Derived Poisson structures and higher character maps

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Plan

1. Classical representation varieties and H_0 Poisson structures.
2. Derived characters.
3. Derived Poisson structures.

Representation schemes

Assumption: k is a fixed field of $char(k) = 0$, all algebras are over k, $\otimes$ denotes $\otimes_k$.

Let $A \in \mathbf{Alg}_k$ be an associative algebra, $V = k^n$ an n-dimensional vector space.

By $\mathrm{Rep}_n(A)$ we denote the moduli space of representation of A in k^n.

Example. $\mathrm{Rep}_n(k\langle x_1, \dots, x_r\rangle) = \mathrm{Mat}_n^{\times r} \simeq \mathbb{A}^{rn^2}$.

Example. $\mathrm{Rep}_n(k[x_1, \dots, x_r]) \subset \mathrm{Mat}_n^{\times r}$ is the closed subscheme, consisting of tuples $(B_1, \dots, B_r)$ of pair-wise commuting matrices.

Character map

Characters define a linear map $\mathsf{Tr}\colon A \to k[\mathrm{Rep}_n(A)]$

$$a \mapsto [\mathsf{Tr}(a)\colon \rho \mapsto \mathsf{tr}(\rho(a))]\,,\ \forall \rho \in \mathrm{Rep}_n(A)$$

This map factors as

$$\begin{tikzcd}
A \arrow[r, "\mathsf{Tr}"] \arrow[d, two heads] & k[\mathrm{Rep}_n(A)] \\
A/[A,A] \arrow[r] & k[\mathrm{Rep}_n(A)]^{\mathsf{GL}_n} \arrow[u, hook, "i"]
\end{tikzcd}$$

The map $A/[A,A] \to k[\mathrm{Rep}_n(A)]^{\mathsf{GL}_n}$ will be called the *character map*.

Procesi theorem

Theorem (Procesi)

The induced homomorphism of algebras

$$\mathsf{Sym}(\mathsf{Tr})\colon\ \mathsf{Sym}(A/[A,A]) \to k[\mathrm{Rep}_n(A)]^{\mathsf{GL}_n}$$

is surjective.

Thus, the characters “capture” representation theory of A:

they determine rings of functions $k[\mathrm{Rep}_n(A)]^{\mathsf{GL}_n}$, which determine the moduli spaces of semi-simple representations.

NC Poisson bracket

Kontsevich–Rosenberg Principle: any "noncommutative" structure on A should induce via the character map its commutative version on $\mathrm{Rep}_n(A)$ for all n.

Definition (Crawley-Boevey)

An H_0-Poisson structure on $A \in \mathbf{Alg}_k$ consists of a Lie bracket $\{-,-\}$ on $A/[A,A]$ s.t. $\{\bar{a},-\}$ is induced by a derivation of A.

Theorem (Crawley-Boevey)

For all n, there is a unique Poisson bracket on $\mathrm{Rep}_n(A)/\!\!/\,\mathsf{GL}_n$ for any n making the map $\mathsf{Tr}\colon A/[A,A] \to k[\mathrm{Rep}_n(A)]^{\mathsf{GL}_n}$ into a map of Lie algebras.

Examples: commutative Poisson algebras, path algebras of doubled quivers, (deformed) preprojective algebras etc.

Extension to DG algebras

In general, $\mathrm{Rep}_n(A)$ is "badly behaved," for example, it is quite singular even for very nice algebras (e.g. polynomial algebras $A = k[x_1, ..., x_d], d > 1$)

One way to "resolve singularities" is to **derive** Rep_n.

Call the functor $(-)_n \colon \mathbf{Alg}_k \to \mathbf{ComAlg}_k$ sending

$$A \mapsto A_n := k[\mathrm{Rep}_n(A)]$$

the **representation functor**.

It extends naturally to $(-)_n \colon \mathbf{DGA}_k \to \mathbf{CDGA}_k$.

Problem: The functor $(-)_n$ is not "exact", i.e. it does not respect quasi-isomorphisms.

Derived representation functor

For $A \in \mathrm{Alg}_k$, resolution is any semi-free DG algebra $R \in \mathrm{DGA}_k$ with a surjective quasi-isomorphism $R \overset{\sim}{\twoheadrightarrow} A$.

Theorem (Berest–Khachatryan–Ramadoss)

The functor $(-)_n$ has a total left derived functor $\mathbb{L}(-)_n$ defined by $\mathbb{L}(A)_n = R_n$ for a resolution $R \overset{\sim}{\twoheadrightarrow} A$. The algebra $\mathbb{L}(A)_n$ **does not depend** *on the choice of resolution, up to quasi-isomorphism.*

Denote $\mathbb{L}A_n$ by $\mathrm{DRep}_n(A)$, call it *derived representation scheme.*
Example: If $A = k[x, y]$, we can take $R = k\langle x, y, \lambda\rangle$ with $\deg(x) = \deg(y) = 0$, $\deg(\lambda) = 1$ and $d\lambda = xy - yx$.
The obvious projection $R \twoheadrightarrow A$ is a quasi-isomorphism.
Then $\mathrm{DRep}_n(A) = k[x_{ij}, y_{ij}, \lambda_{ij}]$ with $\deg(\lambda_{ij}) = 1$ and

$$d\lambda_{ij} = \sum_{k=1}^{n} x_{ik}y_{kj} - y_{ik}x_{kj}$$

Derived character maps

Define n-dimensional **representation homology** by

$$H_\bullet(A, n) := H_\bullet\left[\mathrm{DRep}_n(A)\right]$$

Fact: $H_0(A, n) \simeq k[\mathrm{Rep}_n(A)] =: A_n$.

Proposition (Berest-Khachatryan-Ramadoss)

For any algebra $A \in \mathbf{Alg}_k$ *and any* n *there exists canonical derived character map*

$$\mathrm{Tr}_n(A)_\bullet \colon HC_\bullet(A) \to H_\bullet(A, n)^{\mathrm{GL}_n},$$

lifting the original character map

$$\mathrm{Tr} \colon HC_0(A) = A/[A, A] \to A_n^{\mathrm{GL}_n}$$

General formula for derived characters

There exists an “explicit” formula for derived character maps.

Theorem (Berest–Khachatryan–Ramadoss)

For any $A \in \mathbf{Alg}_k$ *the maps* $\mathrm{Tr}_n(A)_\bullet \colon HC_\bullet(A) \to H_\bullet(A,n)^{\mathrm{GL}_n}$ *are induced by the morphism of complexes* $T \colon CC_\bullet(A) \to R_n^{\mathrm{GL}_n}$ *whose p-th graded component is given by* $T_p \colon A^{\otimes(p+1)}/(1-\tau_n) \to (R_n)_p$

$$T_p(a_0,\dots,a_p) = \sum_{l\in\mathbb{Z}_{p+1}} (-1)^{pl}\,\mathrm{Tr}_n\left[f_{p+1}(a_l,a_{1+l},\dots,a_{p+l})\right]$$

Here $\{f_{p+1} \colon A^{\otimes(p+1)} \to R\}_{p\geq 0}$ are the components of an A_∞ quasi-isomorphism, inverse to the given resolution $R \overset{\sim}{\twoheadrightarrow} A$.

Symmetric algebras

Goal: compute derived character maps for symmetric algebras.

For simplicity, assume that $n = 1$ (i.e. we will only consider 1-dimensional representations).

In general, $Tr(A)_\bullet$ factors through the *reduced* cyclic homology $\overline{HC}_\bullet(A)$.

For $A = \mathrm{Sym}(W)$ a polynomial algebra,

$$\overline{HC}_i(A) \simeq \Omega^i(W)/d\Omega^{i-1}(W).$$

Thus, we can think of $\mathrm{Tr}(A)_i$ as maps

$$\mathrm{Tr}(A)_i \colon \Omega^i(W) \to H_i(A, 1)$$

Example: $A = k[x, y]$

- Recall the resolution $R \overset{\sim}{\twoheadrightarrow} A$ given by $R = k\langle x, y, \lambda\rangle$ with $\deg(x) = \deg(y) = 0$, $\deg(\lambda) = 1$ and $d\lambda = xy - yx$.
- Then $\mathrm{DRep}_1(A) \simeq k[x, y, \lambda]$ with zero differential.
- The character $\mathrm{Tr}_0 \colon k[x, y] \to k[x, y, \lambda]$ is given by

  $$\mathrm{Tr}_0(P) = P$$

  for any $P \in k[x, y]$.
- The character $\mathrm{Tr}_1 \colon \Omega^1(A) \to k[x, y, \lambda]$ is given by

  $$\mathrm{Tr}_1(P(x, y)dx + Q(x, y)dy) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\lambda$$

Koszul resolution of $A = \mathrm{Sym}(W)$

Koszul resolution of $A = \mathrm{Sym}(W) \simeq k[x_1, \dots, x_m]$ is given by $R = \mathbf{\Omega}(C)$, the cobar construction of coalgebra $C = \mathrm{Sym}(W[1])$.

As a graded algebra, it is

$$R = T(s^{-1}\Lambda(sW)) \simeq T(W) * T\left(\Lambda^2(W) \oplus \Lambda^3(W) \oplus \cdots \oplus \Lambda^r(W)\right)$$

We denote generators *of degree* $p-1$ by

$$\lambda(v_1, v_2, \dots, v_p) := s^{-1}(sv_1 \wedge sv_2 \wedge \ldots \wedge sv_p) \in s^{-1}\Lambda^p(sW)$$

Then, $\mathrm{DRep}_1(A)$ is isomorphic to abelianization R_{ab} of R

$$\mathrm{DRep}_1(A) \simeq \mathrm{Sym}(W) \otimes \mathbf{Sym}\left(\Lambda^2(W) \oplus \cdots \oplus \Lambda^r(W)\right).$$

with **zero differential**, so $H_\bullet(A, 1) \simeq \mathrm{DRep}_1(A)$.

The map Tr_1

Proposition

If $A = k[x_1, \ldots, x_m]$*, then* $\mathrm{Tr}(A)_1$ *is given by (shifted) de Rham differential* $s^{-1}d_{dR}$*. Namely, for* $\alpha = \sum P_i dx_i \in \Omega^1(A)$ *we have*

$$\mathrm{Tr}(A)_1(\alpha) = \sum_{i<j} \left(\frac{\partial P_i}{\partial x_j} - \frac{\partial P_j}{\partial x_i} \right) \lambda(x_i, x_j) \in H_\bullet(A, 1)$$

This proposition might suggest that Tr_i for $i \geq 2$ will also be just de Rham differential $d_{dR} \colon \Omega^i(W) \to \Omega^{i+1}(W)$ followed by the embedding (of degree -1)

$$\Omega^{i+1}(W) \hookrightarrow \mathrm{Sym}(W) \otimes \mathbf{Sym}\left(\Lambda^2(W) \oplus \cdots \oplus \Lambda^r(W)\right) = R_{ab}$$

Surprising fact: this is not the case!

Tr_2 for $A = k[x, y, z]$

Take $\omega = Pdx \wedge dy + Qdy \wedge dz + Rdz \wedge dx \in \Omega^2(A)$.
Then $\mathrm{Tr}(A)_2(\omega)$ is given by

$$M\lambda(x, y, z) + M_y\lambda(x, y)\lambda(y, z) + M_z\lambda(y, z)\lambda(z, x) + M_x\lambda(z, x)\lambda(x, y),$$

where

$$M := P_z + Q_x + R_y$$

and for a polynomial F, F_q denotes $\frac{\partial F}{\partial q}$.
$\mathrm{Tr}(A)_2 = D \circ d_{dR}$, where

$$D = s^{-1} + \widetilde{D} \colon \Omega^3 \to H_\bullet(A, 1)$$

is a certain canonical differential operator on differential forms.

Abstract Chern–Simons forms

Let $\mathcal{A}$ be a **cohomologically** graded commutative DG algebra, $\mathfrak{g}$ a finite dimensional Lie algebra.
A $\mathfrak{g}$-valued **connection** is an element $\theta \in \mathcal{A}^1 \otimes \mathfrak{g}$.
It **curvature** is $\Theta := d\theta + \frac{1}{2}[\theta, \theta]$, and Bianchi identity holds:

$$d\Theta = [\Theta, \theta]$$

If $P \in I^r(\mathfrak{g})^{ad\mathfrak{g}}$, for any $\alpha \in \mathcal{A} \otimes \mathrm{Sym}^r(\mathfrak{g})$ we can define $P(\alpha) \in \mathcal{A}$ via

$$\mathcal{A} \otimes \mathrm{Sym}^r(\mathfrak{g}) \xrightarrow{\frac{1}{r!}\,\mathrm{id} \otimes \mathrm{ev}_P} \mathcal{A}$$

Then $P(\Theta^r) \in \mathcal{A}^{2r}$ is exact, and there exists $\mathrm{CS}_P(\theta) \in \mathcal{A}^{2r-1}$ such that $d\,\mathrm{CS}_P(\theta) = P(\Theta^r)$ with $\mathrm{CS}_P(\theta)$ is given explicitly by

$$\mathrm{CS}_P(\theta) = \frac{1}{r!} \int_0^1 P(\theta \wedge \Theta_t^{r-1}) dt$$

where $\Theta_t = t\Theta + \frac{1}{2}(t^2 - t)[\theta, \theta]$.

Derived character maps for polynomial algebras

Let $\mathcal{A} = \underline{\mathrm{hom}}(\Omega^\bullet(W), R_{ab})$. Take $\mathfrak{g} = k$ and let $P_r = x^r \in k[\mathfrak{g}] \simeq k[x]$.

Theorem (Berest-Felder-P-Ramadoss-Willwacher)

There is a canonical k-valued connection θ in $\mathcal{A}$ such that the derived character map $\mathrm{Tr}\colon \Omega^\bullet(A) \to R_{ab} \simeq H_\bullet(A, 1)$ is given by

$$\mathrm{Tr}(A) = \sum_{r=0}^{\infty} \mathrm{CS}_{P_r}(\theta) \circ d.$$

Here, $\theta(P(x_1, \dots, x_m)dx_{i_1} \dots dx_{i_p}) = P(0, \dots, 0)\lambda(x_{i_1}, \dots, x_{i_p})$

Derived Poisson structure

For an associative algebra A we define *derived Poisson structure* to be a graded Lie algebra structure on $HC_\bullet(A)$ induced from a DG Lie algebra structure on $R/[R,R]$ for some resolution $R \overset{\sim}{\twoheadrightarrow} A$ of A.

Theorem (Berest-Chen-F.Eshmatov-Ramadoss)

Given a derived Poisson structure on A, there is unique graded Poisson bracket on $H_\bullet(A,n)^{\mathrm{GL}_n}$ making $\mathrm{Tr}_n(A)_\bullet \colon HC_\bullet(A) \to H_\bullet(A,n)^{\mathrm{GL}_n}$ into a map of graded Lie algebras.

Example: $A = k[x, y]$

- There is a natural Lie bracket on $\overline{HC}_\bullet(A)$ lifting the usual Poisson bracket on the polynomials $\overline{HC}_0(A) = \overline{A}$. For $\overline{f} \in \overline{A}$ and $\overline{\alpha} \in \Omega^1/dA$ it is given by
$$\{\overline{f}, \overline{\alpha}\} = \mathcal{L}_{\theta_f}(\alpha)$$
- This bracket induces a Poisson bracket on $k[x, y, \lambda]$, which is the usual Poisson bracket on deg $= 0$ part:
$$\{P, Q\} = \frac{\partial P}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y}\frac{\partial Q}{\partial x}$$
and $\{P, \lambda\} = 0$.

Derived Poisson structures on cobar constructions

Let C be a DG coalgebra. We say that it is d-cyclic if there is a bilinear form

$$\langle -, - \rangle \colon C \times C \to k[d]$$

satisfying $\langle du, v \rangle \pm \langle u, dv \rangle = 0$ and

$$\langle v', w \rangle \cdot v'' = \pm \langle v, w'' \rangle \cdot w'$$

Theorem (Berest-Chen-F.Eshmatov-Ramadoss)

If C is a d-cyclic DG coalgebra, then there is a derived Poisson structure of degree $d-2$ on the cobar construction $\mathbf{\Omega}(C)$ induced from the map $\mathbf{\Omega}(C) \otimes \mathbf{\Omega}(C) \to \mathbf{\Omega}(C)$ given by

$$\{v, w\} := \sum \langle v_i . w_j \rangle \cdot (w_1, \dots, w_{j-1}, v_{i+1}, \dots, v_p, v_1, \dots, v_{i-1}, w_{j+1}, \dots, w_q)$$

where $v = (v_1, \dots, v_p)$ and $w = (w_1, \dots, w_q)$.

Brackets for polynomial algebras

Theorem (Chen-A.Eshmatov-F.Eshmatov-Yang)

If $A = \mathrm{Sym}(W)$ *with* $\dim W = m$ *, then the Koszul dual coalgebra* $C = \mathrm{Sym}^c(W[1])$ *has* **unique** *cyclic structure, and it is of degree* $-m$*.*

If we denote $\Psi : \Theta_r(W) \xrightarrow{\sim} \Omega^{m-r}(W)$ *the isomorphism* $\Phi(\xi) = \iota_\xi \omega$ *with* ω *being the volume form on* W*, then the cyclic structure above gives derived Poisson bracket on* $\overline{HC}_\bullet(A) = \Omega^\bullet(A)/d\Omega^{\bullet-1}(A)$ *given by*

$$\{\alpha, \beta\} = (-1)^{(m-|\alpha|-1)(m-|\beta|)} \iota_\eta d\alpha$$

with $\eta := \Psi^{-1}(d\beta)$*.*

Bracket on $H_\bullet(k[x,y,z],1)$

There is a natural Poisson bracket of degree -1 on

$$H_\bullet(k[x,y,z],1) \simeq k[x,y,z,\lambda(x,y),\lambda(y,z),\lambda(z,x),\lambda(x,y,z)]$$

determined by

$$\begin{aligned} \{P,\lambda(x,y)\} &= P_z \\ \{P,\lambda(y,z)\} &= P_x \\ \{P,\lambda(z,x)\} &= P_y \end{aligned}$$

for $P \in k[x,y,z]$, and zero for all other generators.

References

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