Sigma Notation

If $a_1, a_2, a_3, \ldots, a_n$ is a sequence, or list, of numbers we want to add, this can be done compactly:

$$a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

The key is that the numbers should follow a pattern so that we have a formula for a_k. For example, suppose we are considering the numbers

$2, 4, 6, \cdots, 98, 100.$

First we note there are 50 numbers in the list so k will start at 1 and end at 50. The hard part is finding a formulat for a_k.. Here, $a_k = 2k$ since the k number in the list is just the k even number and we write

$$2 + 4 + \cdots + 98 + 100 = \sum_{k=1}^{50} 2k$$

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Exercise 1

1. Expand the sum $\sum_{k=1}^{4} 2^k$.
2. Expand the sum $\sum_{k=0}^{3} 2^k$.
3. Write the following sum using sigma notation: 15+20+25+30+35

Algebra with Sigma Notation

- $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

$$(a_1 + b_1) + (a_2 + b_2) + \cdots + (a_{n-1} + b_{n-1}) + (a_n + b_n)$$

$$= a_1 + (b_1 + a_2) + (b_2 + a_3) + \cdots + (b_{n+1} + a_n) + b_n$$

$$= a_1 + (a_2 + b_1) + (a_3 + b_2) + \cdots + a_n + b_{n-1} + b_n$$

$$\cdots = (a_1 + \cdots + a_n) + (b_1 + \cdots b_n)$$

Algebra with Sigma Notation

- $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
- $\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$
- $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$
- $\sum_{k=1}^{n} c = n \cdot c$

Algebra with Sigma Notation

If $\sum_{k=1}^{n} a_k = A$ and $\sum_{k=1}^{n} b_k = B$, then

- $\sum_{k=1}^{n}(a_k + b_k) = A + B$
- $\sum_{k=1}^{n}(a_k - b_k) = A - B$
- $\sum_{k=1}^{n} ca_k = cA$

In reading survey combined 2nd and 3rd:

$\sum_{k=1}^{n}(7a_k - b_k) = \sum_{k=1}^{n} 7a_k - \sum_{k=1}^{n} b_k = 7A - B.$

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Useful Formulas

Sums of integers:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

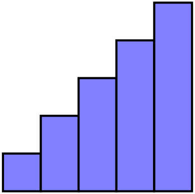
Sums of squares:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Sums of cubes:

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Consider 5 rectangles, each with width 1 and heights $1, \dot{5}$, as pictured.



Let A denote the total area of all of the rectangles. (So, $A = \sum_{k=1}^{5} k$ because all widths are 1.)

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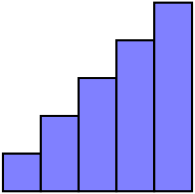

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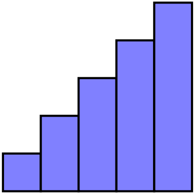

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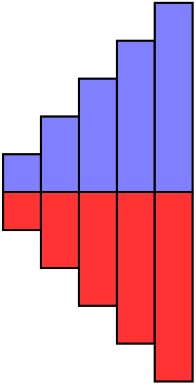
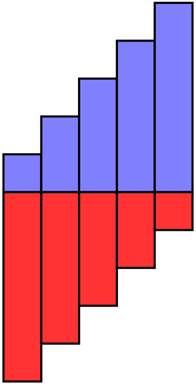


Copy each rectangle so there are two sets:

The total area of the rectangles is now $2A$.

Reverse the order of second set and make 5 rectangles:



Total areas is still $2A$. Notice, each of the rectangles have height, and thus area, 6.

So, $2A = 5(6) \implies A = \frac{5(6)}{2}$. That is $\sum_{k=1}^{5} k = \frac{5(5+1)}{2}$.

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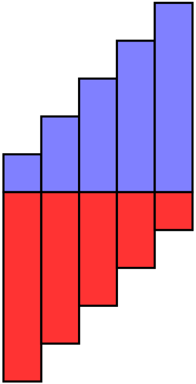

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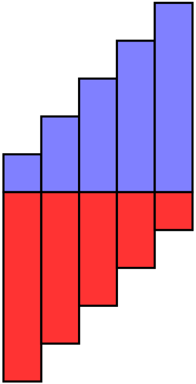

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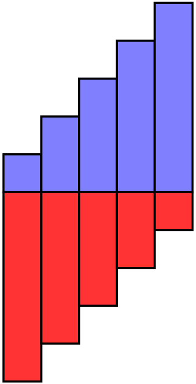

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If we did the same process with n rectangles, all of width 1 and increasing heigths $1, 2, \ldots, n$ copying, reversing and combining would give n rectangles all of the same area $n+1$. This is just a visual reprsenation of the formula

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2},$$

since $A = \sum_{k=1}^{n} k$ is the area of the orignal set of n rectangles, and $2A = n(n+1)$ since each of the n combined rectangles has area $n+1$.

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Exercise 2

a) $$\sum_{k=1}^{4} k^3 - 2k^2 = \sum_{k=1}^{4} k^3 - 2\sum_{k=1}^{4} k^2 = \frac{16(25)}{4} - 2\frac{4(5)(9)}{6} = 40$$

b) $$\sum_{k=4}^{10} k = \sum_{k=1}^{10} k - \sum_{k=1}^{3} k = \frac{10(11) - 3(4)}{2} = 49$$

Exercise 3

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[2, 3]$ into n subintervals of equal length.

a) Find a formula for x_k, $k = 0, 1, \ldots, n$.

b) Find a formula for c_k if we use the midpoints of each subinterval for our subinterval representatives.

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