Math 1710 Class 9

Normal Distributions and Approximation

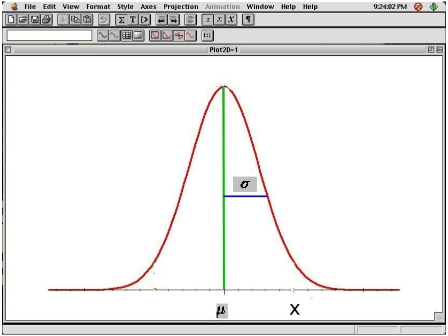
Dr. Back

Sep. 16, 2009

Z-scores

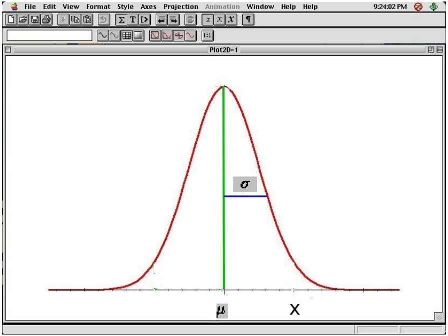
A general normal distribution:

$$N(\mu, \sigma)$$



Z-scores

$N(\mu, \sigma)$



Can convert from a general $N(\mu, \sigma)$ to $N(0, 1)$ via the Z-score.

$$z = \frac{x - \mu}{\sigma}$$

UsingTable Z

For example $P(-.67 < Z < 1) = ?$

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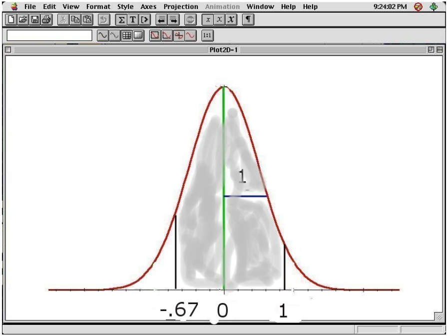
N(0,1)



Using Table Z

For example $P(-.67 < Z < 1) =?$

N(0,1)

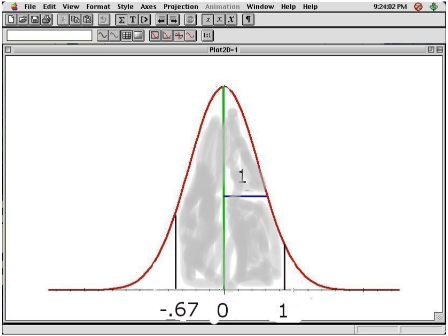


Table Z tells us $P(-.67 < Z < 1) = .8413 - .2514 = .5899.$

Mileage Example

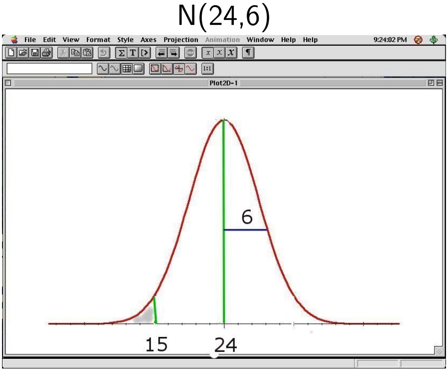
Suppose a normal model N(24 mpg, 6 mpg) describes fuel efficiency of cars in a region:

Mileage Example

Percent of cars with mileage below 15?

Mileage Example

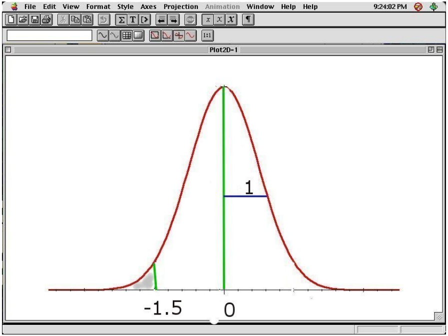
Percent of cars with mileage below 15?



Mileage Example

Percent of cars with mileage below 15?

N(0,1)



Mileage Example

Percent of cars with mileage below 15?

N(0,1)

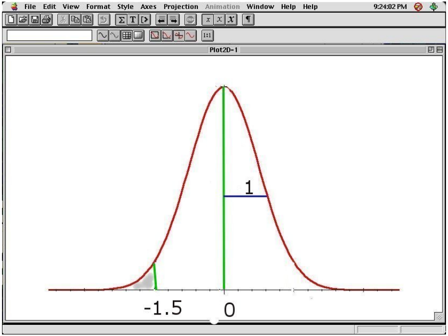


Table Z tells us $P(Z < -1.5) = .0668$.
6.7% of cars will have mileage below 15 mpg.

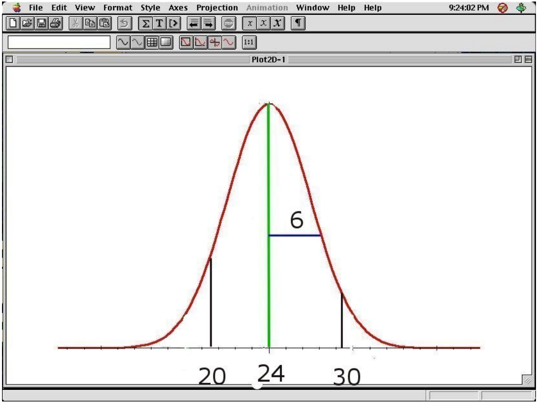
Mileage Example

% between 20 and 30?

Mileage Example

% between 20 and 30?

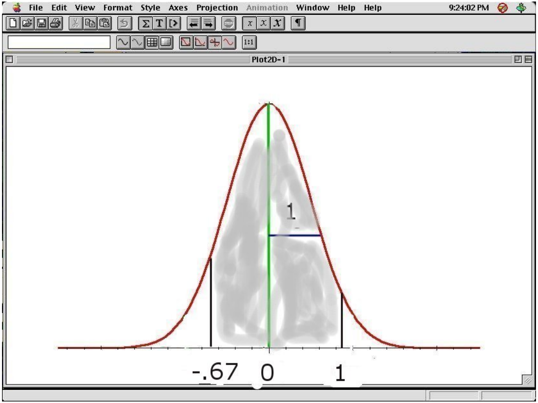
N(24,6)



Mileage Example

% between 20 and 30?

N(0,1)



Mileage Example

% between 20 and 30?

N(0,1)

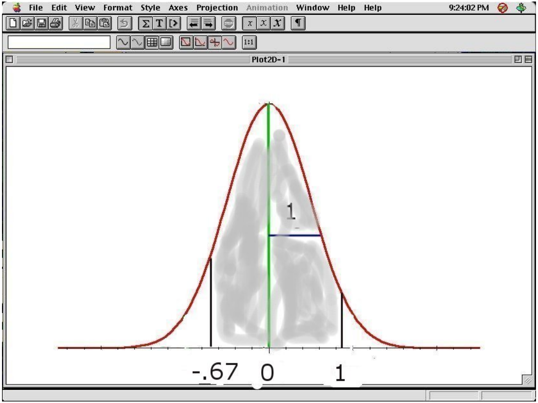


Table Z tells us $P(-.67 < Z < 1) = .8413 - .2514 = .5899$.
58.7% of cars will have mileage between 20 and 30 mpg.

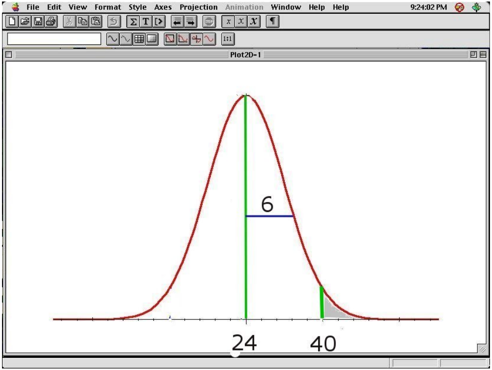
Mileage Example

% of cars above 40?

Mileage Example

% of cars above 40?

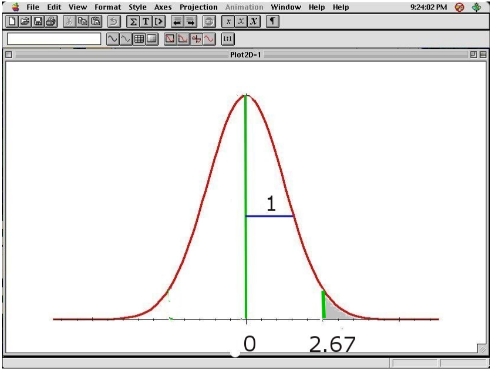
N(24,6)



Mileage Example

% of cars above 40?

N(0,1)



Mileage Example

% of cars above 40?

N(0,1)

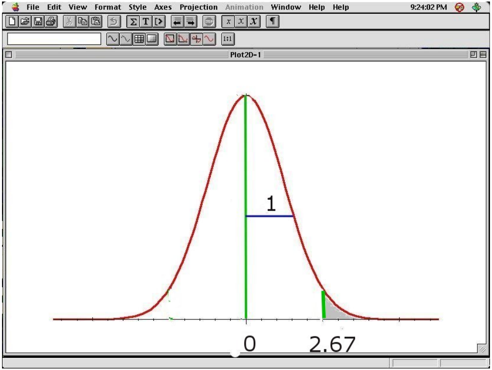


Table Z tells us $P(Z < 2.67) = .9962$.
Therefore $P(Z > -2.67) = 1 - .9962 = .0038$.
0.38% of cars will have mileage above 40 mpg.

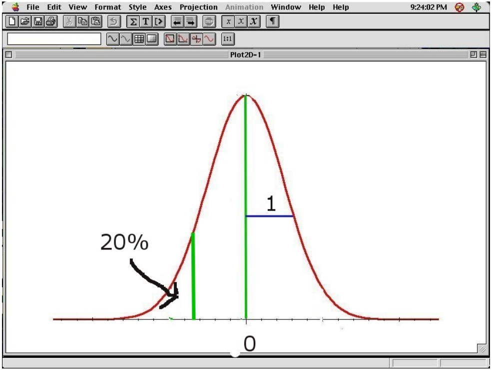
Mileage Example

Worst 20% of cars?

Mileage Example

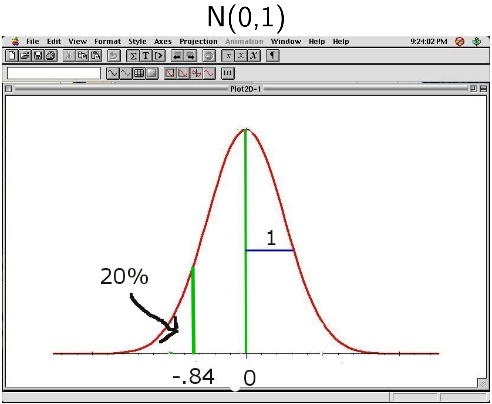
Worst 20% of cars?

N(0,1)



Mileage Example

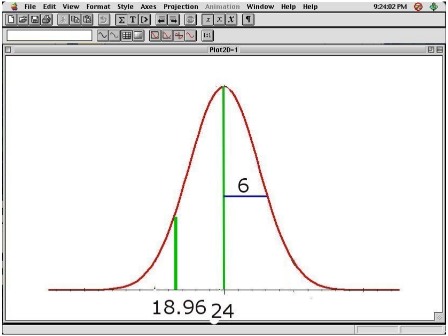
Worst 20% of cars?



Mileage Example

Worst 20% of cars?

$N(24, 6)$:



Having a Z score of $-.84$ means
being .84 std. dev. to the left of 24.
So the value is $24 + (-.84)(6) = 24 - 5.04 = 18.96$.

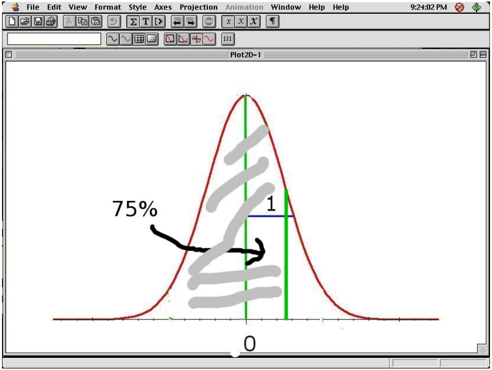
Mileage Example

Third Quartile?

Third Quartile?

Again start with the $N(0,1)$ picture.

N(0,1)



Mileage Example

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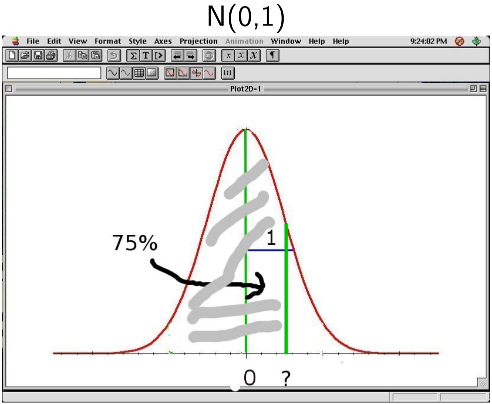


Mileage Example

Third Quartile?

$P(Z < ?) = .75$

Mileage Example

Third Quartile?

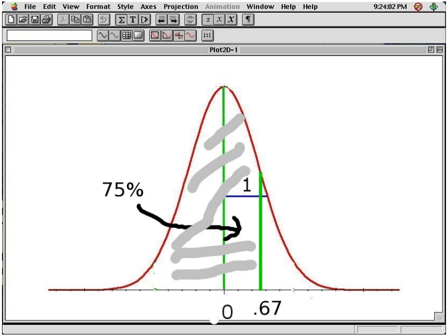
Table Z suggests .67. to 2 decimal places.

$P(Z < .67) = .7486$.

Mileage Example

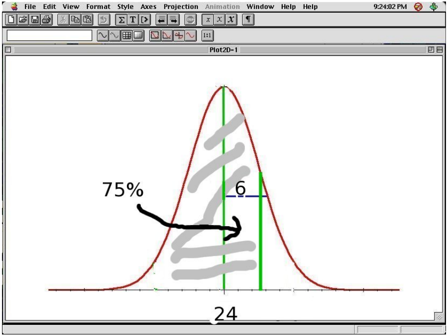
Third Quartile?

N(0,1)



Third Quartile?

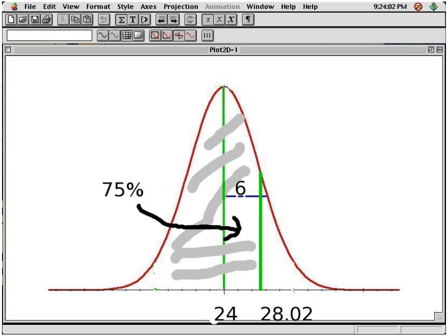
Translating to $N(24, 6)$:



Mileage Example

Third Quartile?

$N(24, 6)$:



Mileage Example

Third Quartile?

Having a Z score of .67 means
being .67 std. dev. to the right of 24.
So the value is $24 + (.67)(6) = 28.02$.

Mileage Example

Third Quartile?

The 3rd quartile of cars are those with a mileage of 28.02 mpg.

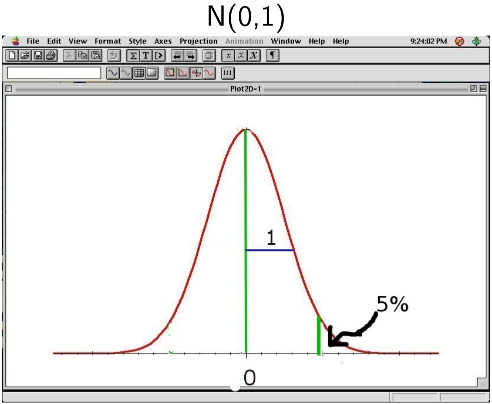
Mileage Example

Gas mileage of the 5% most efficient?

Mileage Example

Gas mileage of the 5% most efficient?

Again start with the $N(0,1)$ picture.

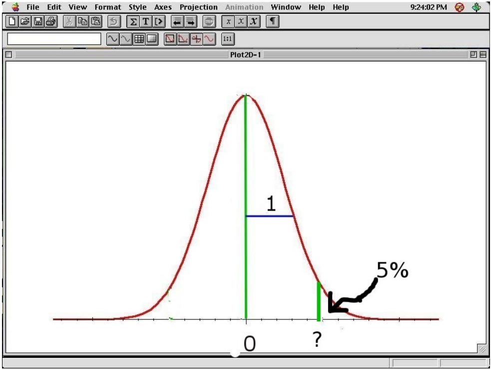


Mileage Example

Gas mileage of the 5% most efficient?

Again start with the $N(0,1)$ picture.

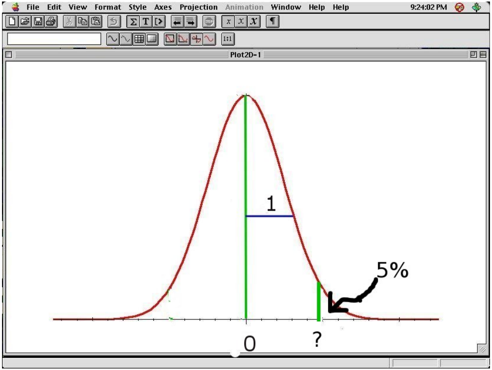
N(0,1)



Mileage Example

Gas mileage of the 5% most efficient?

N(0,1)



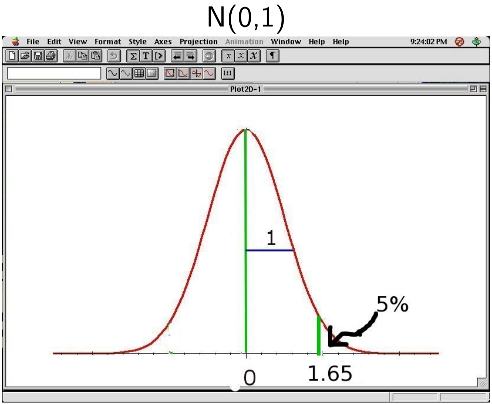
$P(Z > ?) = .05$. That means $P(Z < ?) = .95$.

Table Z suggests 1.65. to 2 decimal places.

$P(Z < 1.65) = .9505$. (1.64 is an equally good choice.)

Mileage Example

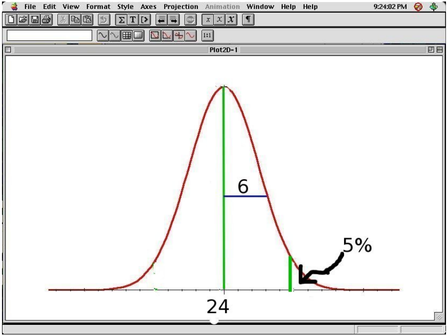
Gas mileage of the 5% most efficient?



Mileage Example

Gas mileage of the 5% most efficient?

Translating to $N(24, 6)$:

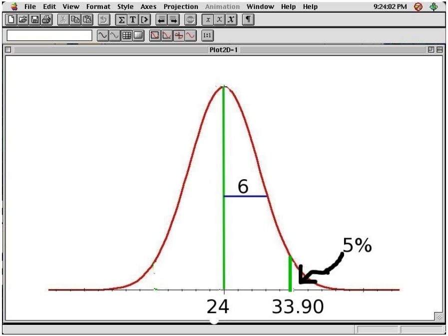


Having a Z score of 1.65 means
being 1.65 std. dev. to the right of 24.
So the value is $24 + (1.65)(6) = 33.90$.

Mileage Example

Gas mileage of the 5% most efficient?

$N(24, 6)$:



Mileage Example

Gas mileage of the 5% most efficient?

The top 5% of cars are those with a mileage of at least 33.90 mpg.

Mileage Example

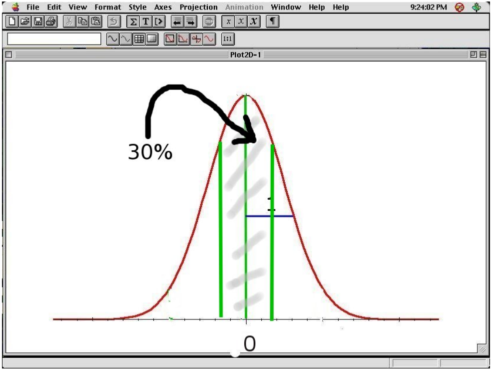
Gas mileage of the middle 30%?

Mileage Example

Gas mileage of the middle 30%?

Again start with the $N(0,1)$ picture.

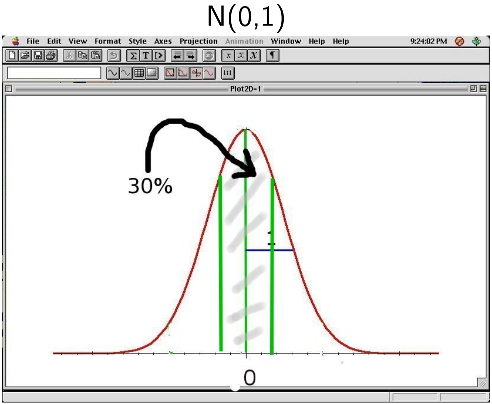
N(0,1)



Mileage Example

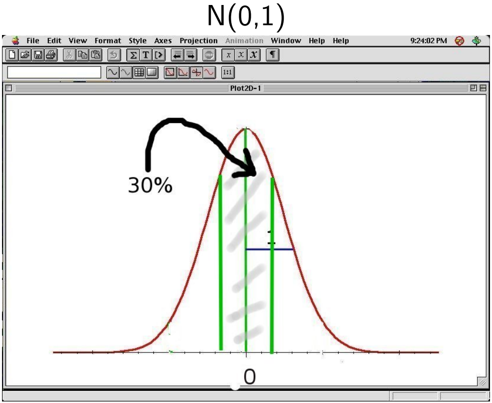
Gas mileage of the middle 30%?

Again start with the $N(0,1)$ picture.



Mileage Example

Gas mileage of the middle 30%?

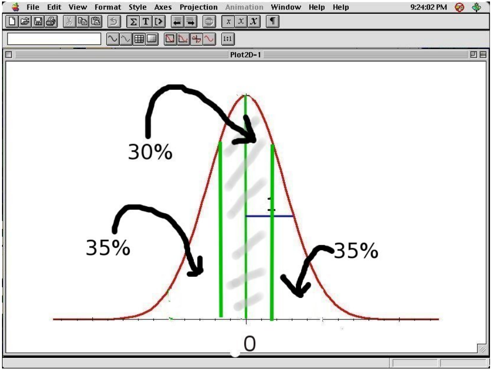


The only trick here is what $P(Z < ?)$ should be.

Mileage Example

Gas mileage of the middle 30%?

N(0,1)



$P(Z <?) = .35 + .30 = .65$. NOT $P(Z <?) = .30$

Mileage Example

Gas mileage of the middle 30%?

N(0,1)

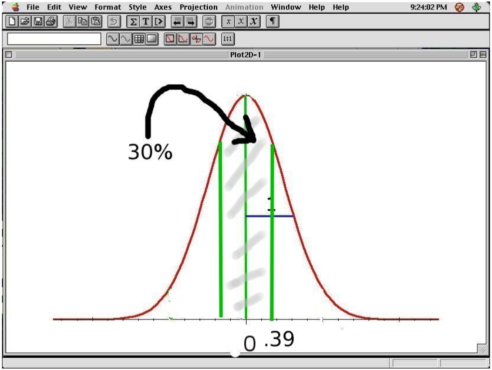


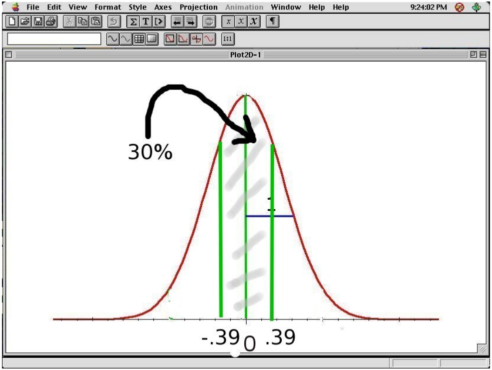
Table Z suggests .39. to 2 decimal places.
$P(Z < .39) = .6517.$

Mileage Example

Gas mileage of the middle 30%?

By symmetry:

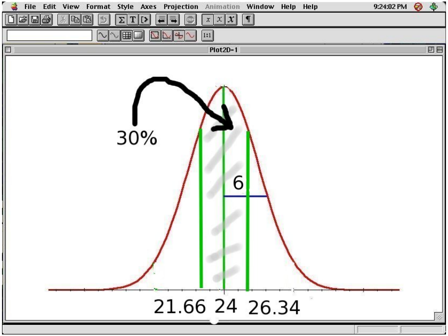
N(0,1)



Mileage Example

Gas mileage of the middle 30%?

Translating to $N(24, 6)$:

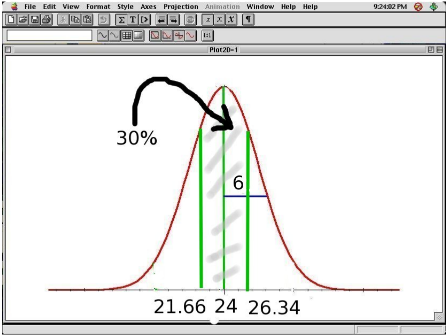


Having a Z score of .39 means
being .39 std. dev. to the right of 24.
So the uppervalue is $24 + (.39)(6) = 26.34$.

Mileage Example

Gas mileage of the middle 30%?

Translating to $N(24, 6)$:

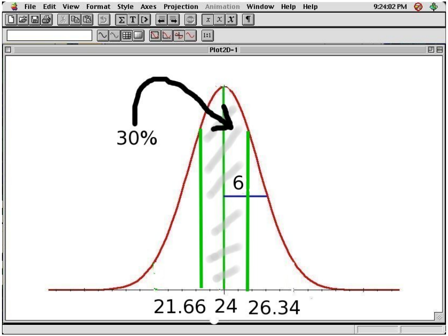


Having a Z score of $-.39$ means
being .39 std. dev. to the left of 24.
So the lower value is $24 - (.39)(6) = 21.66$.

Mileage Example

Gas mileage of the middle 30%?

Translating to $N(24, 6)$:



The middle 30% of cars are those with a mileage between 21.66 and 26.34 mpg.

Suppose 70% approve the President . . .

You poll 100 people.
What is the probability that between 60 and 65 report approval?
(including 60 and 65.)

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Solution: Let $X = \mathit{Binomial}(100, .7)$

X has mean

$$\mu = (100)(.7) = 70$$

and

$$\sigma = \sqrt{(100)(.7)(.3)} = \sqrt{21} = 4.58.$$

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X will then be approximated by a *normal* distribution Y with the same mean and standard deviation.

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$Y = N(70, 4.58)$.

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$$\mu = (100)(.7) = 70$$

and

$$\sigma = \sqrt{(100)(.7)(.3)} = \sqrt{21} = 4.58.$$

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We can approximate $P(60 \leq X \leq 65)$ by $P(60 \leq Y \leq 65)$.

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$Y = N(70, 4.58)$.
We can approximate $P(60 \leq X \leq 65)$ by $P(60 \leq Y \leq 65)$.
The Z score of 60 is $\frac{-10}{4.58} = -2.18$.
The Z score of 65 is $\frac{-5}{4.58} = -1.09$.
So $P(60 \leq Y \leq 65) = P(Z < -1.09) - P(Z < -2.18) = .1379 - .0146 = .1233$.

Suppose 70% approve the President . . .

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So $P(60 \leq Y \leq 65) = P(Z < -1.09) - P(Z < -2.18) = .1379 - .0146 = .1233$.
Actually $P(60 \leq X \leq 65) = .15036$.
So .1233 is not that good an approximation.

Suppose 70% approve the President . . .

You poll 100 people.

What is the probability that between 60 and 65 report approval?

(including 60 and 65.)

We can approximate $P(60 \leq X \leq 65)$ by $P(60 \leq Y \leq 65)$.

The Z score of 60 is $\frac{-10}{4.58} = -2.18$.

The Z score of 65 is $\frac{-5}{4.58} = -1.09$.

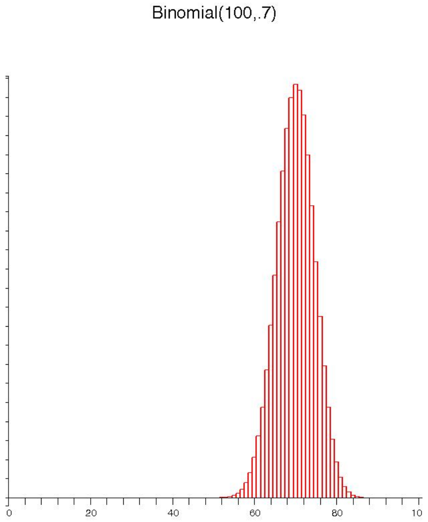
So $P(60 \leq Y \leq 65) = P(Z < -1.09) - P(Z < -2.18) = .1379 - .0146 = .1233$.

Actually $P(60 \leq X \leq 65) = .15036$.

So .1233 is not that good an approximation.

$P(60 < X < 65) = .09509$. would also be approximated by .1233!

Normal Approximation pictures



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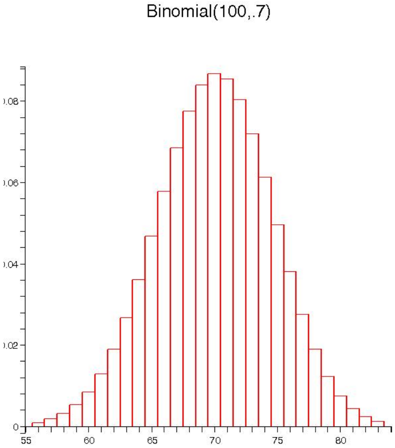

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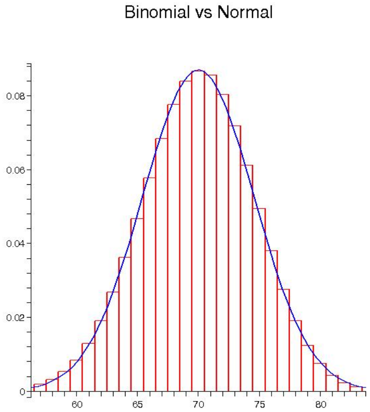

Normal Approximation pictures