Adding all 4 cell contributions together:

$$\left[\frac{n_1 n_2}{n_1+n_2}\right](\hat{p}_2-\hat{p}_1)^2\left[\frac{n_2}{n_1+n_2}\cdot\frac{1}{\bar{p}}+\frac{n_2}{n_1+n_2}\cdot\frac{1}{1-\bar{p}}+\frac{n_1}{n_1+n_2}\cdot\frac{1}{\bar{p}}+\frac{n_1}{n_1+n_2}\cdot\frac{1}{1-\bar{p}}\right]$$

Note $\frac{1}{\bar{p}}+\frac{1}{1-\bar{p}}=\frac{1}{\bar{p}(1-\bar{p})}$

and $\frac{n_1 n_2}{n_1+n_2}=\frac{1}{\frac{1}{n_1}+\frac{1}{n_2}}$ and $\frac{n_2}{n_1+n_2}+\frac{n_1}{n_1+n_2}=1$

So $\chi^2=\dfrac{(\hat{p}_2-\hat{p}_1)^2}{\bar{p}(1-\bar{p})\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}$

which is the square of the z-statistic.