Practice Problem Solutions for 2/20

(1) Find all asymptotes for the graph of $f(x) = \frac{x^3-3x^2}{x^2-1}$ and then draw a picture of the graph.

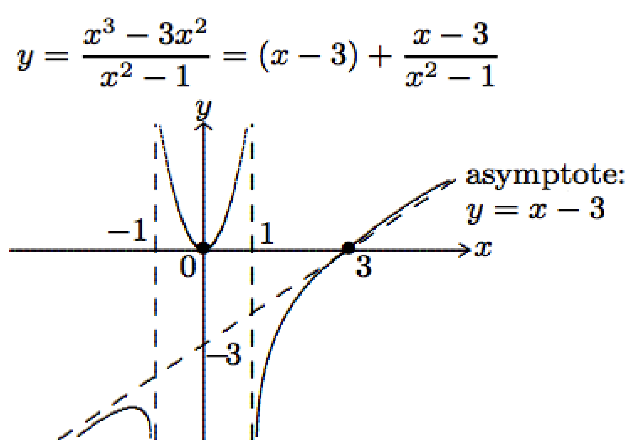
Vertical asymptotes: $f(x)$ is continuous everywhere except for where the denominator equals zero since it is the quotient of polynomials. Thus, there are vertical asymptotes at $x = 1$ and $x = -1$ since $\lim_{x\to 1} \frac{x^3-3x^2}{x^2-1}$ and $\lim_{x\to -1} \frac{x^3-3x^2}{x^2-1}$ do not exist because $f(x)$ tends to positive and negative infinity, respectively.

$$\lim_{x\to\infty} \frac{x^3-3x^2}{x^2-1} = \lim_{x\to\infty} \frac{x-3}{1-1/x^2}$$

The limit does not exist as the denominator tends to 1 and the numerator tends to infinity, and similarly $\lim_{x\to -\infty} \frac{x^3-3x^2}{x^2-1}$ does not exist and tends to negative infinity. Thus, there are no horizontal asymptotes. However, since this is a rational function whose numerator is one degree higher than the denominator, there is an oblique asymptote.

$$\begin{array}{r|l} & x - 3 \\ \hline x^2 - 1 & x^3 - 3x^2 \\ & -x^3 \quad\quad + x \\ \hline & -3x^2 + x \\ & 3x^2 \quad\quad - 3 \\ \hline & x - 3 \end{array}$$

Thus, $f(x) = x - 3 + \frac{x-3}{x^2-1}$. This means that $y = x - 3$ is an oblique asymptote, as $\frac{x-3}{x^2-1}$ tends to 0 as $x \to \pm\infty$. Using the information from the asymptotes, we know the graph looks like:



(2) Find the derivative of $f(x) = x^3$ at the point $x = 5$ using the limit definition we went over in class.

$$\begin{aligned} f'(5) &= \lim_{h\to 0} \frac{f(5+h) - f(5)}{h} = \lim_{h\to 0} \frac{(5+h)^3 - 5^3}{h} \\ &= \lim_{h\to 0} \frac{5^3 + 3(25)h + 3(5)h^2 + h^3 - 5^3}{h} \\ &= \lim_{h\to 0} \frac{75h + 15h^2 + h^3}{h} \\ &= \lim_{h\to 0}\ 75 + 15h + h^2 = 75. \end{aligned}$$