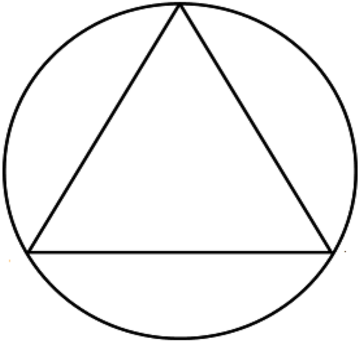
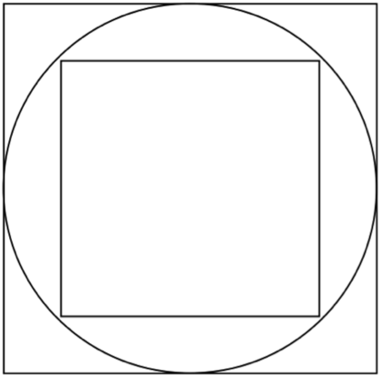
Land of Straight Edges

Suppose you come from a land where there are no curves. You know everything about distances and areas for shapes with straight edges. How would you go about approximating the area of a circle with radius r using this knowledge?

Land of Straight Edges





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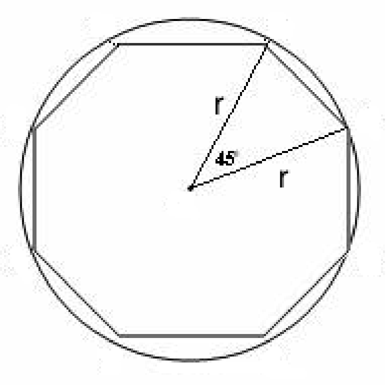
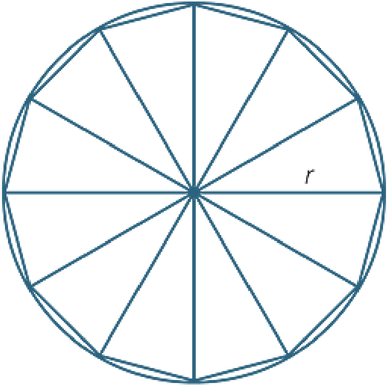
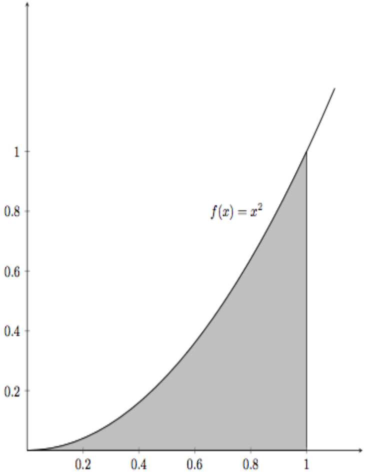




Area Under A Curve

Ex 1. Approximate the area under $f(x) = x^2$ over $[0, 1]$.



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To approximate the area under the curve:

1. Divide the interval $[0, 1]$ into n non-overlapping subintervals. (We'll consider $n = 5$.)
2. Pick one representative point in each subinterval.
3. Look at subintervals individually. Approximate the area under the curve on a single subinterval by using a single rectangle. The height will be dictated by the representative from that subinterval.

   On the subinterval with representative c the area under the curve is approximately (length of subinterval)$\cdot f(c) = \frac{1}{n} f(c)$, where the equality holds if using equal subintervals. (For us $\frac{1}{5}c^2$).
4. Add up the approximation for each subinterval to get an approximation over the interval [0,1].

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What are we really doing?

Representative for subinterval k denoted c_k. We are really approximating $f(x)$ by $f(c_k)$ for all x in the k^{th} subinterval. So the area under $f(x)$ is approximately the area under the horizontal line $y = f(c_k)$.

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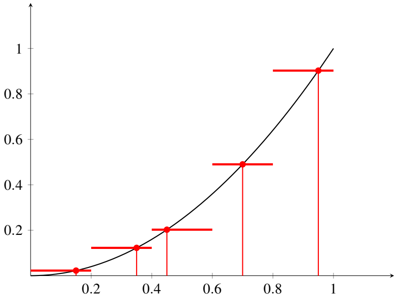

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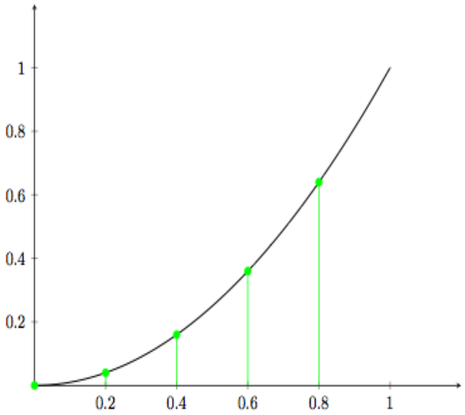
If choose left endpoint of each subinterval:



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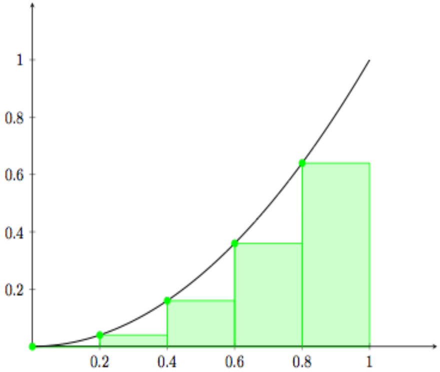

Area $\approx \frac{1}{5}(0^2) + \frac{1}{5}(.2^2) + \frac{1}{5}(.4^2) + \frac{1}{5}(.6^2) + \frac{1}{5}(.8^2) = \frac{6}{25} = .24.$

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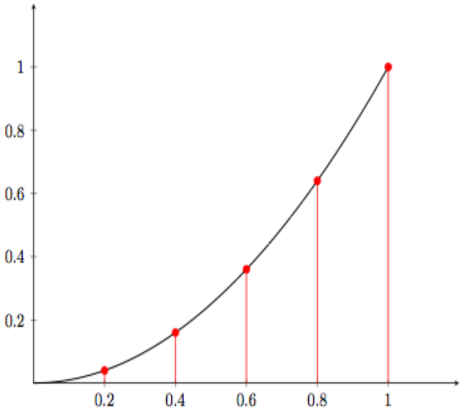

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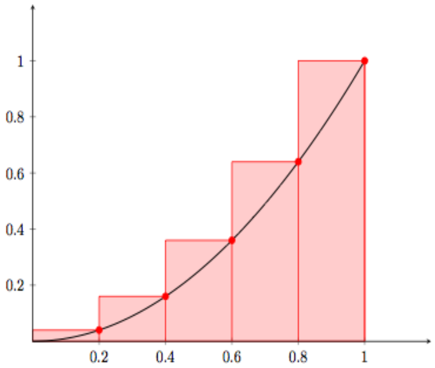

Area $\approx \frac{1}{5}(.2^2) + \frac{1}{5}(.4^2) + \frac{1}{5}(.6^2) + \frac{1}{5}(.8^2) + \frac{1}{5}(1^2) = \frac{11}{25} = .44.$

http://www.personal.psu.edu/dpl14/java/calculus/area.html

Approximating Area Under $f(x)$ on $[a, b]$

If $f(x)$ is increasing on $[a, b]$ then,

- Choosing left endpoints of subintervals for representatives yields an under-approximation.
- Choosing right endpoints of subintervals for representatives yields an over-approximation.

What if $f(x)$ is decreasing on $[a, b]$? (A: Opposite is true.)

Can choose any representative you see fit. Could pick a point from the subinterval completely at random– will get a different estimate if choose different representatives.

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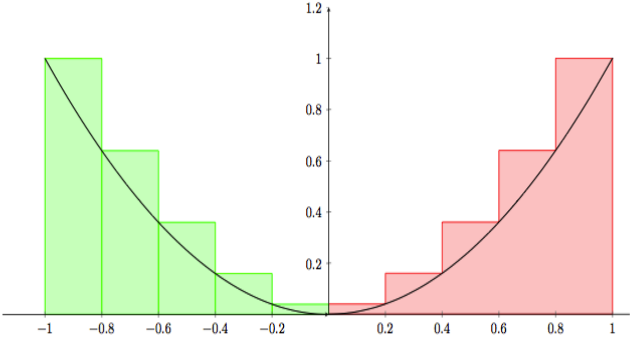
$f(x) = x^2$ on [-1,1]. To get largest possible estimate choose left endpoints for representatives on subintervals where decreasing and right endpoints for representatives on subintervals where increasing.



Ex 2.

a) Suppose we have the following information about a car's velocity:

t (seconds)	0	10	20	30
$v(t)$ (m/s)	5	12	25	30

Approximately how far did the car travel in the first 10 seconds and why?

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t (seconds)	0	10	20	30
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Can approximate the distance traveled on $[10, 20]$ and $[20, 30]$ analogously. Adding up the approximation for each 10 second interval gives:

$$d(30) \approx 5(10) + 12(10) + 25(10) = 420$$

*Units: 5 m/s(10s)+12m/s(10s)+25m/s(10s)=420m

velocity * time=distance

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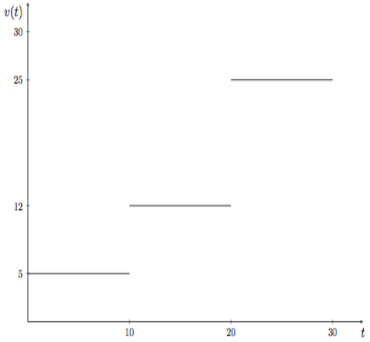
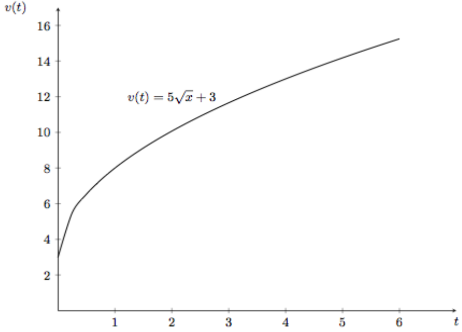
The approximate distance $d(30) \approx 429$ is actually the area under the following function:



Again, velocity $\cdot$ time = distance, so the area under this rate of change (velocity) curve has the unit meters, which is of course the units for position.

Ex 2b

b) Suppose we now have a function describing the velocity of thrown object. Suppose $v(t) = 5\sqrt{t} + 3, \quad 0 \leq t \leq 6.$



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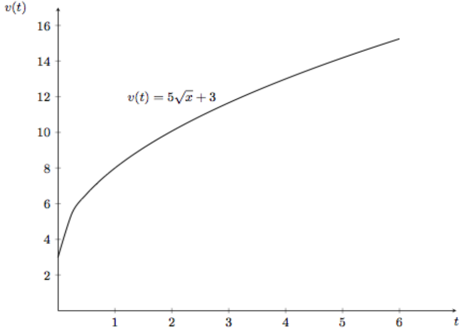

Approximately how far did the object travel in 5 seconds?

...Ex 2b

To approximate, can select n time points in the interval $[0, 5]$ and calculate the velocity at those times. We'll use $n = 5$:

t (seconds)	1	2	3	4	5
$v(t)$ (m/s)	v(1)	v(2)	v(3)	v(4)	v(5)
$v(t)$ (m/s)	8	10.07	11.66	13	14.18

Proceed as in part a:

$v(1)$ will be the approximation for the velocity in the time interval $[0, 1]$ so that the distance traveled in the first second is approximately 8m/s(1s)=8m.

$v(2)$ will be the approximation for the velocity in the time interval $[1, 2]$ so that the distance traveled in the first second is approximately 10.07m/s(1s)=10.07m.

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Similarly, for the remaining three time intervals $[2, 3]$, $[3, 4]$ and $[4, 5]$ we will approximate the distance over the subinterval using the velocity from a single point in that subinterval.

Thus, the total distance traveled in the first 5 seconds is approximately equal to the sum of the subinterval approximations:

$$d(5) \approx 8m + 10.07m + 11.66m + 13m + 14.18m = 56.91m.$$

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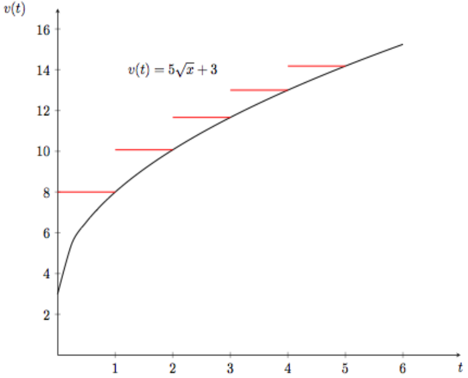
The approximation
$d(5) \approx 8m + 10.07m + 11.66m + 13m + 14.18m = 56.91m$ is equal to the area under the function shown in red.



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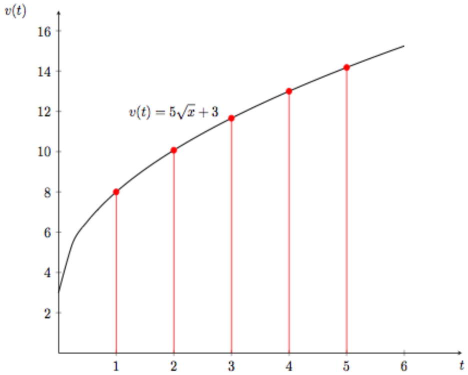

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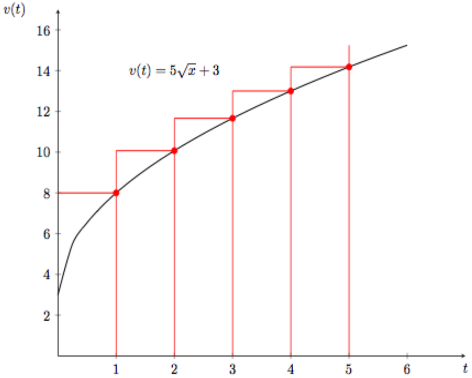

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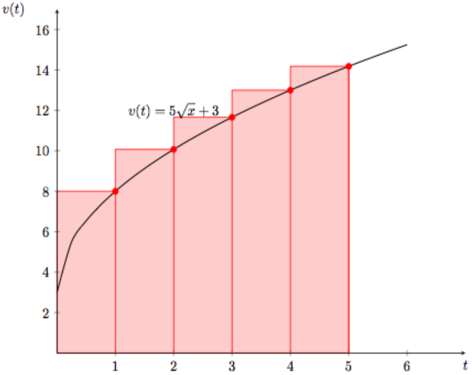

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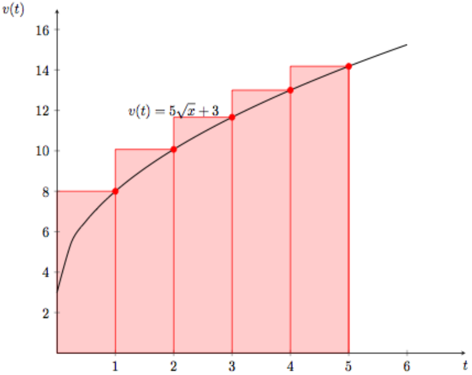

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In both examples, the area under the velocity curve gave the total distance traveled over the interval of time.

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Velocity

Velocity has two components- speed and direction

- speed= magnitude of velocity, i.e. $|v(t)|$
- direction= sign of velocity

  positive vs negative means opposite direction (Think of thrown object- if falling down its velocity is negative.)

Example 3a

Drive from home to the video shop, then to your friends house to watch the video. Your speedometer says 40mph the whole time. You drive 60 mintues to get the video shop, then 15 minutes back in the dirction of your house to get to your friends house.

Q1: How many miles did you travel?

Q2: How far are you from your house?

Q1 has to do with total distance traveled and Q2 has to do with displacement.

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Displacement vs Distance Traveled

$v(t)$= velocity and $s(t)$=position

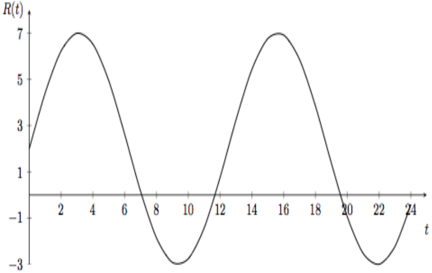
Displacement- Distance from starting point: $s(t) - s(0)$

When you go to work then home your displacement is zero, but you certainly went more than 0 miles.
Total distance traveled depends on *speed* over time, not velocity.

Displacement vs Total Distance traveled- latter doesn't care which direction going during the trip. Just total miles you put on your car when you were out.

Example 3b. Displacement as Net Change

Suppose at the start of the day Pacific Beach has 2500 cubic yards of sand and that the tide adds sand at a *rate* given by $R(t) = 2 + 5\sin\left(\frac{4\pi t}{25}\right)$ yd^3/hr.



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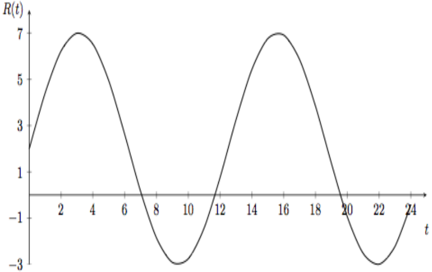

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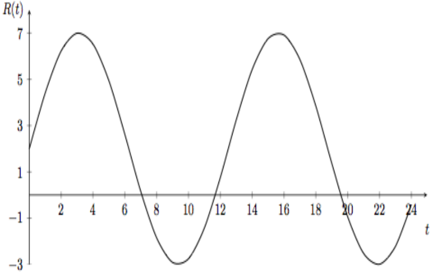

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...Ex 3b

Let $S(t)$ be the amount of sand on the beach at time t.

Then $S(24) - S(0) = S(24) - 2500$ cubic yards is the amount of sand added over the course of the day.

How can we approximate this value using the rate $R(t)$?

This is the area under $R(t)$ on $[0, 24]$. Takes into account sand is added, then removed, then added, then removed due to the cycle in the tides. That is, the **net amount** added to the beach over the course of the day.

(Analogous to displacement.)

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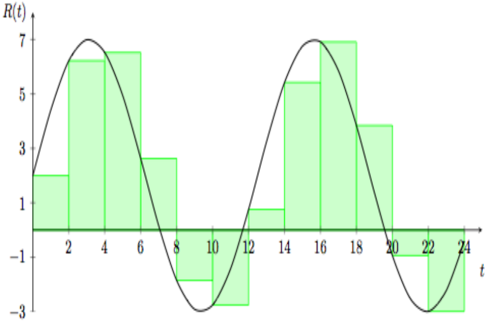

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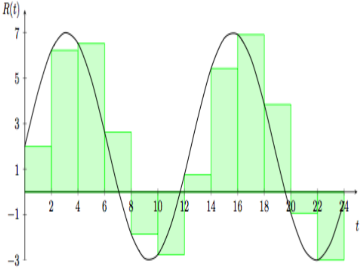

t (hours)	0	2	4	$\cdots$	8	10	12	$\cdots$	22
$R(t)$ (yd^3/hr)	2	6.22	6.52	$\cdots$	-1.85	-2.76	0.76	$\cdots$	-2.99

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The amount of sand added in the first two hours is approximately 2 yd^3/hr (2hr)=4yd^3.

The amount of sand added in the time interval $[8, 10]$ is approximately -1.85 yd^3/hr (2hr)=-3.7 yd^3. So about 3.7 cubic yards have sand were *removed* in this time interval.

Thus, the amount of sand added to the beach in 24 hours is approximately

$$2yd^3/hr(2hr) + 6.22yd^3/hr(2hr) + 6.52yd^3/hr(2hr)$$
$$+ \cdots - 1.85yd^3/hr(2hr) - 2.76yd^3/hr(2hr) + 0.76yd^3/hr(2hr)$$
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$$+ \cdots - 2.99yd^3/hr(2hr)$$

...Ex 3b

Then what does the area under $|R(t)|$ correspond to here?

The total amount of sand *moved* by the tide over the course of the day.

Since taking absolute value, aren't taking into account if the sand is being added or removed from the beach.

(Analogous to total distance traveled.)

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Finite Sum Approximation

-Used to approximate area of circle using an inscribed polygon/triangles

-Have seen can be used to calculate the area under $f(x)$ on $[a,b]$:

Notation:

- $\Delta x = \frac{b-a}{n}$ *is the length of each subinterval if use equal subintervals.*
- c_k *is the representative point lying in the* k^{th} *subinterval of* $[a,b]$
- $f(c_k)$ *is the height of the rectangle approximating the area under the curve on the* k^{th} *subinterval.*
- *Area* $\approx \Delta x f(c_1) + \Delta x f(c_k) + \cdots \Delta x f(c_{n-1}) + \Delta x f(c_n)$.

*This has applications to net and total change if the $f(x)$ is a rate function.

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Finite Sum Approximations

-Can use to approximate other quantities such as:

- length of a curve
- volume of an irregular solid