Limit Laws- Theorem 2.2.1

If $L, \ M, \ c$, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \ \text{then}$$

1. Sum Rule: $\lim_{x\to c}(f(x)+g(x)) = L+M$
2. Difference Rule: $\lim_{x\to c}(f(x)-g(x)) = L-M$
3. Constant Multiple Rule: $\lim_{x\to c}(kf(x)) = kL$
4. Product Rule: $\lim_{x\to c}(f(x)\cdot g(x)) = L\cdot M$
5. Quotient Rule: $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M}, \ M \neq 0$
6. Power Rule: $\lim_{x\to c}[f(x)]^n = L^n, \ n$ a positive integer
7. Root Rule: $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, \ n$ is a positive integer.
   (If n is even, we assume that $L > 0$.)

Exercise

Find the value of

$$\lim_{x\to 5} \frac{x^2 - 5x + 2}{\sqrt{7 + x}}.$$

Carry out a step by step process and explicitly state which of the limit laws you used at each step to arrive at your final answer.

Theorem 2.5.8- Properties of Continuous Functions

If f and g are continuous at $x = c$, then the following functions are also continuous at $x = c$.

1. Sums: $f + g$
2. Differences: $f - g$
3. Constant Multiples: kf, for any real number k
4. Products: $f \cdot g$
5. Quotients: $\frac{f}{g}$, provided $g(c) \neq 0$
6. Powers: f^n, n a positive integer
7. Roots: $\sqrt[n]{f}$, provided it is defined on an open interval containing c, where n is a positive integer.

Compositions

Theorem 2.5.9- Composition of Continuous Functions

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

Theorem 2.5.19-Limits of Compositions

If g is continuous at the point b and $\lim_{x \to c} f(x) = b$, then

$$\lim_{x \to c} g(f(x)) = g(b) = g(\lim_{x \to c} f(x)).$$

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Exercise

Draw a function f which satisfies all of the following.

1. The domain of f is $[1, 5]$.
2. $0 < f(1) < 10 < f(5)$
3. There is no c such that $f(c) = 10$, (i.e. 10 is not in the range of f)

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Theorem 2.5.11-The Intermediate Value Theorem

If f is a continuous function on the closed interval $[a, b]$ and if d is any number between $f(a)$ and $f(b)$, then there exists a $c \in [a, b]$ such that $f(c) = d$.

Existance only, doesn't tell us what the exact value of c is, only that it exists.

Useful for showing a function $f(x)$ has a root. (i.e. the equation f(x)=0 has a solution).

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Example

Let $f(x) = x(x-1)^2$. Show that $f(x) = 1$ has at least one solution.

Let $h(x) = f(x) - 1 = x(x-1)^2 - 1$. Then we must show $h(x)$ has at least one root.

$h(1) = 1(0) - 1 = -1 < 2$, and $h(2) = 2(1) - 1 = 1 > 0$.

Thus, by the I.V.T., there is an $x \in [1, 2]$ such that $h(x) = 0$, which means $f(x) = 1$.

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Jane climbed a mountain, starting at Camp A, which is at an elevation of 1000 ft at 7AM, and ending at Camp B (at the top) which is at an elevation of 5000 ft at 7 PM. The next day she took a different path down the mountain, but started at 7 AM and returned to Camp A at 7 PM. Prove that there was some time between 7 AM and 7 PM for which Jane was at exactly the same elevation on each day at that moment.

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Hint: Let $f(t)$ be her elevation at time t on the *first* day, and $g(t)$ her elevation at time t on the *second* day.

*The key is that both f and g are continuous functions. (The hiker's elevation cannot make any sudden jumps as she walks up or down the mountain.)

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Exercise

$f(0) = g(12) = 1000$ and $f(12) = g(0) = 5000$.

We need to show $f(t) = g(t)$ for some t.

If we let $h(t) = f(t) - g(t)$, this is the same as showing there is a root for $h(t)$ in the interval [0,12].

Using the above facts about f and g we have

- $h(0) = -4000 < 0$
- $h(12) = 4000 > 0$

Since h is continuous, by the I.V.T. we know there is a time t in [0,12] for which $h(t) = 0$. That is there is a time t at which $f(t) = g(t)$.

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