Riemann Sum

A Riemann sum for $f(x)$ on $[a, b]$ with partition $P = \{x_0, x_1, \ldots, x_n\}$ is a sum of the form

$$S_P = \sum_{k=1}^{n} \Delta x_k f(c_k),$$

where $c_k \in [x_{k-1}, x_k]$.

The **norm** of a partition P is

$$||P|| = \max_{1 \leq k \leq n} \Delta x_k.$$

Last time, we found an approximation for the area under $f(x) = x^2$ on [0,1] using n equal subintervals and wrote it in sigma notation. That is, we expressed our approximation as a Reimann sum:

$$Area \approx S_P = \sum_{k=1}^{n} \frac{1}{n} \left(\frac{k}{n}\right)^2,$$

where P is the partition $\{x_k = \frac{k}{n} : k = 0, 1, \ldots, n\}$.

Simplify using sum of squares:

$$Area \approx \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6n^3}.$$

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What is $\lim_{n\to\infty} S_P$ in this case?

$$\begin{aligned}\lim_{n\to\infty} S_P &= \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3}\\ &= \lim_{n\to\infty} \frac{2n^3+3n^2+n}{6n^3} = \frac{2}{6}.\end{aligned}$$

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What if we had used left endpoints in our approximation?

If we approximate the area under $f(x) = x^2$ on [0,1] using the same partition, but left endpoints, then $c_k = x_{k-1} = \frac{k-1}{n}$ so this gives a different Riemann sum:

$$S_P = \sum_{k=1}^{n} \frac{1}{n} \left(\frac{k-1}{n}\right)^2 = \frac{1}{n^3} \left(\sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1\right)$$

$$= \frac{2n^3 + 3n^2 + n}{6n^3} - \frac{n(n+1)}{n^3} + \frac{n}{n^3}$$

Thus, in this case

$$\lim_{n\to\infty} S_P = \lim_{n\to\infty} \frac{2n^3 + 3n^2 + n}{6n^3} - \frac{n(n+1)}{n^3} + \frac{n}{n^3} = \frac{2}{6} + 0 + 0 = \frac{2}{6}.$$

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Remarks

1) The approximations (given by the value of two different Riemann sums) were not equal

$$\frac{2n^3+3n^2+n}{6n^3} \quad \text{vs} \quad \frac{2n^3+3n^2+n}{6n^3} - \frac{n^2}{n^3}$$

2) The *limit* of these different Riemann sums were both equal to $\frac{1}{3}$.

3) Any other choice of the c_ks will give a different approximate value, but the limit of that approximation will also have value $\frac{1}{3}$.

Why? A: Sandwich Theorem

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Definite Integral Properties

Theorem 5.3.2- If f and g are integrable over $[a, b]$ then the definite integral satisfies the following properties.

1. Order of Integration: $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
2. Zero Width: $\int_a^a f(x)\,dx = 0$
3. Constant Multiple: $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, for any constant c
4. Sums and Difference: $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
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5. Additivity: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
6. If f has a maximum value of M and a minimum value of m on $[a, b]$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.
7. If $f(x) \geq g(x)$ for $x \in [a, b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$