Group Theory and the Pyraminx

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May 22, 2016

Introduction

Goals

- To understand the group of symmetries for the Pyraminx and the Two by two Rubik's Cube.
- To create an algorithm of solving both puzzles.

The Two by two Rubik's Cube

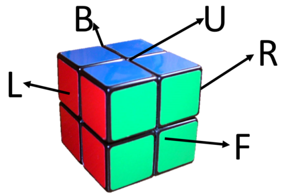
The Two by two cube, or Pocket Cube, is a miniaturized Rubik's Cube with only eight unit cubes (cubits/cubies) and rotatable faces/edges.

Moves

When naming moves, we made sure to:

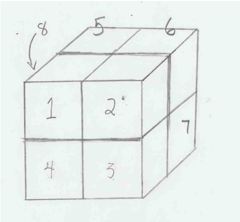
1. never move the puzzle from its original position of one face facing you.
2. keep our movements consistent. A rotation will always be counterclockwise.

There are six possible moves: (F)ront, (B)ack, (R)ight, (L)eft, (U)p and (D)own.



Numbering

We numbered the individual cubits from one to eight.



We can represent how every move affects the position of the cubits.
For example, Front(F) affects the position in the following way:

$$F \mapsto \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 2 & 3 & 5 & 6 & 7 & 8 \end{pmatrix}$$

S_8

This gives a homomorphism σ,

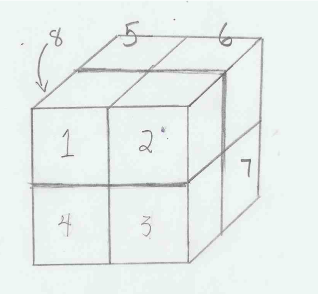
$$\sigma : G \to S_8$$

For example:

$$F \mapsto \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 2 & 3 & 5 & 6 & 7 & 8 \end{pmatrix}$$

$$B \mapsto \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 6 & 7 & 8 & 5 \end{pmatrix}$$

$$\vdots$$



Transposition of Cubits

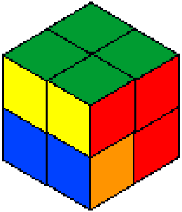
Lemma: σ is a surjective homomorphism.

S_8 is generated by transpositions.
There exists a combination of moves that transposes two cubits.

Written from left to right:

$$(\mathrm{R}\ \mathrm{U})^7\ \mathrm{R}^3\ \mathrm{L}^2\ \mathrm{F}\ \mathrm{B}^3 \mapsto$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 8 & 5 & 6 & 7 & 4 \end{pmatrix}$$



By multiplying versions of that move, you can obtain all of S_8.

Orientation

Notice: σ is not an injective homomorphism.

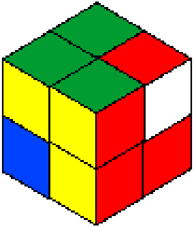
$$\exists \quad g \in G, \quad s.t. \quad \sigma(g) = e, \quad \text{but} \quad g \neq e$$

Changes orientation

(left to right):

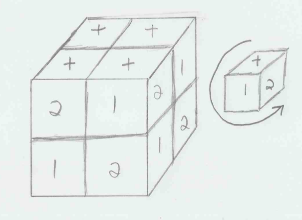
$(\mathrm{R}\ \mathrm{D}^2\ \mathrm{R}^3\ \mathrm{B}\ \mathrm{U}^2\ \mathrm{B}^3)^2 \mapsto$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$$



Orientation

We labeled the orientations of each cubit with a 0(+), 1, and 2 on its three colored faces.



We can represent how every move affects the orientation of the cubits.
For example, Front(F) affects the orientation as follows:

$$F \mapsto \begin{pmatrix} 2 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

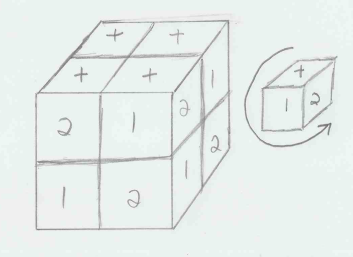
$\left({}^{\mathbb{Z}}/_{3}\right)^8$

This gives a map:

$$\omega : G \to \left({}^{\mathbb{Z}}/_{3}\right)^8$$

For example:

$$F \mapsto \begin{pmatrix} 2 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$B \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 \end{pmatrix}$$
$$\vdots$$

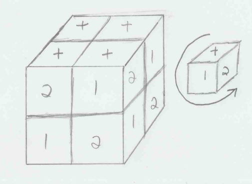


ω(gh)

Notice: ω is not a homomorphism.

$$\omega(gh) \neq \omega(h) + \omega(g)$$

For example, take the move:



$$\omega(F) = \begin{pmatrix} 2 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We know that based on our labeling,

$$\omega(F^2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

However,

$$\omega(F) + \omega(F) = \begin{pmatrix} 1 & 2 & 1 & 2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$(\mathbb{Z}/_3)^8$ cont.

Lemma: Given g, h $\in$ G :

$$\omega(gh) = \sigma(g^{-1})(\omega(h)) + \omega(g)$$

Example: F^2

$$\begin{aligned}
\omega(F) &= \begin{pmatrix} 2 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\sigma(F^{-1}) &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 & 5 & 6 & 7 & 8 \end{pmatrix} \\
\sigma(F^{-1})(\omega(F)) &= \begin{pmatrix} 1 & 2 & 1 & 2 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\sigma(F^{-1})(\omega(F)) + \omega(F) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\omega(F^2) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

It's time to put σ and ω together.
To obtain a homomorphism, we must use a semidirect product.

$$\left({}^{\mathbb{Z}}/_{3}\right)^8 \rtimes S_8$$

$$\rtimes : (x_1, s_1)(x_2, s_2) = (x_1 + s_1^{-1} x_2, s_1 s_2)$$

$$\begin{aligned} (\omega, \sigma) : G &\to \left({}^{\mathbb{Z}}/_{3}\right)^8 \rtimes S_8 \\ g &\mapsto (\omega(g), \sigma(g)) \end{aligned}$$

$$\begin{aligned} &(\omega(g), \sigma(g))(\omega(h), \sigma(h)) = \\ &\qquad (\omega(g) + \sigma(g^{-1})(\omega(h)), \sigma(gh)) = (\omega(gh), \sigma(gh)) \end{aligned}$$

Thus, (ω, σ) is a group homomorphism.

Theorem: $G \simeq \left[(x, x_2, \cdots, x_8, \sigma) \,\middle|\, \begin{matrix} \sigma \in S_8 \\ x_1 + x_2 + \cdots + x_8 = 0 \bmod 3 \end{matrix} \right]$

1. (ω, σ) is injective.
2. $\forall$ g $\in$ G, $\omega(\mathrm{g}) = (g_1, g_2 \cdots g_8)$ satisfies $g_1 + g_2 + \cdots + g_8 = 0$ mod 3.
3. (ω, σ) is surjective.

7 Position Vectors

$$(\mathrm{R}\ \mathrm{D}^2\ \mathrm{R}^3\ \mathrm{B}\ \mathrm{U}^2\ \mathrm{B}^3)^2 \mapsto \begin{pmatrix}0 & 2 & 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}2 & 0 & 0 & 0 & 0 & 0 & 1 & 0\end{pmatrix}$$

$$\begin{pmatrix}0 & 0 & 0 & 2 & 0 & 1 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}0 & 0 & 2 & 0 & 1 & 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}0 & 0 & 2 & 1 & 0 & 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}0 & 2 & 1 & 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}2 & 1 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

Example:

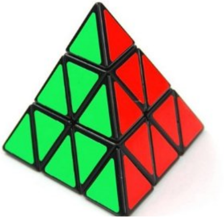
$$\begin{pmatrix} 2 & 1 & 0 & 1 & 1 & 2 & 1 & 1 \end{pmatrix}$$

$$\begin{aligned}
1 \cdot &\begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
1 \cdot &\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \\
2 \cdot &\begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \end{pmatrix} \\
1 \cdot &\begin{pmatrix} 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\
0 \cdot &\begin{pmatrix} 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\
1 \cdot &\begin{pmatrix} 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
0 \cdot &\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

Algorithm

1. Start with scrambled cube.
2. Do transpositions to put the cubits in the correct positions.
3. Read off orientation and decompose it.
4. Apply inverses to correct orientation.

Pyraminx



The Pyraminx is a tetrahedron shaped puzzle with rotatable corners. The tips of the tetrahedron can rotate separately as well.

There are three types of unit pieces we must be aware of.

- Edges - exist as the center of Pyraminx edges
- Tips
- Centers - cannot move, only rotate

Moves

Again, in naming moves, we made sure to:

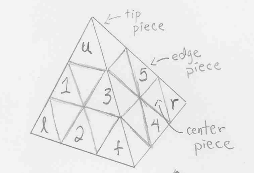
1. never move the Pyraminx from its original position of one corner pointing at you.
2. keep our movements consistent. A rotation will always be counterclockwise.

There are eight possible moves:

- Front, Right, Left, and Top, (uppercase letters) which rotates an entire corner,
- front, right, left, and top, (lowercase letters) which rotates only the tips.

Numbering

For the Pyraminx, we numbered the edge pieces from one to six.

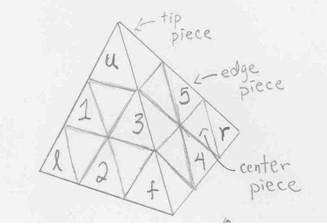


We can represent how every move affects the position of the pieces.
For example, Front(F) affects the position in the following way.

$$F \mapsto \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 4 & 2 & 5 & 6 \end{pmatrix}$$

This gives us a homomorphism σ,

$$\sigma : G \to S_6$$



As an example:

$$F \mapsto \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 4 & 2 & 5 & 6 \end{pmatrix}$$

$$T \mapsto \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 1 & 4 & 3 & 6 \end{pmatrix}$$

$$\vdots$$

Note: All basic moves give even permutations.

Transposition of Edges

Lemma: The im(G) is A_6.

A_6 is generated by cycles of three pieces.
Again, there is a set of moves that allows for the cycles of edge pieces.

Switching 3 edge pieces on same face (left to right):
L' R L R' $\mapsto$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 3 & 4 & 1 & 5 \end{pmatrix}$$

Again, combining cycles can give you all of A_6.

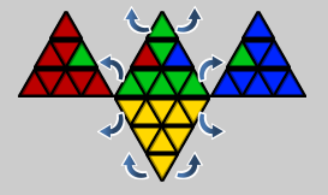
Orientation

Notice: σ is not injective.

Orientation switch

(left to right):

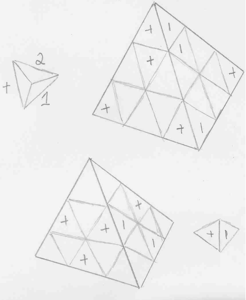
$$(\text{R L' R' L})^2 \text{ T R' T' R} \mapsto \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$



Orientation

This time we:

- labeled the orientations of the tips with 0(+), 1, and 2,
- labeled the orientations of the edge pieces with 0(+) and 1.



We can represent how every move affects the orientation of the cubits.
For example, Front(F) can affect the orientation as follows:

$$F \mapsto \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

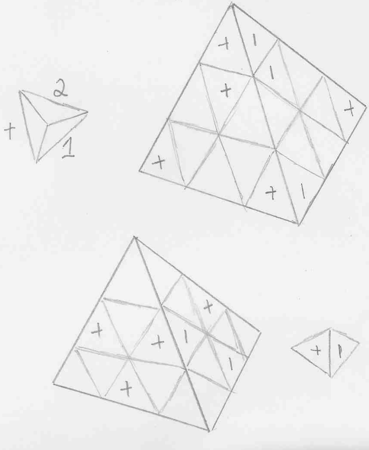
$({}^{\mathbb{Z}}/_{3})^4$ and $({}^{\mathbb{Z}}/_{2})^6$

This gives a map:

$$\nu : G \to ({}^{\mathbb{Z}}/_{3})^4$$

For example:

$$F \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$
$$f \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$
$$\dots$$

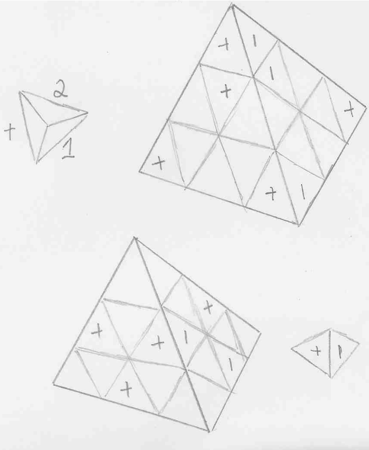


$\left({}^{\mathbb{Z}}/_{3}\right)^4$ and $\left({}^{\mathbb{Z}}/_{2}\right)^6$

Similarly, another map:

$$\omega : G \to \left({}^{\mathbb{Z}}/_{2}\right)^6.$$

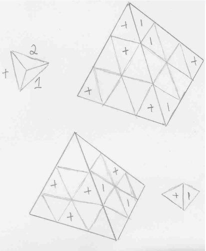
$$F \mapsto \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$
$$T \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$
$$\dots$$



Notice: Unlike ν, ω is not a homomorphism.

$$\omega(gh) \neq \omega(h) + \omega(g)$$

For example, take the move:



$$\omega(F) = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

We know that based on our labeling,

$$\omega(F^2) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

However,

$$\omega(F) + \omega(F) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$(\mathbb{Z}/_3)^4$ and $(\mathbb{Z}/_2)^6$ cont.

Lemma: As before, given g, h $\in$ G:

$$\omega(gh) = \sigma(g^{-1})(\omega(h)) + \omega(g)$$

Example: F^2

$$\begin{aligned}
\omega(F) &= \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \\
\sigma(F^{-1}) &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 3 & 5 & 6 \end{pmatrix} \\
\sigma(F^{-1})(\omega(F)) &= \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \\
\sigma(F^{-1})(\omega(F)) + \omega(F) &= \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \\
\omega(F^2) &= \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

Theorem

Lets put ω, ν and σ together now.

$$\begin{aligned}(\omega, \sigma, \nu) : G &\to \left[\left({}^{Z}/_{2}\right)^{6} \rtimes A_6\right] \times \left({}^{Z}/_{3}\right)^{4} \\ g &\mapsto (\omega(g), \sigma(g), \nu(g))\end{aligned}$$

Theorem:

$$G \simeq \left[\left(x, x_2, \cdots, x_6, \sigma, y, \cdots, y_4\right) \middle| \begin{array}{c} \sigma \in A_6 \\ x_1 + x_2 + \cdots + x_6 = 0 \bmod 2 \end{array}\right]$$

5 Position Vectors

$$
\begin{aligned}
T \mapsto & \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \\
& \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\
& \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\
& \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \\
& \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Algorithm

1. Start with scrambled Pyraminx.
2. Fix tips to centers.
3. Do cycles to put edges in correct positions.
4. Read off orientation of edges and decompose it.
5. Apply inverses to correct orientation.