INTEGRATION BY PARTS

Matrix Basis Elements and Brackets.

1: E_i^j has entry 1 in row i, column j - 0 elsewhere; $E_i^j e_k = \delta_k^i e_j$ where $O_i^j = E_i^j - E_j^i$.
2: $[E_i^j, E_l^k] = -\delta_i^l E_k^j + \delta_k^j E_i^l$.
3: $[O_a^b, O_c^d] = -\delta_{ac} O_b^d + \delta_{ad} O_b^c + \delta_{bc} O_a^d - \delta_{bd} O_a^c$.
4: $[E_j, E_k] = c_{jk}^i E_i \Longleftrightarrow d\theta^i = -\frac{1}{2} c_{jk}^i \theta^j \wedge \theta^k$.

Differential Form Conventions.

1:

$$\theta_1 \wedge \theta_2 = \frac{1}{2}\left(\theta_1 \otimes \theta_2 - \theta_2 \otimes \theta_1\right)$$
$$2d\theta(X,Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y])$$

2: If $\alpha_{ij} = -\alpha_{ji}$

$$\frac{1}{2}\Sigma \alpha_{ij}\theta_i \otimes \theta_j = \alpha_{12}\theta_1 \wedge \theta_2 = \frac{1}{2}\begin{pmatrix} 0 & \alpha_{12} \\ -\alpha_{12} & 0 \end{pmatrix}$$

3:

$$< \theta^1 \wedge \theta^2 \ldots \theta^k, \bar{\theta^1} \wedge \bar{\theta^2} \ldots \bar{\theta^k} > = \det(\theta^i \cdot \theta^j)$$

(So k! times the natural tensor inner product.)

4: $** = (-1)^{p(n-p)}$.
5: $< \alpha, \beta > d\mu = \alpha \wedge *\beta$.
6: $\delta = (-1)^{np+n+1} * d*$ on p-forms since

$$\begin{aligned} d(\alpha^{p-1} \wedge *\beta^p) &= d\alpha^{p-1} \wedge *\beta^p + (-1)^{p-1}\alpha^{p-1} \wedge d * \beta^p) \\ &= d\alpha^{p-1} \wedge *\beta^p + (-1)^{p-1}\alpha^{p-1} \wedge (-1)^{(p-1)(n-p+1)} * *d * \beta^p \end{aligned}$$

and $(p-1)(n-p) \equiv np + n \bmod 2$.
7: Notation and $*$

$$\begin{aligned} \omega &= \begin{pmatrix} \omega_1 & \omega_2 \end{pmatrix} && = \Sigma \omega_i \theta^i \\ *\omega &= \begin{pmatrix} -\omega_2 & \omega_1 \end{pmatrix} \\ \omega &= \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \end{pmatrix} && = \Sigma \omega_i \theta^i. \\ *\omega &= \frac{1}{2}\begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \end{aligned}$$

R^2 or R^3 Case.

1: Notation for vectors (type(1,0)):

$$\vec{v} = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \Sigma v^i e_i.$$

2a: Notation for matrices (type(1,1)):

$$M = \begin{pmatrix} M_1^1 & M_2^1 \\ M_1^2 & M_2^2 \end{pmatrix} = \begin{pmatrix} \vec{M}_1 & \vec{M}_2 \end{pmatrix} = \Sigma M_j^i E_i^j.$$

(*Upper* index of M is the row index.)

2b: Notation for bilinear form (type(0,2)) case:

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} = \Sigma Q_{ij} \theta_i \otimes \theta^j.$$

3: $< \vec{v}, \vec{w} >= v^t w = v_i w^i$.
4: $< A, B >= tr(A^t B) = A_i^j B_j^i$.
5: $\mathrm{d}iv(\vec{v}) = \partial_i v^i = \partial_1 v^1 + \partial_2 v^2$.
6: Divergence on M of type (1,1) giving result of type (1,0):

$$div(M) = \begin{pmatrix} \mathrm{d}iv(\vec{M}_1) & \mathrm{d}iv(\vec{M}_1) \end{pmatrix} = \Sigma \left(\partial_j M_i^j \right) e_i.$$

7: $(\partial_i f) g = \mathrm{d}iv(fge_i) - f\partial_i g$.
8a: $(\partial_i f) v^i = \partial_i (fv^i) - f\partial_i v^i$.
8b: $< \nabla f, \vec{v} >= \mathrm{d}iv(f\vec{v}) - f\mathrm{d}iv(\vec{v})$.
9a: $M_j^i (\partial_i v^j) = \partial_i (M_j^i v^j) - v^j \partial_i M_j^i$.
9b: $\mathrm{tr}(M_j^i D\vec{v}) =< D\vec{v}, M^t >= \mathrm{d}iv(M\vec{v}) - (\mathrm{d}ivM)\vec{v}$.

Interior Product Conventions.

1: $(\theta_1 \wedge \ldots \wedge \theta_p)(e_1, \ldots, e_p) = \dfrac{1}{p!}$.
2: $i_X(\alpha^p)(Y_2, \ldots, Y_p) = p\alpha(X, Y_2, \ldots, Y_p)$.
3: $i_{e_1}(\theta_1 \wedge \ldots \wedge \theta_p) = (\theta_2 \wedge \ldots \wedge \theta_p)$.
4: $i_X(\alpha^p \wedge \beta^q) = i_X(\alpha^p) \wedge \beta^q + (-1)^p \alpha^p \wedge i_X(\beta^q)$. *(An anti-derivation, like d.)*
5: $L_X = i_X \circ d + d \circ i_X$ by checking on $\alpha^p \wedge \beta^q$ and using induction on the total degree.
6: $(L_X \alpha^p)(Y_1, \ldots, Y_p) = (\nabla_X \alpha^p)(Y_1, \ldots, Y_p) + \Sigma_i \alpha^p(Y_1, \ldots, \nabla_{Y_i} X, \ldots, Y_p)$. This is because $\nabla_X Y_i - L_X Y_i = \nabla_{Y_i} X$.
7: With the definition $\mathrm{div}(X) d\mu = L_X d\mu$, we also have

$$\mathrm{div}(X) = \nabla_i X^i = \theta^i(\nabla_{e_i} X) = \mathrm{trace}(Y \to \nabla_Y X)$$

since for an orthonormal coframe field and each i

$$(L_X \theta^i) e_i = (\nabla_X \theta^i) e_i + \theta^i(\nabla_{e_i} X)$$

and $(\nabla_X \theta^i) e_i = -\theta^i(\nabla_X e_i) = - < e_i, \nabla_X e_i >= 0$.

General Case.

1: Besse Formulation is $\nabla : T^r_s \to \Omega^1(M) \otimes T^r_s$ has formal adjoint $\nabla^* : \Omega^1(M) \otimes T^r_s \to T^r_s$ given by

$$(\nabla^*\alpha)(X_1, \ldots, X_r) = -\Sigma(\nabla_{Y_i}\alpha)(Y_i, X_1, \ldots, X_r)$$

for an orthonormal basis Y_i. The "opposite" of the trace of

$$(X, Y) \to (\nabla_X \alpha)(Y, X_1, \ldots, X_r).$$