Math 1710 Class 27

CI's and Hypothesis Testing

Dr. Back

Oct. 30, 2009

Smoke Detectors

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$$\text{So } \hat{p} = \frac{376}{400} = .94.$$

HT Vocabulary List

- Null Hypothesis H
- Alternative Hypothesis H
  (One Sided vs Two Sided)
- $SD(\hat{p})$ vs. $SE(\hat{p})$
- z-statistic
- Tail Probability
- P-value
- Significance Level
- Retain Null H vs. Accept Alt H (equiv. reject Null H)

HT Vocabulary Details

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Or maybe a specification.

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Or maybe a specification.

In the 1-sided case, many would write $p \leq p_0$ or $p \geq p_0$ for H_0.

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Two one-sided possibilities:

- $p < p_0$
- $p > p_0$

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So accepting the alternative means a statistically significant result; something was proven.

Retaining the null just means the evidence was not strong enough to convincingly prove the alternative.

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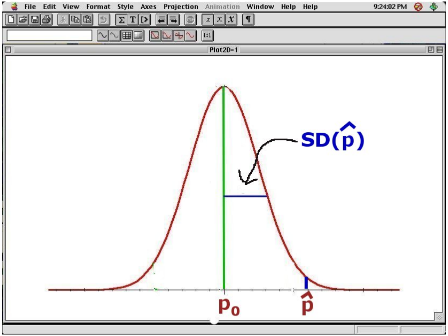

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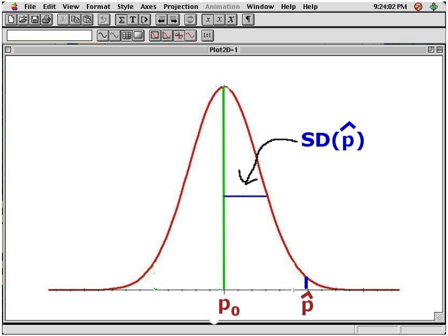

$$z = \frac{\hat{p} - p_0}{SD(\hat{p})}$$

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"...assuming H_0 is true" is why we use $SD(\hat{p})$ rather than $SE(\hat{p})$ in calculating our z-statistic.

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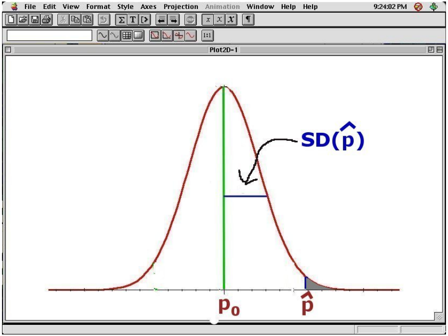
(Tail probability shaded in gray.)



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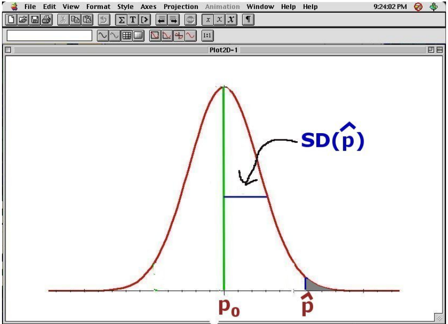
$H_a: \ p > p_0$ P-value=tail probability



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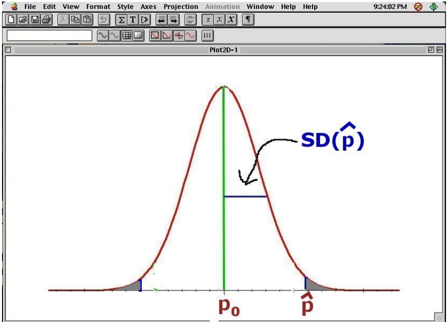
$H_a: \; p \neq p_0$ P-value=2(tail probability)



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Significance Level α: The cutoff between "small" and "big" P-values.
Small P-value means reject the null.
Big P-value means retain the null.

Logic of Hypothesis Testing

Procedure:

1. ...
2. ...
3. Consider the sampling distribution of $\hat{p}$ which would hold if H_0 were true.
4. Retain H_0 if $\hat{p}$ is reasonably consistent with this sampling distribution. Otherwise reject H_0.
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5. ...

Three ways to carry out step 4:

1. Calculate a z–statistic and determine a p-value.
2. Calculate a z-statistic and compare with a critical value z^*.
3. Use a confidence interval.

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z-statistic

$$z = \frac{.94 - .9}{.015} = 2.67.$$

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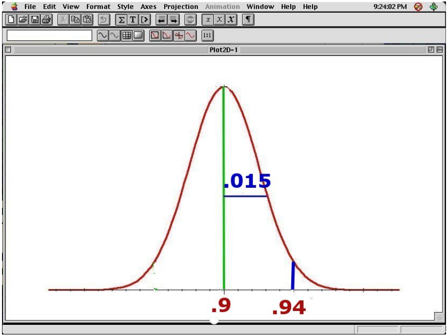
z-statistic is the z-score of $\hat{p}$ on the samp dist centered at the hypothesized value of p



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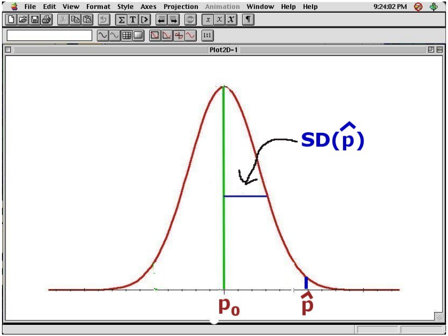

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Method 1: Tail Prob $= P(\hat{p} > .94) = P(Z > 2.67) = .0038$

Since the test is 1-sided, P-value=tail probability=.0038.

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This is small, so conclusion is we reject H_0.

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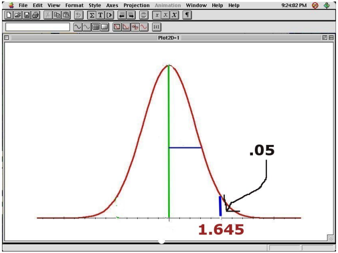
Method 2: Say significance level is $\alpha = .05$.
Since this is a 1-sided test with $\alpha = .05$, the appropriate critical values is $z^* = 1.645$.

N(0,1)



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Two primary ways to continue:
Method 2: Say significance level is $\alpha = .05$.
Since our z-statistic of 2.67 is more extreme than $z^* = 1.645$ (and supports H_a), we reject H_a at $\alpha = .05$.

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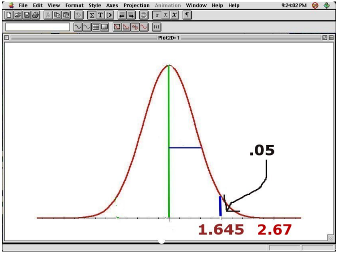
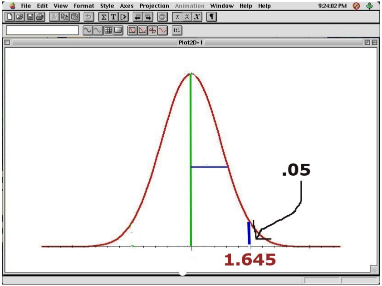


CPSC Example

Method 3: Using a CI. Still $\alpha = .05$.
The picture



shows that a 90% CI has the same critical value as an $\alpha = .05$ HT.

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Since all the values of p in our CI support H_a, the result of the HT is to reject the null.
Had even one been consistent with H_0, we would have retained the null.

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Because of the extra error associated with SE instead of SD, Method 3 is not arithmetically equivalent to Methods 1 and 2. Method 3 could reject the null while method 1 retained or vice-versa.

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This small discrepancy is not of practical importance. It could be corrected by a more elaborate CI formula.

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This small discrepancy is not of practical importance. It could be corrected by a more elaborate CI formula.
The general principle that many HT's can be done via a CI is an important fact.

Three Kinds of $H_a^{\prime}s$

H_a for a prop. HT can be any of the three possibilities

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One chooses among these based on the question being studied.

The value of $\hat{p}$ never plays a role in formulating hypotheses!

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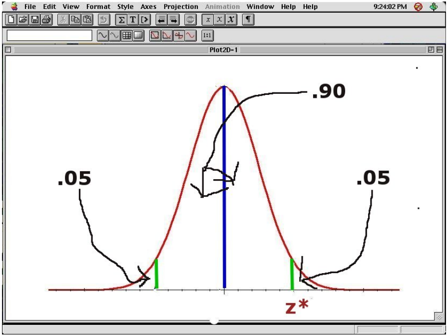
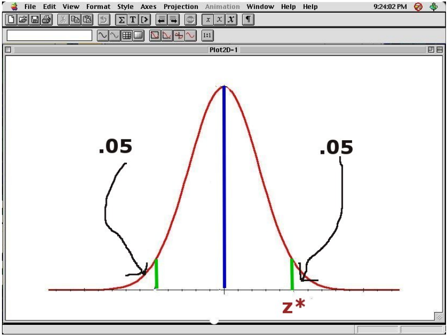
N(0,1) and the critical value z^*



Three Kinds of $H_a's$

N(0,1)

$H_a: \; p \neq p_0$ (2-sided), $\alpha = .10$



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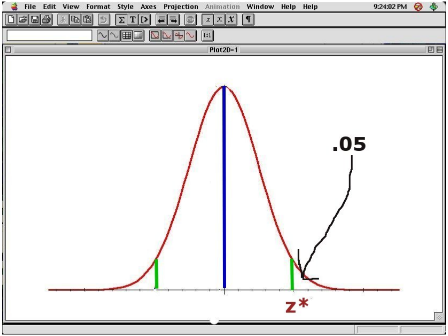

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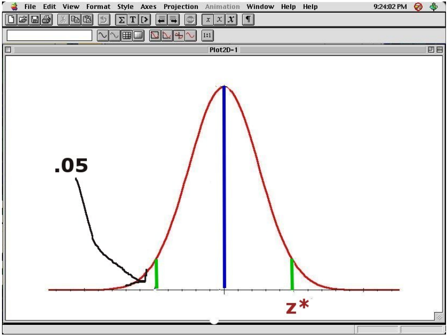

Conditions for Prop Tests

- plausible independence
- random sampling
- 10% condition
- success/failure

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What Happens if Not Satisfied:

random sampling -
could be critical;
might be ok if "representative"
representative hard/impossible to define

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<u>What Happens if Not Satisfied:</u>

10% condition -
results in overestimation of samp. dist. st dev
gradual breakdown in formulas, not method

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What Happens if Not Satisfied:

success/failure -
progressive reduction of accuracy
accuracy varies regardless for smaller values of n

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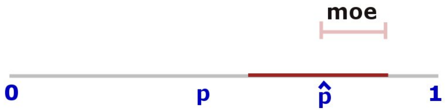
Margin of error picture.



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The challenge is we don't know $\hat{p}$ until we conduct the sample.

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1. Conservative: Just use $\hat{p} = .5$ since that gives the biggest possible value of $\hat{p}\hat{q}$.
(Think about the graph of $y = x(1 - x)$.)

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$$n = 1960^2(.5)(.5) = 980^2 = 960,400$$

which explains why surveys DO NOT seek an MOE of .1%.

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3. If you expect $\hat{p}$ to be in a range, use the most demanding $\hat{p}$ within that range.