Euler characteristic. Orientatability

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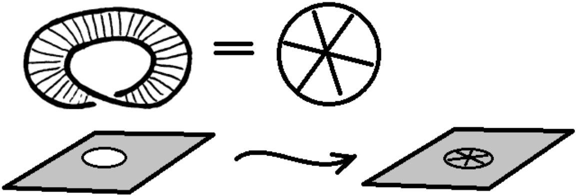
Theorem

Suppose Σ is a surface, and G is an embedded graph. Then the Euler characteristic $\chi(\Sigma) := V - E + F$ is correctly defined.

Exercise: compute Euler characteristic of $\mathbb{R}P^2, K^2, T^2, M^2$.

Attaching a Möbius band:



Euler characteristic and attachments

Question: How does $\chi(\Sigma)$ change when attaching a handle? When attaching a Möbius strip?

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If Σ' is Σ with attached handle, then

$$\chi(\Sigma') = \chi(\Sigma) - 2$$

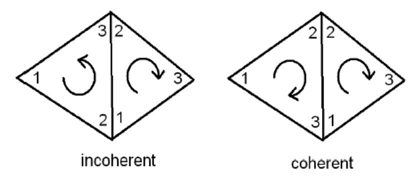
Orientable triangulations

Definition

A **triangulation** of a surface Σ is an embedding of a graph G into Σ such that all faces are triangles.

Definition

A triangulation is **orientable** if all faces can be oriented in a **coherent** way:



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Similarly for any 2-cell decomposition of Σ.

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The following are equivalent:

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2. **any** *triangulation is orientable;*
3. **any** *2-cell decomposition is orientable.*

Super sketchy sketch of proof

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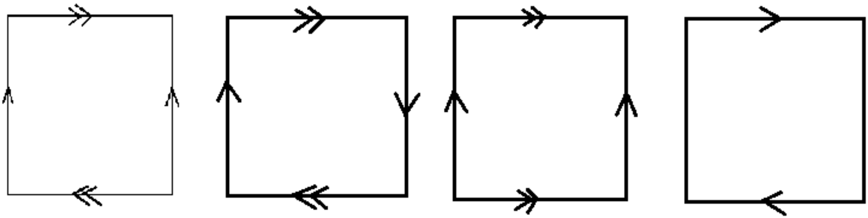
Orientability is invariant under coarsening.

Which of the following surfaces are orientable?

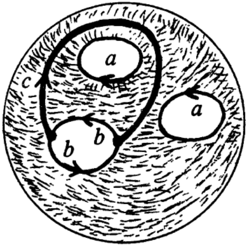


Lemma about attaching

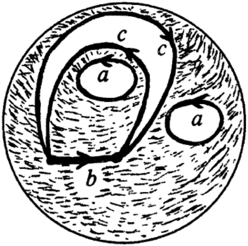
Lemma

Sphere with one handle and one Möbius band is homeomorphic to a sphere with 3 Möbius bands.

Proof:



(a)



(b)

Compact surfaces

A surface embedded in $\mathbb{R}^n$ is called **compact** if it is closed and bounded.
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Give example of a closed surface which is not bounded.

Give example of a bounded surface which is not closed.

Surfaces with and without boundary

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Surfaces are surfaces with (empty) boundary.

Classification theorem

Theorem

Any compact surface Σ is determined uniquely up to homeomorphism by the following data:

$$\chi(\Sigma), \text{ orientability }, \text{ number of boundary components}$$

In particular, orientable surfaces with no boundary are just spheres with handles. Non-orientable surfaces are just spheres with Möbius bands (more than one).