Symmetric groups. Abstract groups.

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Last time

Definition

Let X be a set, and let G be a subset of the set $Bij(X)$ of all bijections $X \to X$. One says G is a **group** if

1. G is closed under composition;
2. $id \in G$;
3. if $g \in G$, then $g^{-1} \in G$.

Example

Symmetries of a triangle, a square and a mattress form a group.

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- Prove that any permutation is a composition of transpositions of neighbors.

Notation

It is convenient to denote permutations by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \dots & \sigma(n) \end{pmatrix}$$

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- Verify that $(\sigma_2 \circ \sigma_1)^{-1} = \sigma_1^{-1} \circ \sigma_2^{-1}$.

Cycle decomposition

- Let $a_1, \dots, a_m \in \{1, 2, \dots, n\}$ distinct elements. Denote by $(a_1 \dots a_m)$ the cyclic permutation $a_1 \mapsto a_2 \mapsto \dots \mapsto a_m \mapsto a_1$.

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- Prove that **any** permutation can be represented by composition of cycles on disjoint collections of elements.
- Note: such a decomposition is unique up to the order of composition.
- Note: two disjoint cycles commute.

Sign of a permutation

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For $\sigma \in S_n$ define $inv(\sigma)$ to be the number of pairs (ij) such that $i < j$ but $\sigma(i) > \sigma(j)$. This number $inv(\sigma)$ is called the **number of inversions** of σ.

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Definition

If $sgn(\sigma) = 1$, σ is called an **even permutation**, otherwise it's called **odd**.