Math 1710 Class 7

Binomial Distribution, Normal Approximation

Dr. Back

Sep. 11, 2009

Two Independent RV's

X	**probability**	**Y**	**probability**
x_1	p_1	y_1	q_1
x_2	p_2	y_2	q_2
$\dots$	$\dots$	$\dots$	$\dots$
x_n	p_n	y_m	q_m

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Independence means $P\{X = x_1 \;\&\; Y = y_1\} = p_1 q_1$.

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Independence means $P\{X = x_1 \ \& \ Y = y_1\} = p_1 q_1$.
In this case $X + Y$ would be $x_1 + y_1$.

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$x_1 + y_1$	$p_1 q_1$
$x_1 + y_2$	$p_1 q_2$
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(Why *roughly:* Some value c of $X + Y$ may show up several times in the above table and then more precisely $p_i q_j$ is $P\{X = x_i \text{ \& } Y = y_j\}$. And $P(\{X + Y = c\})$ would be the sum of several entries. But computing μ,σ won't be affected by not combining such rows.)

Variance

Assuming we know E(X+Y)=EX+EY, then $Var(X+Y)$

$$
\begin{aligned}
&= \Sigma_{i,j} p_i q_j (x_i + y_j - \mu_X - \mu_Y)^2 \\
&= \Sigma_{i,j} p_i q_j (x_i - \mu_X)^2 + \Sigma_{i,j} p_i q_j (y_j - \mu_Y)^2 \\
&\quad + 2\Sigma_{i,j} p_i q_j (x_i - \mu_X)(y_j - \mu_Y) \\
&= (\Sigma_j q_j)\, Var(X) + (\Sigma_i p_i)\, Var(Y) \\
&\quad + 2\,(\Sigma_i p_i (x_i - \mu_X))\,(\Sigma_j q_j (y_j - \mu_Y)) \\
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where we used $(a+b)^2 = a^2 + 2ab + b^2$,

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and $\Sigma p_i (x_i - \mu_X) = 0$.

Proof of Expectation of Mean Formula

$E(X+Y)$

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\begin{aligned}
=&\Sigma_{i,j}(x_i + y_j)P(X = x_i \text{ and } Y = y_j)\\
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Unsurprising, but not obvious looking.

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Did not require X and Y to be independent!

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What is the probability that the first box remains empty?

What is the expected number of empty boxes?

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What does $Y = X_1 + X_2 + X_3 + X_4 + X_5$ represent?

Three Girls Out of Five Children

Suppose P(Girl)=.6 and gender of births independent.

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But this undercounts the answer.

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How many such orders?

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10 Possible Orders:

- *First child a girl:* GGGBB GGBGB GGBBG GBGGB GBGBG GBBGG
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So answer is $10(.6)^3(.4)^2 = .3456$.

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5 birth positions, 3 of which girls

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Above showed

$$\binom{5}{3} = \binom{5}{2}$$

which works in general as well.

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$$\binom{5}{3} = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = \frac{5 \cdot 4}{1 \cdot 2} = 10$$

If one tacks a 2! onto both the numerator and denominator,

$$\frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3!2!}$$

showing

$$\binom{5}{3} = \frac{5!}{3!2!}$$

which is also a general formula.

Why $C_{5,3} = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3}$?

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Suppose the three girls are named Abby, Betty, and Carla.

Then 5 birth positions for Abby.

4 for Betty.

3 for Carla.

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$$C_{5,3} = \binom{5}{3} = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3}$$

Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

$(a+b)^0 = 1$
$(a+b)^1 = 1a + 1b$
$(a+b)^2 = 1a^2 + 2ab + 1b^2.$
$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3.$
...

Pascal's Triangle

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

10 Possible Orders: (Two points of view.)

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Pascal's Triangle

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1		5		10		<u>10</u>		5		1

10 Possible Orders: (Three points of view.)

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$$\binom{5}{3} = \binom{4}{2} + \binom{4}{3}.$$

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For Y we know $\mu = np$, $\sigma = \sqrt{npq}$, and

$$P(Y = k) = \binom{n}{k} p^k q^{n-k}.$$

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A substitute for a big table giving the prob. dist. of Y.

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You poll 100 people.

What is the probability that exactly 65 report approval?

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$$P(Y = 65) = ?$$

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TI-83,84	100 $Math \to Prb \to nCr$ 65 (*Math* is at the left of row 3.)
TI-89	$Math \to Probability \to nCr(100, 65)$ (*Math* is above the 5.)
TI-30	100 nCr 65 (nCr is above the 8 on my TI-30.)

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($Math$ is above the 5.)

TI-30 100 nCr 65
(nCr is above the 8 on my TI-30.)

An answer like $1.095067153E27$ means 1.095×10^{27} and so

$$P(Y = 65) = \binom{100}{65} .7^{65} .3^{35} = .04678.$$

Tedious

Notice that calculating $P(60 \leq Y \leq 65)$ by the above method would not be pleasant.

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TI-8x calculators have a binomialcdf function which can do this. Please don't use that function to supply any homework or exam answers in this course.

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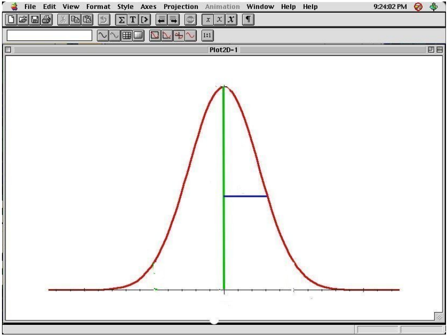
$$N(\mu, \sigma)$$



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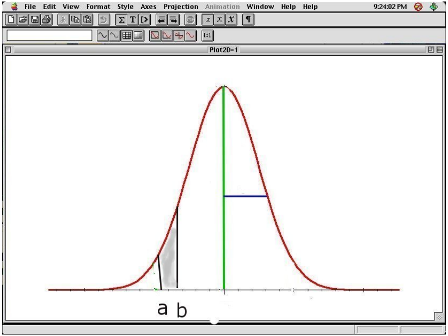

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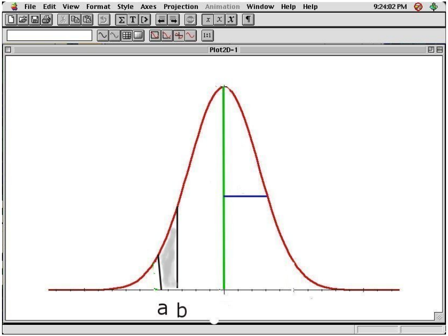

Area corresponds to probability.

A Continuous Distribution

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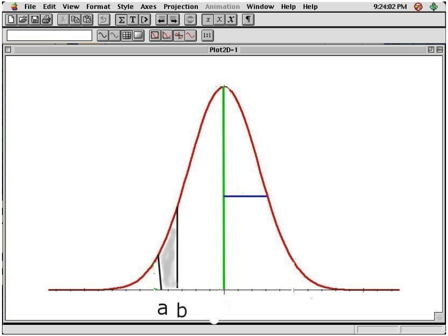

The entire area under the curve is 1.

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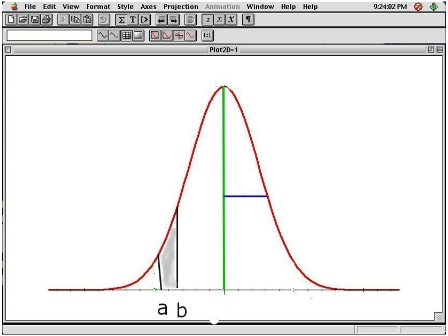

The area between a and b is the probability of a value x falling within that range.

The 68 – 95 – 99.7 Rule

68% within 1 standard deviation of the mean.

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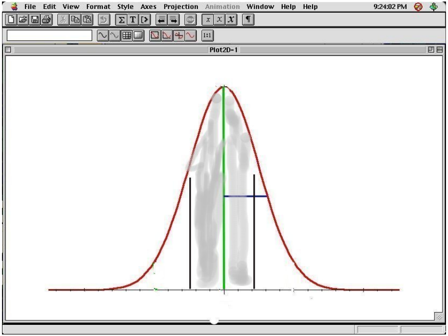
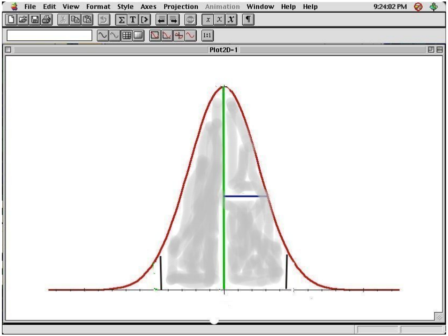
$N(\mu,\sigma)$



The 68 — 95 — 99.7 Rule

95% within 2 standard deviations of the mean.

$$N(\mu, \sigma)$$



The 68 – 95 – 99.7 Rule

99.7% within 3 standard deviations of the mean.

$N(\mu, \sigma)$