SYMMETRY GROUPS OF PUZZLES

The Pyraminx is a puzzle similar in idea to the famous Rubik's Cube. It looks like this:

Problem 1. Play with the puzzle for a bit to convince yourself that the problem is non-trivial.

We assume that the puzzle is standing on a flat surface, with one of the facets facing us. We denote the vertices of the tetrahedron by u, l, r, b, for "up," "left," "right" and "back." The basic transformations are the counter-clockwise rotations by 120° of the tips (we denote these transformations by u, l, r, b), and of the middle layers (we dete these transformations by U, L, R, B). Thus the group G we are interested in is generated by u, l, r, b, U, L, R, B.

There is a natural action of G on the set of edge elements E, the set of tips V and the set of center elements C. Enumerate the edges. Denote $\sigma\colon G \to S_6$ the corresponding action homomorphism.

Problem 2. Prove that $\mathrm{Im}(\sigma) \subseteq A_6 \subset S_6$.

Both tips and edge elements can be given an *orientation.* That is, we mark one of the facets of each tip by '+', and one of the facets of each edge element by '+'.

For each corner piece, assign to a move $g \in G$ either

- a '0' if the '+ facet' for that piece when it was in the solved position is sent to the '+ facet' for that piece in the present position,
- a '1' if the '+ facet' for that piece when it was in the solved position is sent to the facet which is a 120 degrees rotation about its vertex from the '+ facet' for that piece in the present position,
- a '2' otherwise.

Problem 3. Prove that the assignment described above gives a surjective homomorphism $v\colon G \twoheadrightarrow (\mathbb{Z}/3)^4$.

Problem 4. In a similar fashion, assign to each $g \in G$ a 6-tuple of elements of $\mathbb{Z}/2$, using the orientation of the edge element.

Let's call this assignment $w\colon G \to (\mathbb{Z}/2)^6$.

Problem 5. Prove that for any $g, h \in G$,

$$w(gh) = \sigma(g^{-1})\,(w(h)) + w(g)$$

Problem 6. Prove that the sum of entries of $w(g)$ is always $0 \mod 2$.

Now, consider the semi-direct product $(\mathbb{Z}/2)^6 \rtimes S_6$ where S_6 acts on $(\mathbb{Z}/2)^6$ by permuting coordinates. Such a semidirect product is called *wreath product* and is denoted by $\mathbb{Z}/2 \wr S_6$.

Problem 7. Prove that the assignment $g \mapsto (v(g), w(g), \sigma(g))$ defines an group homomorphism

$$G \to (\mathbb{Z}/3)^4 \times (\mathbb{Z}/2 \wr S_6)$$

Problem 8. Prove that this homomorphism is injective.

Problem 9. Prove that the image of σ is precisely A_6. For this, you can use the following lemma.

Lemma 0.0.1. *Any alternating group A_n is generated by the 3-cycles $(1, 2, i)$.*

Problem 10. Prove that the image of w is precisely the set of tuples $(w_1, w_2, \ldots, w_6)$ with $\sum w_i \equiv 0 \mod 2$.

Thus, we have proved the following theorem.

Theorem 0.0.2. *The group of symmetries of the Pyraminx is the subgroup of $(\mathbb{Z}/3)^4 \times (\mathbb{Z}/2 \wr S_6)$ consisting of tuples (v, w, s) with $s \in A_6$ and $\sum w_i \equiv 0 \mod 2$.*

Problem 11. Based on the gained information, try to design an algorithm of solving the Pyraminx.

Problem 12*. Try to describe in a similar fashion, and design an algorithm of solving, of the Pocket cube: