Math 1710 Class 4

Random Variables

Dr. Back

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Two Fair Dice

Roll two fair dice (one die white, one die black)
Keep track of the sum of the numbers on the dice:

Two Fair Dice

T	**probability**
2	$\frac{1}{36}$
3	$\frac{2}{36}$
4	$\frac{3}{36}$
5	$\frac{4}{36}$
6	$\frac{5}{36}$
7	$\frac{6}{36}$
8	$\frac{5}{36}$
9	$\frac{4}{36}$
10	$\frac{3}{36}$
11	$\frac{2}{36}$
12	$\frac{1}{36}$

Two Fair Dice

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And each of these has probability $\frac{1}{36}$.

One writes this more formally as

$$P\{T = 4\} = \frac{3}{36}.$$

Two Fair Dice

T is a random variable.
The above table is the

probability distribution

of T, or as your textbook prefers to say, a

probability model

for T.

Definition

A variable describing some number associated with a random phenomenon is termed a random variable.

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The probability model of an RV X might be given by a table:

X	**probability**
x_1	p_1
x_2	p_2
$\dots$	$\dots$
x_n	p_n

A Random Variable for 1 Die

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Important idea:

Relatively complicated RV's

like T describing what happens with 2 dice

built up out of simpler RV's

like U describing one die.

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If U_1 describes the white die and U_2 the black one, then

$$T = U_1 + U_2.$$

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like T describing what happens with 2 dice
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This is a good example of one RV being the sum of two *independent* RV's.

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i.e. For any numbers x and y, the events

$$X \text{ takes the value x}$$

and

$$Y \text{ takes the value y}$$

are independent events.

Independent Copies of the Same RV

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Your textbook: Writing $U_1 + U_2$ means this.

Useful!

We'll see shortly that $T = U_1 + U_2$ in fact gets us a lot.

Mean of an RV

The RV X with probability distribution

X	**probability**
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x_n	p_n

has mean defined by

$$\mu_X = \Sigma p_i x_i.$$

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of X. Sometimes we just write μ if which RV is clear.

Mean of U?

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U	probability
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6	$\frac{1}{6}$

$$\begin{aligned}\mu =& \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 5 + \frac{1}{6}\cdot 6 \\ =& \frac{1}{6}(1+2+3+4+5+6) = 3.5\end{aligned}$$

Meaning of EX?

Experiment of rolling a die 1000 times:

Outcome	**Frequency**
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2	179
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Law of Large Numbers implies this will generally happen.

Standard Deviation of an RV

The RV X with probability distribution

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has variance defined by

$$\mathrm{Var}(X) = \Sigma p_i (x_i - \mu)^2$$

and standard deviation

$$\sigma_X = \sqrt{\mathrm{Var}(X)}.$$

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Sometimes we just write σ if which RV is clear.

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\begin{aligned}
Var(U) =& \frac{1}{6}(1-3.5)^2 + \frac{1}{6}(2-3.5)^2 + \frac{1}{6}(3-3.5)^2 \\
&+ \frac{1}{6}(4-3.5)^2 + \frac{1}{6}(5-3.5)^2 + \frac{1}{6}(6-3.5)^2 \\
=& \frac{1}{6}((1-3.5)^2 + (2-3.5)^2 + (3-3.5)^2 \\
&+ (4-3.5)^2 + (5-3.5)^2 + (6-3.5)^2) = 2.917.
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and $\sigma = \sqrt{2.917} = 1.708$.

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There's a *sample standard deviation* s which is a natural measure of the spread in data.

Meaning of σ_X?

For data $x_1, x_2, \ldots, x_n$,

$$\text{sample variance } s^2 = \frac{1}{(n-1)}\left(\Sigma(x_i - \bar{x})^2\right)$$

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Close to *random variable standard deviation* $\sigma = 1.708$.

Advanced Point of View

RV's are simply number valued functions on a sample space.

Advanced Point of View

For 2 dice:

$$S = \begin{Bmatrix} W1B1 & W1B2 & W1B3 & W1B4 & W1B5 & W1B6 \\ W2B1 & W2B2 & W2B3 & W2B4 & W2B5 & W2B6 \\ W31B & W3B2 & W3B3 & W3B4 & W3B5 & W3B6 \\ W4B1 & W4B2 & W4B3 & W4B4 & W4B5 & W4B6 \\ W5B1 & W5B2 & W5B3 & W5B4 & W5B5 & W5B6 \\ W6B1 & W6B2 & W6B3 & W6B4 & W6B5 & W6B6 \end{Bmatrix}$$

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e.g. for the outcome W3B2, $T = 5, U_1 = 3$, and $U_2 = 2$.

$T = U_1 + U_2$ just as in addition of functions.

Another sample space for U

For just one die, the simpler sample space

$$S = \{1, 2, 3, 4, 5, 6\}$$

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Another sample space for U

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would do.
Computing the mean or standard deviation of U would be the same on this simpler sample space; we'll just take that for granted.

Three Important Operations

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Addition of RV's as in $T = U_1 + U_2$.
Two simpler but also important operations:
Adding a constant: $Y = X + c$.
Multiplying by a constant: $Y = cX$.
And we can keep track of what happens to μ and σ under these.

Thinking about cX and $X + c$

A simple die game:
1, 2, or 3 gives \$0; 4 or 5 gives \$5; 6 gives \$50.

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$\mu = \frac{5}{3} + \frac{25}{3} = 10$; $\sigma = 18.03$.

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Doubling the stakes: $Y = 2X$

$2X$	**probability**
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100	$\frac{1}{6}$

$\mu = 20$; $\sigma = 36.06$.

Thinking about cX and $X + c$

X	**probability**
0	$\frac{1}{2}$
5	$\frac{1}{3}$
50	$\frac{1}{6}$

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Including a Payment of 12$ to Play the Game: $W = X - 12$

$X - 12$	**probability**
-12	$\frac{1}{2}$
-7	$\frac{1}{3}$
38	$\frac{1}{6}$

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$\mu = -2$; $\sigma = 18.03$.