Math 1710 Class 5

Random Variable Operations, Reverse Conditioning

Dr. Back

Sep. 7, 2009

Announcements

Please write prominently on your HW which Th Recitation you are *attending*.

Please bring two pennies (or other coins) to class on Wed.

Mean of an RV

The RV X with probability distribution

X	**probability**
x_1	p_1
x_2	p_2
$\dots$	$\dots$
x_n	p_n

Mean:

$$\mu = EX = \Sigma p_i x_i .$$

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Mean:

$$\mu = EX = \Sigma p_i x_i.$$

Variance:

$$\mathrm{Var}(X) = \Sigma p_i (x_i - \mu)^2$$

Standard Deviation:

$$\sigma_X = \sqrt{\mathrm{Var}(X)}.$$

Two Fair Dice

T	**probability**
2	$\frac{1}{36}$
3	$\frac{2}{36}$
4	$\frac{3}{36}$
5	$\frac{4}{36}$
6	$\frac{5}{36}$
7	$\frac{6}{36}$
8	$\frac{5}{36}$
9	$\frac{4}{36}$
10	$\frac{3}{36}$
11	$\frac{2}{36}$
12	$\frac{1}{36}$

Two Fair Dice

$$T = U_1 + U_2$$

U	**probability**
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

Two Fair Dice

U	**probability**
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

$\mu = \text{EU} = 3.5$

$$\begin{aligned} Var(U) =& \frac{1}{6}(1-3.5)^2 + \frac{1}{6}(2-3.5)^2 + \frac{1}{6}(3-3.5)^2 \\ &+ \frac{1}{6}(4-3.5)^2 + \frac{1}{6}(5-3.5)^2 + \frac{1}{6}(6-3.5)^2 \\ =& 2.917. \end{aligned}$$

and $\sigma = \sqrt{2.917} = 1.708.$

Three Important Operations

- Addition of RV's as in $T = U_1 + U_2$.
- Adding a constant: $Y = X + c$.
- Multiplying by a constant: $Y = cX$.

Thinking about cX and $X + c$

A simple die game:
1, 2, or 3 gives \$0; 4 or 5 gives \$5; 6 gives \$50.

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$\mu = \frac{5}{3} + \frac{25}{3} = 10$; $\sigma = 18.03$.

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0	$\frac{1}{2}$
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Doubling the stakes: $Y = 2X$

Y	**probability**
0	$\frac{1}{2}$
10	$\frac{1}{3}$
100	$\frac{1}{6}$

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$\mu = 20$; $\sigma = 36.06$.

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Including a Payment of 12$ to Play the Game: $W = X - 12$

W	**probability**
-12	$\frac{1}{2}$
-7	$\frac{1}{3}$
38	$\frac{1}{6}$

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$\mu = -2$; $\sigma = 18.03$.

Means are very intuitive!

$$\mu = \Sigma p_i x_i$$

$$\mu_{X+Y} = \mu_X + \mu_Y$$

$$\mu_{X+c} = \mu_X + c$$

$$\mu_{cX} = c\mu_X$$

(Handy: $\Sigma p_i = 1$.)

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Std. Dev. for cX and $X + c$ are pretty natural.

$$\mathrm{Var}(X) = \Sigma p_i(x_i - \mu)^2$$

$$\mathbf{Var}(X + c) = \mathbf{Var}(X).$$

$$\sigma_{X+c} = \sigma_X$$

$$\mathrm{Var}(cX) = c^2\mathrm{Var}(X).$$

$\sigma_{cX} = |c|\sigma_X$. So $\boldsymbol{\sigma_X = \sigma_{-X}}$.

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Variance of a Sum of *Independent* RV's

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Very important in Statistics!

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But if X and Y are independent RV's:

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

Very important in Statistics!
Without independence, this often fails; e.g.

$$\begin{aligned} \mathrm{Var}(X+X) = \mathrm{Var}(2X) = 2^2\mathrm{Var}(X) \quad &= 4\mathrm{Var}(X) \\ \neq \mathrm{Var}(X) + \mathrm{Var}(X) \quad &= 2\mathrm{Var}(X) \end{aligned}$$

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$\sigma = \sqrt{pq}.$

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The X_i are independent copies of X with

$$\mu_X = p, \ \textbf{Var}(X) = pq.$$

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And $\mathbf{Var}(Y) = pq + pq + \dots + pq = npq$.
Making $\sigma_Y = \sqrt{npq}$.

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5300 is 6 std dev from the mean.

Joan concludes her coin is probably not fair.

Used Car Preparation

Suppose a 2-step process:

Mechanical Checkup: \$50 per hour, 90 min average, 15 min standard deviation.

Appearance Prep: \$6 per hour, 60 min average, 5 min standard deviation.

Find μ and σ for:

- Total Expense.
- Difference in Expense of Phases.

Solution:

Let M be an RV describing the time for the mechanical part.

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We're given (in hours):

	μ	σ
M	$\frac{3}{2}$	$\frac{1}{4}$
A	1	$\frac{1}{12}$

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We're given (in hours):

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$T = 50M + 6A.$

Solution Continued:

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$\mu_T = 50\mu_M + 6\mu_A.$

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$T = 50M + 6A.$

$\mu_T = 50\mu_M + 6\mu_A.$

$\mu_T = 50\mu_M + 6\mu_A = 50(\frac{3}{2}) + 6(1) = 75 + 6 = 81.$

Solution Continued:

We're given (in hours):

	μ	σ
M	$\frac{3}{2}$	$\frac{1}{4}$
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$T = 50M + 6A$.
Assuming independence of M and A,
$\mathrm{Var}(T) = 50^2\mathrm{Var}(M) + 6^2\mathrm{Var}(A)$.

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$\mathrm{Var}(T) = 50^2(\frac{1}{4})^2 + 6^2(\frac{1}{12})^2$.
$\mathrm{Var}(T) = (\frac{25}{2})^2 + (\frac{1}{2})^2 = \frac{626}{4} = 156.5$.

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We're given (in hours):

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$\sigma_T = \sqrt{156.5} = 12.51.$

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Is the independence of M and A natural to assume?

The Difference D in Expense of Phases:

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$\mu_D = 50\mu_M - 6\mu_A = 50(\frac{3}{2}) - 6(1) = 75 - 6 = 69.$

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$\sigma_D = \sqrt{156.5} = 12.51.$
Same std. dev. for D and T!

Two Independent RV's

X	**probability**
x_1	p_1
x_2	p_2
$\cdots$	$\cdots$
x_n	p_n

Y	**probability**
y_1	q_1
y_2	q_2
$\cdots$	$\cdots$
y_m	q_m

Two Independent RV's

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x_1	p_1	y_1	q_1
x_2	p_2	y_2	q_2
$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_n	p_n	y_m	q_m

Independence means $P\{X = x_1 \ \& \ Y = y_1\} = p_1 q_1$.

Two Independent RV's

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x_1	p_1	y_1	q_1
x_2	p_2	y_2	q_2
$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_n	p_n	y_m	q_m

Independence means $P\{X = x_1 \;\&\; Y = y_1\} = p_1 q_1$.
In this case $X + Y$ would be $x_1 + y_1$.

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X + Y	**probability**
$x_1 + y_1$	$p_1 q_1$
$x_1 + y_2$	$p_1 q_2$
$\cdots$	$\cdots$
$x_n + y_m$	$p_n q_m$

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$\mathbf{X + Y}$	**probability**
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$x_1 + y_2$	$p_1 q_2$
$\dots$	$\dots$
$x_n + y_m$	$p_n q_m$

(Why *roughly:* Some value c of $X + Y$ may show up several times in the above table and then more precisely $p_i q_j$ is $P\{X = x_i \;\&\; Y = y_j\}$. And $P(\{X + Y = c\})$ would be the sum of several entries. But computing μ,σ won't be affected by not combining such rows.)

Variance

Assuming we know E(X+Y)=EX+EY, then $Var(X+Y)$

$$
\begin{aligned}
&= \Sigma_{i,j} p_i q_j (x_i + y_j - \mu_X - \mu_Y)^2 \\
&= \Sigma_{i,j} p_i q_j (x_i - \mu_X)^2 + \Sigma_{i,j} p_i q_j (y_j - \mu_Y)^2 \\
&\quad + 2\Sigma_{i,j} p_i q_j (x_i - \mu_X)(y_j - \mu_Y) \\
&= (\Sigma_j q_j)\, Var(X) + (\Sigma_i p_i)\, Var(Y) \\
&\quad + 2\left(\Sigma_i p_i (x_i - \mu_X)\right)\left(\Sigma_j q_j (y_j - \mu_Y)\right) \\
&= 1 \cdot Var(X) + 1 \cdot Var(Y) + 2 \cdot 0 \cdot 0 \\
&= Var(X) + Var(Y).
\end{aligned}
$$

Variance

Assuming we know E(X+Y)=EX+EY, then $Var(X+Y)$

$$
\begin{aligned}
&= \Sigma_{i,j} p_i q_j (x_i + y_j - \mu_X - \mu_Y)^2 \\
&= \Sigma_{i,j} p_i q_j (x_i - \mu_X)^2 + \Sigma_{i,j} p_i q_j (y_j - \mu_Y)^2 \\
&\quad + 2\Sigma_{i,j} p_i q_j (x_i - \mu_X)(y_j - \mu_Y) \\
&= (\Sigma_j q_j)\, Var(X) + (\Sigma_i p_i)\, Var(Y) \\
&\quad + 2\,(\Sigma_i p_i (x_i - \mu_X))\,(\Sigma_j q_j (y_j - \mu_Y)) \\
&= 1 \cdot Var(X) + 1 \cdot Var(Y) + 2 \cdot 0 \cdot 0 \\
&= Var(X) + Var(Y).
\end{aligned}
$$

where we used $(a+b)^2 = a^2 + 2ab + b^2$,

Variance

Assuming we know E(X+Y)=EX+EY, then $Var(X+Y)$

$$
\begin{aligned}
&= \Sigma_{i,j} p_i q_j (x_i + y_j - \mu_X - \mu_Y)^2 \\
&= \Sigma_{i,j} p_i q_j (x_i - \mu_X)^2 + \Sigma_{i,j} p_i q_j (y_j - \mu_Y)^2 \\
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&= (\Sigma_j q_j)\, Var(X) + (\Sigma_i p_i)\, Var(Y) \\
&\quad + 2\left(\Sigma_i p_i (x_i - \mu_X)\right)\left(\Sigma_j q_j (y_j - \mu_Y)\right) \\
&= 1 \cdot Var(X) + 1 \cdot Var(Y) + 2 \cdot 0 \cdot 0 \\
&= Var(X) + Var(Y).
\end{aligned}
$$

$\Sigma p_i = 1,$

Variance

Assuming we know E(X+Y)=EX+EY, then $Var(X+Y)$

$$
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&= \Sigma_{i,j} p_i q_j (x_i - \mu_X)^2 + \Sigma_{i,j} p_i q_j (y_j - \mu_Y)^2 \\
&\quad + 2\Sigma_{i,j} p_i q_j (x_i - \mu_X)(y_j - \mu_Y) \\
&= (\Sigma_j q_j)\, Var(X) + (\Sigma_i p_i)\, Var(Y) \\
&\quad + 2\,(\Sigma_i p_i (x_i - \mu_X))\,(\Sigma_j q_j (y_j - \mu_Y)) \\
&= 1 \cdot Var(X) + 1 \cdot Var(Y) + 2 \cdot 0 \cdot 0 \\
&= Var(X) + Var(Y).
\end{aligned}
$$

and $\Sigma p_i (x_i - \mu_X) = 0.$

Proof of Expectation of Mean Formula

$E(X+Y)$

$$
\begin{aligned}
&= \Sigma_{i,j}(x_i+y_j)P(X=x_i \text{ and } Y=y_j) \\
&= \Sigma_{i,j}x_iP(X=x_i \mid Y=y_j)P(Y=y_j) \\
&\quad + \Sigma_{i,j}y_jP(Y=y_j \mid X=x_i)P(X=x_i) \\
&= \Sigma_i x_i \left(\Sigma_j P(X=x_i \mid Y=y_j)P(Y=y_j)\right) \\
&\quad + \Sigma_j y_j \left(\Sigma_i P(Y=y_j \mid X=x_i)P(X=x_i)\right) \\
&= \Sigma_i x_i P(X=x_i) \\
&\quad + \Sigma_j y_j P(Y=y_j) \\
&= E(X)+E(Y).
\end{aligned}
$$

Unsurprising, but not obvious looking.

Proof of Expectation of Mean Formula

$E(X+Y)$

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=&\Sigma_{i,j}(x_i+y_j)P(X=x_i \text{ and } Y=y_j)\\
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&+\Sigma_j y_jP(Y=y_j)\\
=&E(X)+E(Y).
\end{aligned}$$

Unsurprising, but not obvious looking.

Did not require X and Y to be independent!

7 Balls Randomly into 5 boxes

What is the probability that the first box remains empty?

What is the expected number of empty boxes?

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The 1st question is a great hint for easily doing the 2nd!

Triple blood test for Down Syndrome

Suppose

- 1 in 800 pregnant women affected.
- $\frac{8}{9}$ of affected women identified by the test
- $\frac{1}{4}$ of unaffected women show up as false positives.

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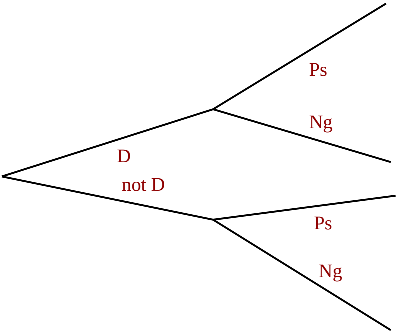
P(Down syndrome if test is positive)?

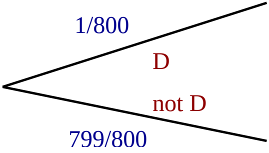
Notation

D: foetus affected by Downs.
Ps: positive test result reports Downs.
Ng: not Ps. (Negative)

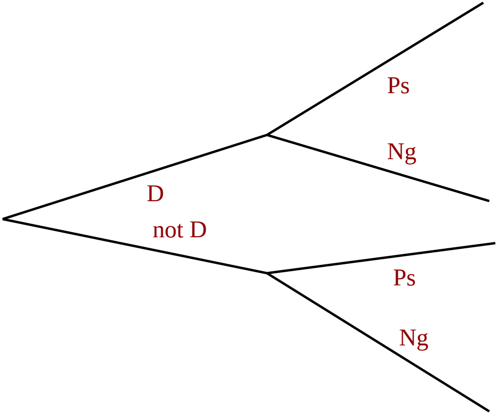


With Probabilities

$$P(D) = \frac{1}{800}$$

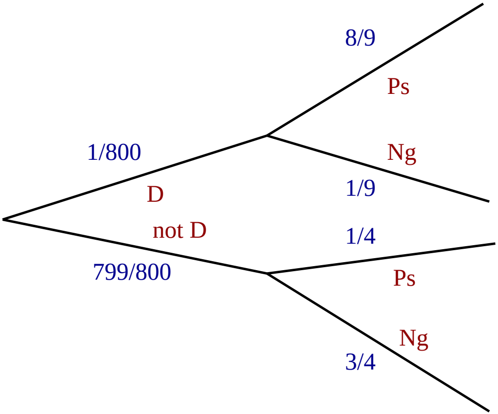


With the Conditional Probabilities

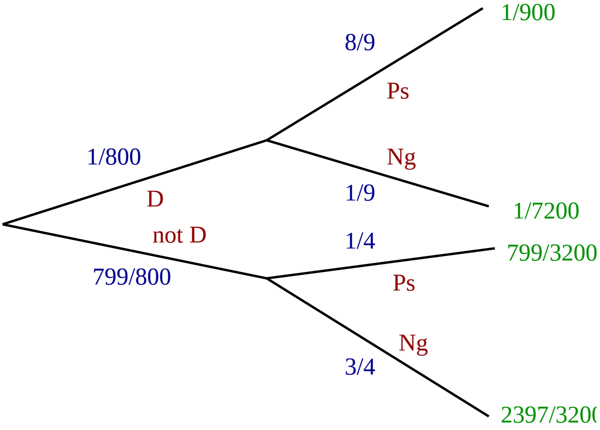


$$P(Ps|D) = \frac{8}{9}, \qquad P(Ps|\text{not D}) = \frac{1}{4}$$

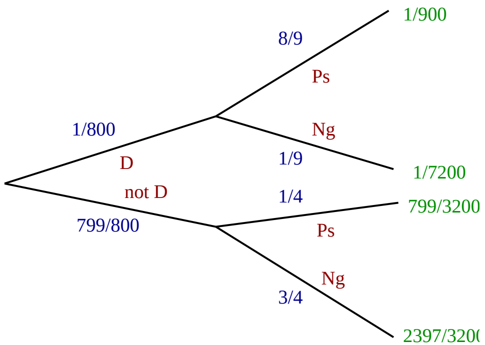
With the Conditional Probabilities



With the Conditional Probabilities

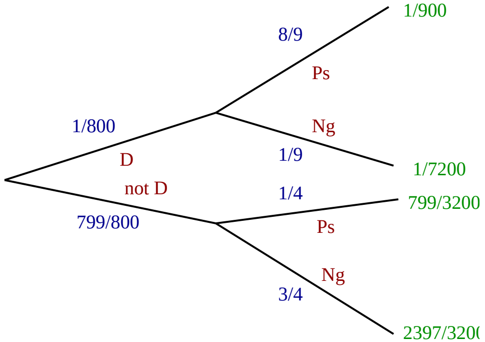


With the Conditional Probabilities



$$P(D|Ps) = \frac{P(D \text{ and } Ps)}{P(Ps} = \frac{\frac{1}{900}}{\frac{1}{900} + \frac{799}{3200}} \sim .0044.$$

With the Conditional Probabilities



$$P(D|Ps) = \frac{P(D\ and\ Ps)}{P(Ps)} = \frac{\frac{1}{900}}{\frac{1}{900}+\frac{799}{3200}} \sim .0044.$$

Not very helpful!.