Derived characters of finite-dimensional representations

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References

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Convenient: consider the set of all representations of algebra A in the vector space W, denoted by $\mathrm{Rep}_W(A)$.

We will identify vector space W with $k^{\dim W}$. Write $\mathrm{Rep}_n(A)$ for $\mathrm{Rep}_W(A)$ if $\dim W = n$.

Examples of $\mathrm{Rep}_n(A)$

- If $A_1 = k\langle x_1, \ldots, x_r\rangle$ then $\mathrm{Rep}_n(A_1) = \mathrm{Mat}_n^{\times r} \simeq \mathbb{A}^{rn^2}$.

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- If $A_2 = k[x_1, \dots, x_r]$ then $\mathrm{Rep}_n(A_2) \subset \mathrm{Rep}_n(A_1)$ is closed subscheme, consisting of tuples $(B_1, \dots, B_r)$ of pair-wise commuting matrices.

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- In general, if $A = \langle a_1, \ldots, a_m \mid r_1, \ldots, r_l\rangle$ then $\mathrm{Rep}_n(A)$ is given by tuples $(B_1, \ldots, B_m)$ of matrices satisfying $r_i(B_1, \ldots, B_m) = 0$.

Character map

Way to think about characters: they associate to any $a \in A$ a function $\hat{a}$ on $\mathrm{Rep}_n(A)$:

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Correspondence $a \mapsto \hat{a}$ gives a linear map $\mathrm{Tr} \colon A \to k[\mathrm{Rep}_n(A)]$.

This map factors as

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{Tr}} & k[\mathrm{Rep}_n(A)] \\ \downarrow & & \uparrow i \\ A/[A,A] & \longrightarrow & k[\mathrm{Rep}_n(A)]^{\mathrm{GL}_n} \end{array}$$

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The map $A/[A,A] \to k[\mathrm{Rep}_n(A)]^{\mathrm{GL}_n}$ will be called the *character map* and will also be denoted by Tr.

Procesi theorem

Theorem (Procesi)

The induced homomorphism of algebras

$$\mathsf{Sym}(\mathrm{Tr})\colon\ \mathsf{Sym}(A/[A,A]) \to k[\mathrm{Rep}_n(A)]^{\mathsf{GL}_n}$$

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is surjective.

Thus, characters "capture" representation theory of any algebra A: they determine rings of functions $k[\mathrm{Rep}_n(A)]^{\mathsf{GL}_n}$, which determine the moduli spaces of semi-simple representations.

Extension to DG algebras

Suppose $A \in \mathsf{DGA}_k$ – category of DG algebras. Can still consider $\mathrm{Rep}_n(A)$.

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Main idea: in DG algebras it is natural to replace *isomorphisms* by *quasi-isomorphisms*.

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Problem: The functor sending $A \mapsto k[\mathrm{Rep}_n(A)] = A_n$ is not "exact", i.e. it does not respect quasi-isomorphisms.

Solution: We need to *derive* the functor $(-)_n \colon A \mapsto A_n$.

Derived representation functor

For $A \in \mathrm{Alg}_k \subset \mathrm{DGA}_k$, to compute the *derived functor* $\mathbb{L}A_n$ we need to pick a *resolution* for A and apply $(-)_n$.

We denote $\mathbb{L}A_n$ by $\mathrm{DRep}_n(A)$ and call it *derived representation scheme* (in the sense of Kapranov).

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Example: If $A = k[x, y]$, we can take $R = k\langle x, y, t\rangle$ with $\deg(x) = \deg(y) = 0$, $\deg(t) = 1$ and $dt = xy - yx$.

The obvious projection $R \twoheadrightarrow A$ is a quasi-isomorphism.

Then $\mathrm{DRep}_n(A) = k[x_{ij}, y_{ij}, t_{ij}]$ with $\deg(t_{ij}) = 1$ and

$$dt_{ij} = \sum_{k=1}^{n} x_{ik}y_{kj} - y_{ik}x_{kj}$$

Classical definition of cyclic homology

For an algebra A define $\mathrm{CC}_n(A) = A^{\otimes n}/(1-t)$ where

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$$d(a_1, \ldots, a_n) = (-1)^n (a_n a_1, a_2, \ldots, a_{n-1}) + \sum_{i=1}^{n-1} (-1)^{i+1} (a_1, \ldots, a_i a_{i+1}, \ldots, a_n)$$

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For example, $d(a, b) = ab - ba$. Thus $\mathrm{HC}_0(A) = A/[A, A]$.
It gives: the source of our character map Tr was actually $\mathrm{HC}_0(A)$.

Derived character maps

Proposition ([BKR])

For any algebra $A \in \mathrm{Alg}_k$ *and any* n *there exists canonical derived character map*

$$\mathrm{Tr}_n(A)_\bullet \colon \mathrm{HC}_\bullet(A) \to \mathrm{H}_\bullet(\mathrm{DRep}_n(A))^{\mathrm{GL}_n},$$

lifting the original character map $\mathrm{Tr} \colon \mathrm{HC}_0(A) \to A_n^{\mathrm{GL}_n}$.

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Remark: Procesi theorem does not hold for $\mathbf{Sym}(\mathrm{Tr}_1(A)_\bullet)$.
Counter-example: $A = k[x]/(x^2)$, see:

Yu. Berest, A. Ramadoss *Stable representation homology and Koszul duality*, Journal fur die reine und angewandte Mathematik, March 2014.

Cyclic homology of $A = \mathrm{Sym}(V)$

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Fact: For a *smooth* commutative algebra A, there is a canonical isomorphism

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Thus, we can think of $\mathrm{Tr}_n(A)_i$ as maps

$$\mathrm{Tr}_n(A)_i \colon \Omega^i(A) \to \mathrm{H}_i(A, n)^{\mathrm{GL}_n}$$

(Minimal) resolution of $A = \mathrm{Sym}(V)$

As a graded algebra, we have

$$R = T(s^{-1}\Lambda(sV)) \simeq T(V) * T\left(\Lambda^2(V) \oplus \Lambda^3(V) \oplus \cdots \oplus \Lambda^r(V)\right)$$

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We denote generators *of degree* $p-1$

$$\lambda(v_1, v_2, \ldots, v_p) := s^{-1}(sv_1 \wedge sv_2 \wedge \ldots \wedge sv_p) \in s^{-1}\Lambda^p(sV)$$

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Lemma

The differential $d\lambda(v_1, \ldots, v_n)$ *on* R *is a sum of certain commutators:*

$$\sum_{\substack{p+q=n \\ \sigma \in \mathrm{Sh}(p,q)}} \pm \left[\lambda(v_{\sigma(1)}, \ldots, v_{\sigma(p)}),\, \lambda(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)})\right] ,$$

where $\mathrm{Sh}(p, q)$ *denotes the set of* (p, q)*-shuffles.*

Representation homology of $A = \mathrm{Sym}(V)$

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Now we have all the components of the derived character map. It is a map

$$\Omega^\bullet(A) = \mathrm{Sym}(V) \otimes \Lambda^\bullet(V) \to \mathrm{Sym}(V) \otimes \mathbf{Sym}\left(\Lambda^2(V) \oplus \cdots \oplus \Lambda^r(V)\right)$$

The map Tr_1

Proposition

Assume that we fixed a basis in V, so that $A = k[x_1, \ldots, x_r]$. Then $\mathrm{Tr}_1(A)_1$ *is given by de Rham differential $s^{-1}d_{dR}$. Namely, for* $\alpha = \sum P_i dx_i \in \Omega^1(A)$ *we have*

$$\mathrm{Tr}_1(A)_1(\alpha) = \sum_{i<j} \left(\frac{\partial P_i}{\partial x_j} - \frac{\partial P_j}{\partial x_i} \right) \lambda(x_i, x_j) \in \mathrm{DRep}_1(A)$$

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Surprising fact: this is not the case!

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Take $\omega = Pdx \wedge dy + Qdy \wedge dz + Rdz \wedge dx \in \Omega^2(A)$.

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Then $\mathrm{Tr}_1(A)_2(\omega)$ is given by

$$M\lambda(x, y, z) + M_y\lambda(x, y)\lambda(y, z) + M_z\lambda(y, z)\lambda(z, x) + M_x\lambda(z, x)\lambda(x, y),$$

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and for a polynomial F, F_q denotes $\frac{\partial F}{\partial q}$.

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Remark 2: It has $s^{-1}d_{dR}$ as a summand.

Some linear algebra

- For each $1 \leq p \leq q \leq r = \dim(V)$ define a map $V \otimes \Lambda^q(V) \to \Lambda^p(V) \otimes \Lambda^{q+1-p}(V)$ sending $u \otimes v_1 \wedge \cdots \wedge v_q$ to

$$\sum_{j_1 < \cdots < j_{p-1}} (-1)^{1+\sum j_s} (u \wedge v_{j_1} \wedge \cdots \wedge v_{j_{p-1}}) \otimes (v_1 \wedge \cdots \wedge \hat{v}_{j_1} \wedge \cdots \wedge \hat{v}_{j_{p-1}} \wedge \cdots \wedge v_q).$$

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- By duality, we get

$$\Delta_q^{(p,q+1-p)} \colon \Lambda^q(V) \to V^* \otimes \Lambda^p(V) \otimes \Lambda^{q+1-p}(V).$$

Some linear algebra

- For each $1 \leq p \leq q \leq r = \dim(V)$ define a map $V \otimes \Lambda^q(V) \to \Lambda^p(V) \otimes \Lambda^{q+1-p}(V)$ sending $u \otimes v_1 \wedge \cdots \wedge v_q$ to

$$\sum_{j_1 < \cdots < j_{p-1}} (-1)^{1+\sum j_s} (u \wedge v_{j_1} \wedge \cdots \wedge v_{j_{p-1}}) \otimes (v_1 \wedge \cdots \wedge \hat{v}_{j_1} \wedge \cdots \wedge \hat{v}_{j_{p-1}} \wedge \cdots \wedge v_q).$$

- By duality, we get

$$\Delta_q^{(p,q+1-p)} \colon \Lambda^q(V) \to V^* \otimes \Lambda^p(V) \otimes \Lambda^{q+1-p}(V).$$

- For any multi-index $(i_1, \ldots, i_m)$ such that $1 \leq i_1 \leq \cdots \leq i_m$ and $i_1 + \cdots + i_m = q + m - 1$, we can construct

$$\Delta_q^{(i_1,\ldots,i_m)} \colon \Lambda^q(V) \to \mathrm{Sym}^{m-1}(V^*) \otimes \Lambda^{i_1}(V) \otimes \cdots \otimes \Lambda^{i_m}(V)$$

by iterating the above map, and composing it with the projection map $(V^*)^{\otimes(m-1)} \to \mathrm{Sym}^{m-1}(V^*)$.

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Here the projection is given by act $\otimes (s^{-1})^{\otimes m}$, where act: $\mathrm{Sym}(V) \otimes \mathrm{Sym}^{m-1}(V^*) \to \mathrm{Sym}(V) \hookrightarrow \mathrm{DRep}_1(A)$ is the action map.

General formula for derived characters

Theorem

For a finite dimensional vector space V, let $A = \mathrm{Sym}(V)$. Then for each $q \geq 1$, the q-th trace $\mathrm{Tr}(A)_q \colon \Omega^q(A) \to \mathrm{DRep}_1(A)$ is given by

$$\mathrm{Tr}(A)_q = \mathrm{D}_{q+1} \circ d_{dR}$$

where the differential operator D_{q+1} is given by

$$\mathrm{D}_{q+1} = \sum_{\substack{2 \leq i_1 \leq \cdots \leq i_m \\ i_1 + \cdots + i_m = q+m}} c_{(i_1,\ldots,i_m)} \, \mathrm{D}_{q+1}^{(i_1,\ldots,i_m)} \, .$$

Here $\mathrm{D}_{q+1}^{(i_1,\ldots,i_m)}$ is the differential operator of order $m-1$ defined above and $c_{(i_1,\ldots,i_m)}$ are scalar factors.

Combinatorial description (due to T.Willwacher)

The value $\mathrm{D}_q^{(i_1,\dots,i_m)}(y_1 y_2 \dots y_p dy_{p+1} \dots dy_{p+q})$ can be described as

$$(const) \cdot \sum_f \pm y_{f^{-1}(0)} y_{\{1\} \cup f^{-1}(1)} y_{\{2\} \cup f^{-1}(2)} \cdots y_{\{p\} \cup f^{-1}(p)}.$$

Summation over all $f \colon \{p+1, \dots, p+q\} \to \{0, 1, \dots, p\}$ s.t.

- exactly $m-1$ sets among $f^{-1}(1), \dots, f^{-1}(p)$ are non-empty;
- among $f^{-1}(1), \dots, f^{-1}(p)$, there are as many of cardinality j as there are numbers j amond $i_1, \dots, i_m$, for any $j \geq 1$.

Here y_I for a set $I = \{n_1, \dots\}$ denotes the element $\lambda(x_{n_1}, \dots)$. For example, $y_{\{13\}} = \lambda(y_1, y_3)$.

NC Poisson bracket

W. Crawley-Boevey *Poisson structures on moduli spaces of representations*, J.Algebra **325** (2011), 205 – 215.

There is a notion of NC Poisson structure on an algebra $A \in \mathrm{Alg}_k$ (introduced by Crawley-Boevey). By definition, it consists of a Lie bracket $\{-,-\}$ on $A/[A,A]$ s.t. $\{\bar{a},-\}$ is induced by a derivation of A.

Theorem (Crawley-Boevey)

For all n, there is unique Poisson bracket on $\mathrm{Rep}_n(A)$ making the map Tr *into a map of Lie algebras.*

We will later that this situation generalizes to the derived case.

Symplectic geometry in derived case

Yu. Berest, X.Chen, F. Eshmatov and A. Ramadoss *Noncommutative Poisson structures, derived representation schemes and Calabi-Yau algebras*, Cont. Math. Volume **583**, 2012.

For an associative algebra A we define NC *derived* Poisson structure to be a DG Lie algebra structure on $\mathrm{HC}_\bullet(A)$ induced from a DG Lie algebra structure on $R/[R,R]$ for some resolution $R \overset{\sim}{\twoheadrightarrow} A$ of A.

Theorem (Berest-Chen-Eshmatov-Ramadoss)

Given a derived Poisson structure on A, there is unique graded Poisson bracket on $\mathrm{H}_\bullet(\mathrm{DRep}_n(A))^{\mathrm{GL}_n}$ making $\mathrm{Tr}_n(A)_\bullet$ into a map of Lie algebras.

Example of derived Poisson bracket

Suppose $A = k[x, y]$, and consider the case $n = 1$. Then we have a derived Poisson structure on $\overline{\mathrm{HC}}_\bullet(A) \simeq \overline{A} \oplus \Omega^1(A)/dA$.

- If $\bar{f}, \bar{g} \in \overline{A}$ then
$$\{\bar{f}, \bar{g}\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$$
is the usual Poisson bracket of polynomials.
- If $\bar{\alpha} \in \Omega^1(A)/dA$ and $\bar{f} \in \overline{A}$, we have
$$\{\bar{f}, \bar{\alpha}\} = \mathcal{L}_{\theta_f}(\alpha),$$
where $\mathcal{L}$ means Lie derivative and θ_f is the Hamiltonian vector field corresponding to f.