From Math 2220 Class 40

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Dec. 3, 2014

Interpretations of Div and Curl

Recall that a vector field $\vec{F} = (P, Q, R)$ has associated with it a flow line (or integral curve) through each point.
For example, the linear vector field

$$\vec{F}(x, y, z) = (ax, by, cz)$$

has integral curve

$$c(t) = (x_0 e^{at}, y_0 e^{bt}, z_0 e^{ct})$$

satisfying $c(0) = (x_0, .y_0, z_0.)$
You can think of this path as describing where a particle in a fluid will be at time t given its position at time 0. (Assuming this vector field describes the flow . . .)

Interpretations of Div and Curl

What happens to a little rectangular prism of fluid under this flow?

Let its sides be $\Delta x, \Delta y, \Delta z$.

For $a, b, c > 0$, the sides of the prism occupied by the particles will expand as t increases. In fact its volume will be

$$V(t) = e^{(a+b+c)t}\ \Delta x\ \Delta y\ \Delta z.$$

Interpretations of Div and Curl

So we have $\vec{F}(x, y, z) = (ax, by, cz)$. And $V(t) = e^{(a+b+c)t}\ \Delta x\ \Delta y\ \Delta z$.

That means the logarithmic derivative

$$\left(\frac{1}{V(t)}\right) \frac{dV}{dt}\bigg|_{t=0} = div(\vec{F}).$$

Interpretations of Div and Curl

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That actually works in complete generality for any vector field! (Using diagonalization ideas, it's not hard to essentially reduce the case of any linear vector field to the above diagonal case. And one can show that both the divergence and the "first order" change of $V(t)$ are the same at one point for an arbitrary vector field and for its linearization.)

Interpretations of Div and Curl

One consequence of this is that vector fields with divergence 0 lead to flows that preserve volume.
This happens for example in an "incompressible fluid."

Interpretations of Div and Curl

Now let's look at a first interpretation of the curl.
A simple approximate model of water circling as it waits to flow out of the bottom of a bathtub is "constant angular velocity" (about the z-axis) with $\vec{v}(x, y, z) = (-\omega y, \omega x, 0)$ and flowlines

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}.$$

Interpretations of Div and Curl

The velocity above at the point $\vec{r} = (x, y, z)$ can be written as

$$\vec{v} = (\omega\hat{k}) \times \vec{r}$$

This same kind of rotational motion about an arbitrary axis $\vec{\omega}$ leads to

$$\vec{v} = \vec{\omega} \times \vec{r}$$

and $curl(\vec{v}) = 2\vec{\omega}$.
(Think of the magnitude of $\vec{\omega}$) as giving the angular velocity of the rotation; the direction gives the axis)
It's easy to verify that $curl(\vec{v}) = 2\vec{\omega}$ in this case.
And while the match of curl to the angular velocity of rotation is not perfect, the curl of a general velocity vector field does capture a large part of the rotational aspects of the flow.

Interpretations of Div and Curl

Stokes and Gauss give some immediate interpretations of these operators!

For example Gauss tells us

$$div(\vec{F}) = \lim_{\Delta V \to 0} \frac{1}{\text{Volume}(\Delta V)} \iint_{\partial \Delta V} \vec{F} \cdot \hat{n} \; dS.$$

In words, the flux of a vector field over the boundary of a small volume is approximately the divergence of the vector field times the volume.

Interpretations of Div and Curl

And Stokes says that for any unit vector $\hat{e}$ (e.g. $\hat{e} = \hat{k}$),

$$curl(\vec{v}) \cdot \hat{e} = \lim_{\Delta S \to 0} \frac{1}{\text{Area}(\Delta S)} \int_{\partial \Delta S} \vec{v} \cdot d\vec{s}.$$

In words, the circulation of a vector field over the boundary of a small area enclosed by a loop $\mathcal{C}$ in a plane perpendicular to $\hat{e}$ is approximately the curl of the vector field times the area.

Interpretations of Div and Curl

One way to read this is that for a small disk of radius r perpendicular to $\hat{e}$, the average tangential velocity of the fluid along the boundary is approximately half the $\hat{e}$ component of the curl times the distance from the center.

(Which actually means the angular velocity around the $\hat{e}$ axis is approximately half the $\hat{e}$ component of the curl. This is also sometimes expressed in terms of how fast a "paddle wheel" in a fluid will rotate.)

Interpretations of Div and Curl

"Lines of force" are flowlines of vector fields such as the electric (force) field $\vec{E}$ felt by a unit charge.
It is popular to draw them so that number crossing a piece of surface (with its chosen orientation) is proportional to the surface integral of the vector field over that piece of surface.
Physicists also like to say that flowlines can only end at points where $div(\vec{F}) \neq 0$.
Sometimes they are thinking about the Maxwell's equation relating the total flux of the electric field over the boundary of a region to the charge inside; one can also construct Gauss theorem arguments suggesting this without using the laws of electromagnetism.

$d\theta$

$$\int_C -y\ dx + x\ dy$$

for C the unit circle traversed counterclockwise.

$d\theta$

$$\int_C x \; dx + y \; dy$$

for C the unit circle traversed counterclockwise.

$d\theta$

$$\int_C \frac{-y\ dx + x\ dy}{x^2+y^2}$$

is still nonzero for C a circle of radius R centered at the origin traversed counterclockwise.

This is remarkable since

$$\nabla \tan^{-1}\frac{y}{x} = \frac{1}{x^2+y^2}\left(-y, x\right).$$

$d\theta$

Why does this not contradict $\int_C \nabla f \cdot d\vec{s} = 0$ for a closed curve (i.e. start=end) C?

Conservative Fields More Carefully

In order that a vector field $(P(x,y), Q(x,y))$ be $\nabla f = (f_x, f_y)$ for a C^2 function f, the equality of mixed partials shows that we have a necessary condition

$$P_y = Q_x.$$

(This can also be phrased as the scalar curl $Q_x - P_y = 0$.)

Conservative Fields More Carefully

The "$d\theta$" example $\int_{\mathcal{C}} \frac{-y\ dx + x\ dy}{x^2 + y^2} \neq 0$ for a circle enclosing the origin shows that this necessary condition is not always sufficient to guarantee the existence of such an f.
Here the vector field has domain $R^2 - \{(0,0)\}$, a region with a (tiny) hole in it.

Conservative Fields More Carefully

Similarly in R^3, the identity $curl(\nabla f) = 0$ for a C^2 vector field makes $curl(\vec{F}) = 0$ a necessary condition for being able to find a C^2 f with

$$\nabla f = \vec{F}.$$

Conservative Fields More Carefully

The vector field

$$\vec{F}(x,y,z) = \left(\frac{1}{x^2+y^2}\right)(-y,x,0)$$

with domain $R^3 - \{(0,0,z)\}$ (a region with a (thin) hole in it) shows that again $curl(\vec{F}) = 0$ is not sufficient for the existence for such an f.

Conservative Fields More Carefully

There is a *topological condition* on the domain $\mathcal{D}$ that does guarantee these necessary conditions are sufficient; namely that the domain of the vector field be simply-connected.

Intuitively, $\mathcal{D}$ being simply connected means any closed curve $\mathcal{C}$ can be continuously shrunk down to a point (entirely within $\mathcal{D}$.) (The region $\mathcal{D}$ is also assumed as part of simple connectivity to have just one piece; i.e. to be connected.)

Regions like all of R^n, a ball, a rectangle, or any "convex" set are simply connected.

But a ring, $R^2 - \{(0,0)\}$, and $R^3 - \{(0,0,z)\}$ are not.

Conservative Fields More Carefully

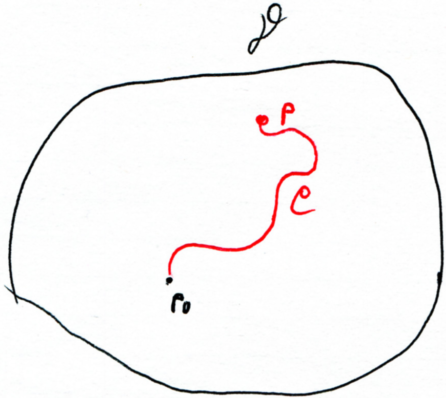
For a simply connected region, $\mathcal{D}$, the necessary conditions *are* sufficient for the existence of such an f with $\nabla f = \vec{F}$.
Though doing this rigorously requires more topology than we have, the basic idea is to use Stokes theorem.

Conservative Fields More Carefully

The idea is to pick a point $p_0 \in \mathcal{D}$ and define $f(p)$ as follows: For any $p \in \mathcal{D}$, pick a curve $\mathcal{C}$ from p_0 to p. Define

$$f(p) = \int_{\mathcal{C}} \vec{F} \cdot d\vec{s}.$$

Conservative Fields More Carefully

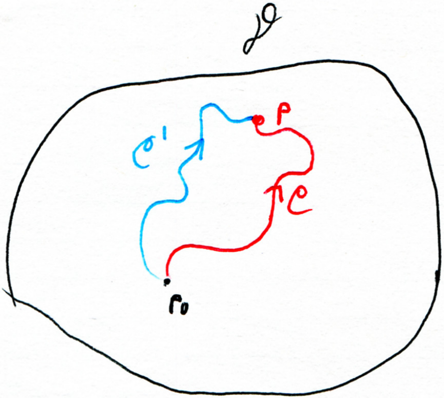


Conservative Fields More Carefully

This apparently strange definition works better than you would at first think because if we choose a different path $\mathcal{C}'$ from p_0 to p, for a simply connected region, we have the "path independence" property

$$\int_{\mathcal{C}'} \vec{F} \cdot d\vec{s} = \int_{\mathcal{C}} \vec{F} \cdot d\vec{s}.$$

Conservative Fields More Carefully

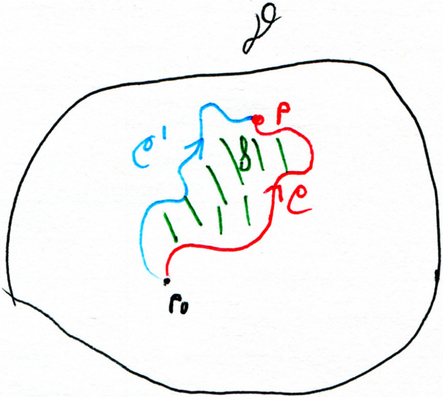


Conservative Fields More Carefully

The idea is, for a simply connected region, we can, in a sense good enough for Stokes, find a surface $\mathcal{S}$ filling in the inside of the closed loop $\mathcal{C}$ followed by $\mathcal{C}'$ in the reverse direction. Applying Stokes to this surface in the presence of $curl(\vec{F}) = 0$ gives

$$0 = \iint_{\mathcal{S}} curl(\vec{F}) \cdot \hat{n}\ dS = \int_{\mathcal{C}} \vec{F} \cdot d\vec{s} - \int_{\mathcal{C}'} \vec{F} \cdot d\vec{s}.$$

Conservative Fields More Carefully



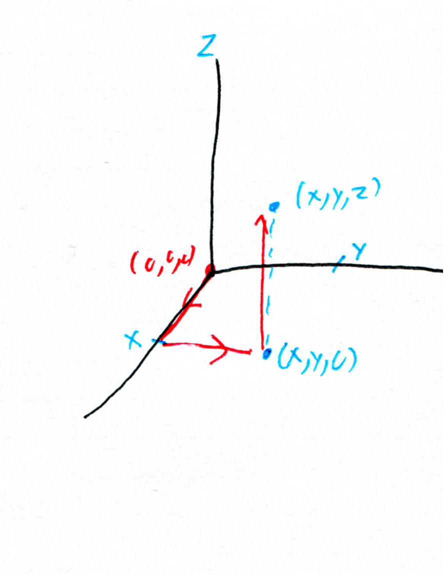
Conservative Fields More Carefully

In the presence of path independence, we can argue (by using well chosen paths) why $\nabla f = \vec{F}$ with this definition of f :

Conservative Fields More Carefully

The computational scheme we looked at before Thanksgiving for finding potentials is an instance of this procedure, even though it didn't look that way.
For example, with $p_0 = (0, 0, 0)$, picking the path $\mathcal{C}$ to consist of 3 segments parallel to the coordinate axis,makes the line integral computation a sequence of 3 one-variable anti-differentiations.

Conservative Fields More Carefully



Conservative Fields More Carefully

There is a similar story with the converse to $div(curl(\vec{A})) = 0$ for C^2 vector fields.
A $\vec{A}$ satisfying $curl(\vec{A}) = \vec{F}$ is called a *vector potential* for $\vec{F}$.
They are widely used in studying magnetic fields $\vec{B}$, which by one of Maxwell's equations satisfy $div(\vec{B}) = 0$.

Conservative Fields More Carefully

The point charge field

$$\vec{F}(x,y,z) = \frac{\vec{r}}{\|\vec{r}\|^3}$$

is an example of a vector field on R^3 minus the origin whose divergence vanishes, yet (because of Gauss' theorem applied to a ball around the origin) cannot be $curl(\vec{A})$.

Conservative Fields More Carefully

Once again some topological condition is needed to guarantee that $div(\vec{F}) = 0$ is sufficient for the existence of a vector potential. Simple connectivity won't do; what is actually relevant is something call the second cohomology group (with real coefficients.) The group being zero is equivalent to all divergence free vector fields having vector potentials.

Integral Theorem Problems

Problem: Let

$$\vec{F}(x, y, z) = (y^2 + z^2, x^2 + z^2, x^2).$$

Find

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{s}$$

where $\mathcal{C}$ is the boundary of the plane $x + 2y + 2z = 2$ intersected with the first octant, oriented counterclockwise from above.

Integral Theorem Problems

Problem: Find the flux of the vector field

$$\vec{F}(x, y, z) = (xy, yz, xz)$$

through the boundary of the unit cube $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$ where the boundary of the cube has its usual outward normal.

Integral Theorem Problems

Problem: Find

$$\iint_{\mathcal{S}} \vec{F} \cdot \hat{n} \, dS$$

for $\vec{F}(x, y, z) = (0, yz, z^2)$ and $\mathcal{S}$ the portion of the cylinder $y^2 + z^2 = 1$ with $0 \le x \le 1$, $z \ge 0$, and the positive orientation chosen to be a radial outward (from the axis of the cylinder) normal.

Integral Theorem Problems

$\frac{1}{2}\int_{\mathcal{C}} x\ dy - y\ dx$ for $\mathcal{C}$ the boundary of the ellipse

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$$

oriented counterclockwise.

Integral Theorem Problems

Let

$$\vec{F} = \frac{1}{x^2 + y^2}(-y, x).$$

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two simple closed curves enclosing the origin (and oriented with the usual inside to the left), can you say whether one of $\int_{\mathcal{C}_1} \vec{F} \cdot d\vec{s}$ and $\int_{\mathcal{C}_2} \vec{F} \cdot d\vec{s}$ is bigger than the other?

Conservative Vector Fields

A vector field $\vec{F}(x, y, z)$ which can be written as

$$\vec{F} = \nabla f$$

is called conservative. We already know

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{s} = 0$$

for any *closed* curve.

Conservative Vector Fields

The origin of the term is physics (I think) where in the case of $\vec{F}$ a force, it does no work (and so saps/adds no energy) as a particle traverses the closed curve.

Conservative Vector Fields

A vector field $\vec{F}(x, y, z)$ which can be written as

$$\vec{F} = \nabla f$$

is called conservative. We already know

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{s} = 0$$

for any *closed* curve.
In physics, the convention is to choose ϕ so that

$$\vec{F} = -\nabla \phi$$

and ϕ is referred to as the potential energy.

Conservative Vector Fields

Conservation of energy (in e.g mechanics) becomes a theorem in multivariable calculus combining the definition of a flow line with the computation of a line integral.

Newton's 2nd law ($\vec{F} = m\vec{a}$ and other versions) is also key . . . .

Conservative Vector Fields

The concept of voltage arises here too; it is just a potential energy per unit charge.

Conservative Vector Fields

The theorem (vector identity) $curl(\nabla f) = 0$ means the

$$curl(\vec{F}) = 0$$

is a necessary condition for the existence of a function f satisfying

$$\nabla f = \vec{F}.$$

Conservative Vector Fields

It turns out that for vector fields defined on e.g. all of R^2 or R^3, the converse of the theorem $curl(\nabla f) = 0$ is true. (For R^2, we're thinking of the scalar curl.)
In other words, in such a case, if $curl(\vec{F}) = 0$, (for a C^1 vector field), there is guaranteed to be a function $f(x, y, z)$ such that

$$\nabla f = \vec{F}.$$

Conservative Vector Fields

While this hold for vector fields with domains R^2, R^3, or more generally any "simply connected" region, the example $d\theta$ below shows this converse does not hold in general.

Conservative Vector Fields

Time permitting, we'll talk about simple connectivity next week.

Systematic Method of Finding a Potential

Finding a potential by inspection is fine when you can, but it is not systematic.
I often ask on a final exams for this.

Systematic Method of Finding a Potential

Problem: Use a systematic method to find a function $f(x, y, z)$ for which

$$\nabla f = (2xy, x^2 + z^2, 2yz + 1).$$

Systematic Method of Finding a Potential

Problem: Use a systematic method to find a function $f(x, y, z)$ for which

$$\nabla f = \left(y^2 z e^{xyz} + \frac{1}{y}, (1 + xyz)e^{xyz} - \frac{x}{y^2}, -(\cos^2(xyz))e^z + xy^2 e^{xyx} - e^z \sin^2(xyz)\right).$$

Systematic Method of Finding a Potential

Problem: Use a systematic method to find a function $f(x, y, z)$ for which

$$\nabla f = (2xz, 2y, x^2 + 6e^z).$$

Stokes and Gauss

Integration of a conservative vector field cartoon.

$$\int_C \nabla F \cdot d\vec{s} = F(\text{end}) - F(\text{start})$$

Stokes and Gauss

Green's Theorem cartoon.

$$\iint_R Q_x - P_y \, dA = \int_{\partial R} P\,dx + Q\,dy$$

Stokes and Gauss

Stokes' Theorem cartoon.

$$\iint_S (\nabla \times \vec{v}) \cdot \hat{n} \, dS = \int_{\partial S} \vec{v} \cdot d\vec{s}$$

Stokes and Gauss

Both sides of Stokes involve integrals whose signs depend on the orientation, so to have a chance at being true, there needs to be some compatibility between the choices.

The rule is that, from the "positive" side of the surface, (i.e. the side chosen by the orientation), the positive direction of the curve has the inside of the surface to the left.

As with all orientations, this can be expressed in terms of the sign of some determinant. (Or in many cases in terms of the sign of some combination of dot and cross products.)

Stokes and Gauss

Problem: Let $\mathcal{S}$ be the portion of the unit sphere $x^2 + y^2 + z^2 = 1$ with $z \geq 0$. Orient the hemisphere with an upward unit normal. Let $\vec{F}(x, y, z) = (y, -x, e^{z^2})$. Calculate the value of the surface integral

$$\iint_{\mathcal{S}} \nabla \times \vec{F} \ \cdot \hat{n} \ dS.$$

Stokes and Gauss

Gauss' Theorem field cartoon.

$$\iiint_W \operatorname{div}(\vec{v})\, dV = \iint_{\partial W} (\vec{v} \cdot \hat{n})\, dS$$

Stokes and Gauss

The surface integral side of Gauss depends on the orientation, so there needs to be a choice making the theorem true.

The rule is that the normal to the surface should point outward from the inside of the region.

(For the 2d analogue of Gauss (really an application of Green's)

$$\int_{\mathcal{C}} \vec{F} \cdot \hat{n} = \iint_{\text{inside}} (P_x + Q_y)\ dx\ dy$$

we also use an outward normal, where here $\mathcal{C}$ must of course be a closed curve.

Stokes and Gauss

Problem: Let $\mathcal{W}$ be the solid cylinder $x^2 + y^2 \leq 3$ with $1 \leq z \leq 5$. Let $\vec{F}(x, y, z) = (x, y, z)$. Find the value of the surface integral

$$\iint_{\partial \mathcal{W}} \vec{F} \cdot \hat{n} \, dS.$$

Surface of Revolution Case

This is not worth memorizing!

If one rotates about the z-axis the path (curve) $z = f(x)$ in the xz-plane for $0 \leq a \leq x \leq b$, one obtains a surface of revolution with a parametrization

$$\Phi(u, v) = (u \cos v, u \sin v, f(u))$$

and $dS = ?$

Surface of Revolution Case

$$dS = u\sqrt{1+(f'(u))^2}\ du\ dv.$$

Graph Case

This is not worth memorizing!

For the graph parametrization of $z = f(x, y)$,

$$\Phi(u, v) = (u, v, f(u, v))$$

and $dS = ?$

Graph Case

$$dS = \sqrt{1 + f_u^2 + f_v^2}\; du\; dv.$$

Graph Case

For such a graph, the normal to the surface at a point $(x, y, f(x, y))$ (this is the level set $z - f(x, y) = 0$) is

$$(-f_x, -f_y, 1)$$

so we can see that

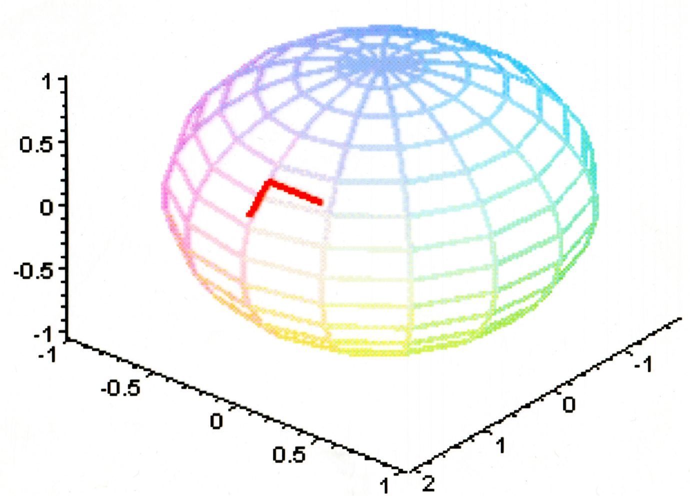
$$\cos\gamma = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}}$$

determines the angle γ of the normal with the z-axis. And so at the point $(u, v, f(u, v))$ on a graph,

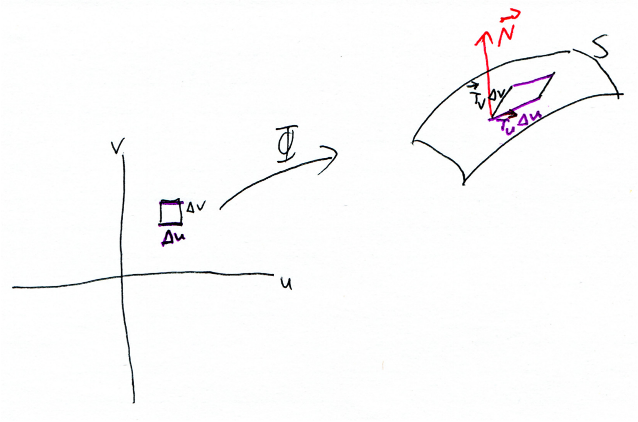
$$dS = \frac{1}{\cos\gamma}\ du\ dv.$$

(Note that $du\ dv$ is essentially the same as $dx\ dy$ here.)

Picture of $\vec{T}_u$, $\vec{T}_v$ for a Lat/Long Param. of the Sphere.



Basic Parametrization Picture



Surface Integrals

Parametrization $\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$

Tangents $T_u = (x_u, y_u, z_u)$ $\quad T_v = (x_v, y_v, z_v)$

Area Element $dS = \|\vec{T}_u \times \vec{T}_v\|\ du\ dv$

Normal $\vec{N} = \vec{T}_u \times \vec{T}_v$

Unit normal $\hat{n} = \pm \dfrac{\vec{T}_u \times \vec{T}_v|}{\|\vec{T}_u \times \vec{T}_v\|}$

(Choosing the $\pm$ sign corresponds to an *orientation* of the surface.)

Surface Integrals

Two Kinds of Surface Integrals

Surface Integral of a scalar function $f(x, y, z)$:

$$\iint_{\mathcal{S}} f(x, y, z)\ dS$$

Surface Integral of a vector field $\vec{F}(x, y, z)$:

$$\iint_{\mathcal{S}} \vec{F}(x, y, z) \cdot \hat{n}\ dS.$$

Surface Integrals

Surface Integral of a scalar function $f(x, y, z)$ calculated by

$$\iint_{\mathcal{S}} f(x, y, z)\, dS = \iint_{\mathcal{D}} f(\Phi(u, v))\, \|\vec{T}_u \times \vec{T}_v\|\, du\, dv$$

where $\mathcal{D}$ is the domain of the parametrization Φ.

Surface Integral of a vector field $\vec{F}(x, y, z)$ calculated by

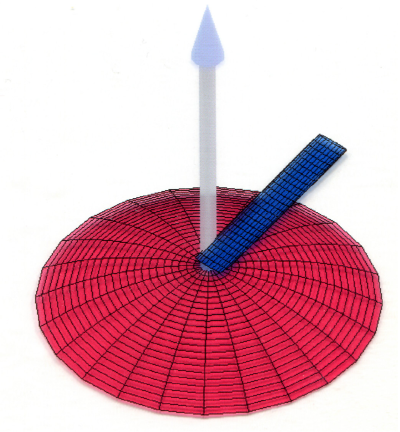
$$\iint_{\mathcal{S}} \vec{F}(x, y, z) \cdot \hat{n}\, dS$$

$$= \pm \iint_{\mathcal{D}} \vec{F}(\Phi(u, v)) \cdot \left(\frac{\vec{T}_u \times \vec{T}_v|}{\|\vec{T}_u \times \vec{T}_v\|} \right) \|\vec{T}_u \times \vec{T}_v\|\, du\, dv$$

where $\mathcal{D}$ is the domain of the parametrization Φ.

Surface Integrals

3d Flux Picture



Surface Integrals

The preceding picture can be used to argue that if $\vec{F}(x, y, z)$ is the velocity vector field, e.g. of a fluid of density $\rho(x, y, z)$, then the surface integral

$$\iint_{\mathcal{S}} \rho\vec{F} \cdot \hat{n}\ dS$$

(with associated Riemann Sum

$$\sum \rho(x_i^*, y_j^*, z_k^*)\vec{F}(x_i^*, y_j^*, z_k^*) \cdot \hat{n}(x_i^*, y_j^*, z_k^*)\ \Delta S_{ijk})$$

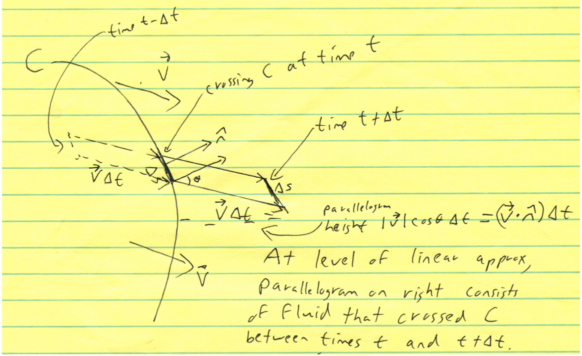
represents the rate at which material (e.g. grams per second) crosses the surface.

Surface Integrals

From this point of view the orientation of a surface simple tells us which side is accumulatiing mass, in the case where the value of the integral is positive.

Surface Integrals

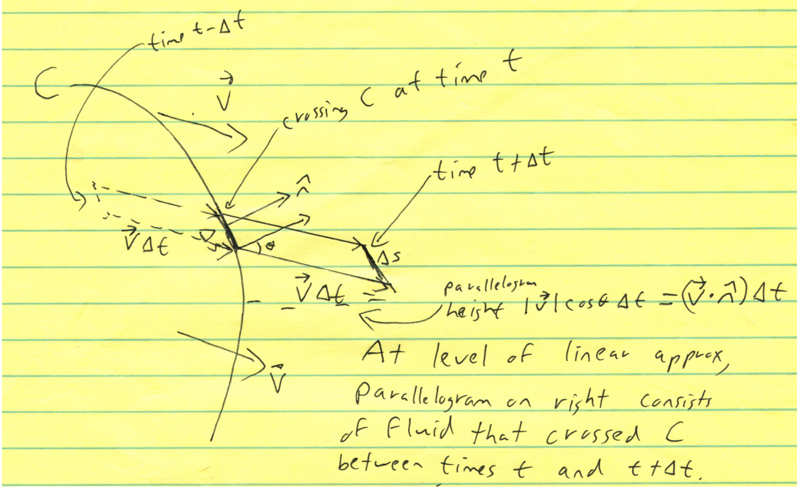
2d Flux Picture



There's an analagous 2d Riemann sum and interp of

$$\int_{\mathcal{C}} \vec{F} \cdot \hat{n}\, ds.$$

Surface Integrals



Surface Integrals

Problem: Calculate

$$\iint_{\mathcal{S}} \vec{F}(x, y, z) \cdot \hat{n} \; dS$$

for the vector field $\vec{F}(x, y, z) = (x, y, z)$ and $\mathcal{S}$ the part of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane. Choose the positive orientation of the paraboloid to be the one with normal pointing downward.

Surface Integrals

Problem: Calculate the surface area of the above paraboloid.