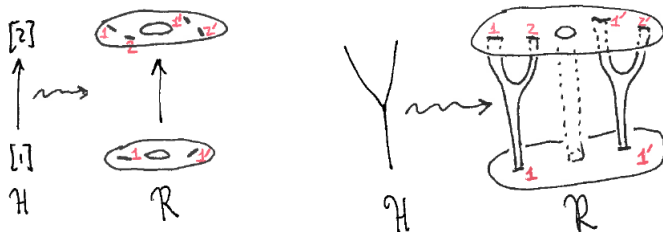
$\mathbb{K}\{n\text{-tuples of ribbons in } \Sigma \times (-\infty, 0] \text{ with ends in a small fixed disk on } \Sigma \times \{0\}\}$

Replace: $\mathbb{K}(G) \rightsquigarrow \mathbb{K}_q(G)$, the corresponding quantum group.

Replace: $\mathcal{H} \rightsquigarrow \mathcal{R}$ a certain category with objects being slits in an annulus and morphisms being ribbons in the cylinder, connecting the slits.

Category \mathcal{R}

Morphisms in \mathcal{R} are ribbon analogs of the morphisms in \mathcal{H} :



Σ gives a functor $F_\Sigma: \mathcal{R} \rightarrow \text{Vect}_{\mathbb{K}}$, and $\mathbb{K}_q(G)$ gives $E_{\mathbb{K}_q(G)}: \mathcal{R}^{\text{op}} \rightarrow \text{Vect}$.

Theorem (Kassabov-P)

$F_\Sigma \otimes_{\mathcal{R}} E_{\mathbb{K}_q(G)}$ is a (non-commutative) algebra "quantizing" $\mathbb{K}(\mathcal{X}(\Gamma, G))$.