Symmetries of a cube. Group actions

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Last time

Definition

Let G, H be two groups. Define the **product** $G \times H$ of group G, H to be the set of all pairs (g, h) with $g \in G, h \in H$ and with operation $*$ on it given by

$$(g, h) * (g', h') := (gg', hh')$$

- $(G \times H, *)$ is again a group.
- The subsets $G \times \{e\} = \{(g, e) \mid g \in G\}$ and $\{e\} \times H$ are two subgroups of $G \times H$, isomorphic to G and H respectively.
- The two subgroups commute with each other.

Product of groups

Theorem

Let G be a group, and $H, K \subseteq G$ be two subgroups satisfying the following three properties:

1. $H \cap K = \{e\}$;
2. *elements of H commute with elements of K, i.e. $hk = kh$ for any $h \in H, k \in K$;*
3. $G = HK$, *i.e. any g can be represented as $g = hk$ with $h \in H, k \in K$.*

Then $G \simeq H \times K$.

Symmetries of a cube

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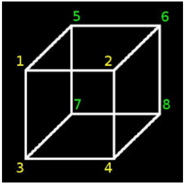
The group G of symmetries of a cube is isomorphic to $S_4 \times \mathbb{Z}/2$.



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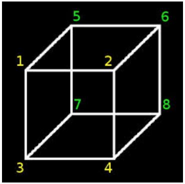

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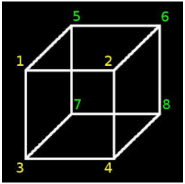

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- There are $|G| = 48$ symmetries.

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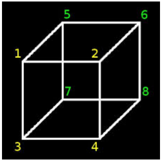

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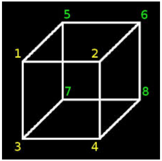

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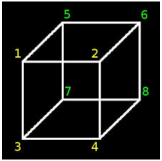

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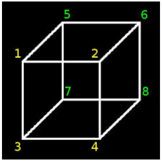

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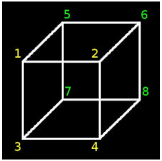

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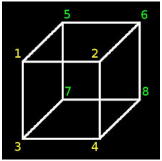

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- By the order consideration, $G = RS$, and so $G \simeq R \times S$ by the previous theorem.

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- In particular any subgroup of G also acts on whatever G acts on.

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- Note that the circle S^1 is itself a group. It acts on $\mathbf{R}^2$ by rotations. In other words, e^{ix} rotates $\mathbb{R}^2$ around the origin by the angle x counter-clockwise.