INTEGRATION BY PARTS V3

Much of this is making explicit how divergence computations (easy in index notation) look in more general notations. The index point of view is built on the fact that the integral of the divergence of a vector field over a manifold without boundary is zero.

Matrix Basis Elements and Brackets.

1: E_i^j has entry 1 in row i, column j - 0 elsewhere; $E_i^j e_k = \delta_k^i e_j$ where $O_i^j = E_i^j - E_j^i$.
2: $[E_i^j, E_l^k] = -\delta_i^l E_k^j + \delta_k^j E_i^l$.
3: $[O_a^b, O_c^d] = -\delta_{ac} O_b^d + \delta_{ad} O_b^c + \delta_{bc} O_a^d - \delta_{bd} O_a^c$.
4: $[E_j, E_k] = c_{jk}^i E_i \Longleftrightarrow d\theta^i = -\frac{1}{2} c_{jk}^i \theta^j \wedge \theta^k$.

Differential Form Conventions.

1:

$$\begin{aligned} \theta_1 \wedge \theta_2 &= \frac{1}{2}\left(\theta_1 \otimes \theta_2 - \theta_2 \otimes \theta_1\right) \\ 2d\theta(X,Y) &= X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y]) \end{aligned}$$

2: If $\alpha_{ij} = -\alpha_{ji}$

$$\frac{1}{2}\Sigma \alpha_{ij} \theta_i \otimes \theta_j = \alpha_{12} \theta_1 \wedge \theta_2 = \frac{1}{2}\begin{pmatrix} 0 & \alpha_{12} \\ -\alpha_{12} & 0 \end{pmatrix}$$

3:

$$< \theta^1 \wedge \theta^2 \ldots \theta^k, \bar{\theta^1} \wedge \bar{\theta^2} \ldots \bar{\theta^k} > = \det(\theta^i \cdot \theta^j)$$

(So k! times the natural tensor inner product.)

4: $** = (-1)^{p(n-p)}$.
5: $< \alpha, \beta > d\mu = \alpha \wedge *\beta$.
6: $\delta = (-1)^{np+n+1} * d*$ on p-forms since

$$\begin{aligned} d(\alpha^{p-1} \wedge *\beta^p) &= d\alpha^{p-1} \wedge *\beta^p + (-1)^{p-1} \alpha^{p-1} \wedge d * \beta^p) \\ &= d\alpha^{p-1} \wedge *\beta^p + (-1)^{p-1} \alpha^{p-1} \wedge (-1)^{(p-1)(n-p+1)} * *d * \beta^p \end{aligned}$$

and $(p-1)(n-p) \equiv np + n$ mod2.

7: Notation and $*$

$$\begin{aligned} \omega &= \begin{pmatrix} \omega_1 & \omega_2 \end{pmatrix} && = \Sigma \omega_i \theta^i \\ *\omega &= \begin{pmatrix} -\omega_2 & \omega_1 \end{pmatrix} && \\ \omega &= \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \end{pmatrix} && = \Sigma \omega_i \theta^i. \\ *\omega &= \frac{1}{2} \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} && \end{aligned}$$

R^2 or R^3 Case.

1: Notation for vectors (type(1,0)):

$$\vec{v} = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \Sigma v^i e_i.$$

2a: Notation for matrices (type(1,1)):

$$M = \begin{pmatrix} M^1_1 & M^1_2 \\ M^2_1 & M^2_2 \end{pmatrix} = \begin{pmatrix} \vec{M}_1 & \vec{M}_2 \end{pmatrix} = \Sigma M^i_j E^j_i.$$

(*Upper* index of M is the row index.)

2b: Notation for bilinear form (type(0,2)) case:

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} = \Sigma Q_{ij} \theta_i \otimes \theta^j.$$

3: $< \vec{v}, \vec{w} >= v^t w = v_i w^i$.

4: $< A, B >= tr(A^t B) = A^j_i B^i_j$.

5: $\mathrm{d}iv(\vec{v}) = \partial_i v^i = \partial_1 v^1 + \partial_2 v^2$.

6: Divergence on M of type (1,1) giving result of type (1,0):

$$div(M) = \begin{pmatrix} \mathrm{d}iv(\vec{M}_1) & \mathrm{d}iv(\vec{M}_1) \end{pmatrix} = \Sigma \left(\partial_j M^j_i \right) e_i.$$

7: $(\partial_i f) g = \mathrm{d}iv(f g e_i) - f \partial_i g$.

8a: $(\partial_i f) v^i = \partial_i (f v^i) - f \partial_i v^i$.

8b: $< \nabla f, \vec{v} >= \mathrm{d}iv(f\vec{v}) - f\mathrm{d}iv(\vec{v})$.

9a: $M^i_j (\partial_i v^j) = \partial_i (M^i_j v^j) - v^j \partial_i M^i_j$.

9b: $\mathrm{tr}(M^i_j D\vec{v}) =< D\vec{v}, M^t >= \mathrm{d}iv(M\vec{v}) - (\mathrm{d}iv M)\vec{v}$.

Interior Product and Lie Derivative Conventions.

1: $(\theta_1 \wedge \ldots \wedge \theta_p)(e_1, \ldots, e_p) = \dfrac{1}{p!}$.

2: $i_X(\alpha^p)(Y_2, \ldots, Y_p) = p\alpha(X, Y_2, \ldots, Y_p)$.

3: $i_{e_1}(\theta_1 \wedge \ldots \wedge \theta_p) = (\theta_2 \wedge \ldots \wedge \theta_p)$.

4: $i_X(\alpha^p \wedge \beta^q) = i_X(\alpha^p) \wedge \beta^q + (-1)^p \alpha^p \wedge i_X(\beta^q)$. *(An anti-derivation, like d.)*

5: $L_X = i_X \circ d + d \circ i_X$ by checking on $\alpha^p \wedge \beta^q$ and using induction on the total degree.

6: $(L_X\alpha^p)(Y_1,\ldots,Y_p) = (\nabla_X\alpha^p)(Y_1,\ldots,Y_p) + \Sigma_i\alpha^p(Y_1,\ldots,\nabla_{Y_i}X,\ldots,Y_p)$. This is because $\nabla_X Y_i - L_X Y_i = \nabla_{Y_i}X$.

7: The formulas

$$\begin{aligned} L_X Y = &\quad \nabla_X Y - \nabla_Y X \\ L_X(\theta)(Y) = &\quad (\nabla_X\theta)(Y) + \theta(\nabla_Y X) \end{aligned}$$

and their extensions to tensors of type T^s_r correspond to

$$\begin{aligned} (L_X T)^I_J = &\quad X^c\nabla_c T^I_J - \Sigma^s_{k=1} T^{i_1 i_2 \ldots c \ldots i_s}_J \nabla_c X^{i_k} \\ &+ \Sigma^r_{k=1} T^I_{j_1 j_2 \ldots c \ldots j_r}\nabla_{j_k} X^c \end{aligned}$$

Divergences in Special Cases.

1: With the definition $\mathrm{div}(X)d\mu = L_X d\mu$, we also have

$$\mathrm{div}(X) = \nabla_i X^i = \theta^i(\nabla_{e_i}X) = \mathrm{trace}(Y \to \nabla_Y X)$$

since for an orthonormal coframe field and each i

$$(L_X\theta^i)e_i = (\nabla_X\theta^i)e_i + \theta^i(\nabla_{e_i}X)$$

and $(\nabla_X\theta^i)e_i = -\theta^i(\nabla_X e_i) = -<e_i, \nabla_X e_i> = 0$.

2: Above div is the negative of the formal adjoint to grad because

$$\mathrm{div}(X) = \mathrm{trace}(Y \to \nabla_Y X) = \Sigma < \nabla_{e_i}X, e_i >$$

implies

$$\mathrm{div}(fX) = <\nabla f, X> + f\mathrm{div}(X).$$

(This used $\Sigma(e_i(f)) < X, e_i > = Xf$.)

3: For a 1-form ω, the definition

$$\delta\omega = -\mathrm{trace}(Y \to \nabla_Y\omega) = -\Sigma(\nabla_{e_i}\omega)(e_i)$$

leads to

$$\delta(f\omega) = f\delta\omega - < df, \omega >$$

since $< df, \omega > = \Sigma(e_i(f))\omega(e_i) = Xf$, X being the vector field dual to ω. Thus all the following are equal in this case:

$$Xf = < df, \omega > = < \nabla f, X > = \omega(\nabla f).$$

Also

$$\delta\omega = -\mathrm{div}X$$

since starting with $\omega = < X, \ >$, we have

$$\begin{aligned} \nabla_{e_i}\omega = &\quad < \nabla_{e_i}X, \ > \\ \Sigma\left(\nabla_{e_i}\omega\right)(e_i) = &\quad \Sigma < \nabla_{e_i}X, e_i > . \end{aligned}$$

4: With K-N conventions of

$$\begin{aligned} \nabla_{e_i}e_j = &\quad \Gamma^k_{ij}e_k \\ \nabla_{e_i}\theta^k = &\quad -\Gamma^k_{ij}\theta^j \\ \Gamma^k_{ij} = &\quad -\Gamma^j_{ik} \end{aligned}$$

for an orthonormal basis, we have

$$\begin{aligned}\nabla_{e_i}\Sigma_j e_j\otimes e_j = &\quad \Sigma_{j,k}\Gamma^k_{ij}e_k\otimes e_j+\Gamma^k_{ij}e_j\otimes e_k\\ = &\quad \Sigma_{j,k}\Gamma^k_{ij}e_k\otimes e_j+\Gamma^j_{ik}e_k\otimes e_j\\ = &\quad 0\end{aligned}$$

since $\Gamma^k_{ij}=-\Gamma^j_{ik}$.

5: So starting with $X=\Sigma_j\left(\omega(e_j)\right)e_j$, we have

$$\begin{aligned}\nabla_{e_i}X = &\quad \Sigma_j < \nabla_{e_i}\left(\omega(e_j)e_j\right)\\ \mathrm{div}X=\Sigma_i<\nabla_{e_i}X,e_i> = &\quad \Sigma_{i,j}<\left(\left(\nabla_{e_i}\omega\right)(e_j)\right)e_j,e_i> .\\ &+\Sigma_i<C^1_1\left(\omega\left(\nabla_{e_i}\left(\Sigma_je_j\otimes e_j\right)\right)\right),e_i>\\ = &\quad \Sigma_i\left(\nabla_{e_i}\omega\right)(e_i)+0\\ = &\quad -\delta\omega\end{aligned}$$

where C^1_1 is contraction on the first two slots and the second term vanishes because $\nabla_{e_i}\Sigma_je_j\otimes e_j=0$.

6: As in Besse (but with the K-N T^r_s convention which is oposite to Besse), view

$$\nabla : T^r_s\to\Omega^1(M)\otimes T^r_s$$

so that the "slot" for the covariant derivative comes first. Then

$$\nabla(A\otimes B)=(\nabla A)\otimes B+A\otimes(\nabla B).$$

7: Use the notation

$$C_{i_1,j_1}\dots C_{i_r,j_r}T$$

for contraction pairing covariant indices i_1 with j_1, etc. Similarly for contravariant or mixed contractions. Note with this notation

$$C_{1,2}C_{3,4}T=C_{1,2}\left(C_{1,2}T\right)=C_{1,2}\left(C_{3,4}T\right)$$

The above would also be $C_{1,2}\left(C_{3,4}T\right)$. This notation lines up with natural contractions to write in index notation.

8: We use the non-standard notation $I_{i,j}$ to indicate the interchange of covariant slots i and j in a tensor. With this notation, for a 1-form ω, the symmetrized covariant derivative is

$$\delta^*\omega=\frac{1}{2}\left(\nabla\omega+I_{1,2}\left(\nabla\omega\right)\right).$$

9: The symmetrized covariant derivative from 1-forms to symmetric covariant 2-tensors (sections of) $S^2(T^*M)$ is defined by

$$(\delta^*\omega)(X,Y)=\frac{1}{2}\left(\left(\nabla_X(\omega)\right)+\left(\nabla_Y(\omega)\right)\right)$$

is the dual to the divergence $\delta : S^2(T^*M) \to T^*(M)$ defined by

$$\begin{aligned} \delta h = & \ -\text{trace}((Y,Z) \to (\nabla h)(Y,Z,\ )) \\ (\delta h)(X) = & \ -\text{trace}((Y,Z) \to (\nabla h)(Y,Z,X)) \\ (\delta h)(X) = & \ -\Sigma_i (\nabla_{e_i} h)(e_i, X). \end{aligned}$$

To see this, note

$$\begin{aligned} -\delta\left(C_{1,2}\left(\omega \otimes h\right)\right) = & \quad C_{1,4}C_{2,3}\left(\left(\nabla\omega\right)\otimes h\right) \\ & +C_{1,4}C_{2,3}\left(\omega \otimes \left(\nabla h\right)\right) \\ = & \quad \frac{1}{2}\left(C_{1,4}C_{2,3} + C_{1,3}C_{2,4}\right)\left(\left(\nabla\omega\right)\otimes h\right) \\ & - < \omega, \delta h > \\ = & \quad \frac{1}{2}\left(C_{2,4}C_{1,3}\right)\left(\left(I_{1,2}\left(\nabla\omega\right)\right)\otimes h\right) \\ & +\frac{1}{2}\left(C_{2,4}C_{1,3}\right)\left(\left(\nabla\omega\right)\otimes h\right) \\ & - < \omega, \delta h > \\ = & \ - < \delta^*\omega, h > \\ & - < \omega, \delta h > \end{aligned}$$

where we have used the argument interchange operator $I_{1,2}$ as well as the symmetry of h.

10: For a 1-form ω,

$$\delta^*\omega = \frac{1}{2} L_X g$$

where X is the 1-form dual to ω and $g =< \ , \ >$ is the inner product. This is because

$$\begin{aligned} \left(L_X(g)\right)(U,V) = & \ \left(\nabla_X g\right)(U,V) + < \nabla_U X, V > + < U, \nabla_V X > \\ = & \ < \nabla_U X, V > + < U, \nabla_V X > \\ = & \ 2\left(\nabla\omega + I_{1,2}\left(\nabla\omega\right)\right)(U,V) \end{aligned}$$

General Case.

1: Besse Formulation is $\nabla : T^s_r \to \Omega^1(M) \otimes T^s_r$ has formal adjoint $\nabla^* : \Omega^1(M) \otimes T^s_r \to T^s_r$ given by

$$(\nabla^*\alpha)(X_1, \ldots, X_r) = -\Sigma(\nabla_{Y_i}\alpha)(Y_i, X_1, \ldots, X_r)$$

for an orthonormal basis Y_i. The "opposite" (as in minus sign) of the trace of

$$(X,Y) \to (\nabla_X \alpha)(Y, X_1, \ldots, X_r).$$

The trace here might also be referred to as the contraction $C_{1,2}$.

2: To prove this, for $\alpha \in T^s_{r+1}$ and $\beta \in T^s_r$,

$$\begin{aligned} -\delta\left(C_1\left(\beta \otimes \alpha\right)\right) = &\ C_{1,1}\left(\nabla\left(C_1\left(\beta \otimes \alpha\right)\right)\right) \\ = &\ C_2\left(\left(\nabla\beta \otimes \alpha\right) + \left(\beta \otimes \nabla\alpha\right)\right) \\ = &\ < \nabla\beta, \alpha > + < \beta, C_{1,2}\nabla\alpha > \\ = &\ < \nabla\beta, \alpha > - < \beta, \nabla^*\alpha > \end{aligned}$$

where the C_i are contractions. Specifically

$$\begin{aligned} C_1 = &\ C_{1,r+2}\dots C_{r,2r+1}C^{1,s+1}\dots C^{s+1,2s} \\ C_2 = &\ C_{1,r+2}\dots C_{r+1,2r+2}C^{1,s+1}\dots C^{s+1,2s} \end{aligned}$$

So $C_{1,2}\left(C_1 T\right) = C_2 T$ for a tensor field $T \in T^s_{r+1}$.

3: ∇^* is also the divergence δ on forms since

$$\begin{aligned} < d\alpha, \beta > = &\ < \mathrm{Alt}(\nabla\alpha), \beta > \\ = &\ < \nabla\alpha, \mathrm{Alt}(\beta) > \\ = &\ < \nabla\alpha, \beta > \\ = &\ < \alpha, \nabla^*\beta > \end{aligned}$$

using the facts that β and $\nabla^*\beta$ are skew symmetric, and the fact that the inner product on forms is the same (or a multiple of) the inner product on tensors.