From Math 2220 Class 16

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1 Variable Taylor Series

Multivariable Taylor approximation follows (using the chain rule) easily from the 1-variable case, so we'll start by reviewing how the results there work out.

1 Variable Taylor Series

Taylor Series of $f(x)$ about $x = a$:

$$\Sigma_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n.$$

1 Variable Taylor Series

Partial sums (the $k+1$'st) give the k'th Taylor Polynomial.

$$P_k(x) = \Sigma_{n=0}^{k} \frac{f^n(a)}{n!}(x-a)^n.$$

(P_k is of degree k but has $k+1$ terms because we start with 0.)

So $P_1(x)$ is the linear approximation to $f(x)$ at $x = a$.

1 Variable Taylor Series

Examples:

$$(a) e^x \text{ about } x = 0 \qquad (b) e^x \text{ about } x = 2$$

1 Variable Taylor Series

Where does the formula for P_k come from heuristically?

1 Variable Taylor Series

Where does the formula for P_k come from heuristically?
The k'th Taylor polynomial $P_k(x)$ is the unique polynomial of degree k whose value, derivative, 2nd derivative, $\ldots$ k'th derivative all agree with those of $f(x)$ at $x = a$.

1 Variable Taylor Series

Where does the formula for P_k come from heuristically?

Or integration by parts: $u = f'(x), v' = 1, v = (x - b)$

$$f(b) = f(a) + \int_a^b f'(x)\, dx$$

$$f(b) = f(a) + \left((x-b)f'(x)\Big|_a^b - \int_a^b (x-b)f''(x)\, dx \right)$$

$$f(b) = f(a) + (b-a)f'(a) + \int_a^b (b-x)f''(x)\, dx$$

The last term is one form of the "remainder" in approximating $f(b)$ by the first Taylor polynomial (aka linear approximant) $P_1(b)$.

1 Variable Taylor Series

Where does the formula for P_k come from heuristically?

Or integration by parts: $u = f''(x), v' = (b-x), v = \frac{-(b-x)^2}{2}$

$$f(b) = \quad f(a) + (b-a)f'(a) + \qquad \int_a^b (b-x)f''(x)\ dx \Bigg)$$

$$f(b) = \quad f(a) + (b-a)f'(a) + \qquad \left(\frac{-(b-x)^2}{2} f''(x) \Big|_a^b \right. - \quad \int_a^b \frac{-(b-x)^2}{2} f'''(x)\ dx \Bigg)$$

1 Variable Taylor Series

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$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2} f''(a) + \int_a^b \frac{(b-x)^2}{2} f'''(x)\ dx$$

1 Variable Taylor Series

The above “integral formula” for the remainder can be turned into something simpler to remember but sufficient for most applications:
The next Taylor term except with the higher derivative evaluated at some unknown point c between a and b.

1 Variable Taylor Series

Taylor's Theorem with Remainder

If $f(x), f''(x), \dots f^{(k)}(x)$ all exist and are continuous on $[a, b]$ and $f^{(k+1)}(x)$ exists on (a, b), then there is a $c \in (a, b)$ so that

$$f(b) = P_k(b) + R_k(b)$$

where P_k is the k'th Taylor polynomial of f about $x = a$ and

$$R_k(b) = \frac{f^{k+1}(c)}{(k+1)!}(b-a)^{k+1}.$$

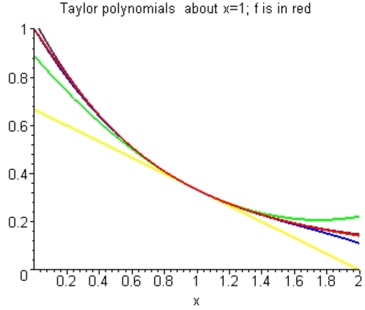
Please remember this!

Example: Accuracy of $\sin x$ by$P_1(x)$ (or $P_3(x)$) (about $x = 0$) for $|x| < .1$? Or $x < .01$?

1 Variable Taylor Series

$f(x) = \frac{1}{1+x+x^2}$ about $x = 1$.

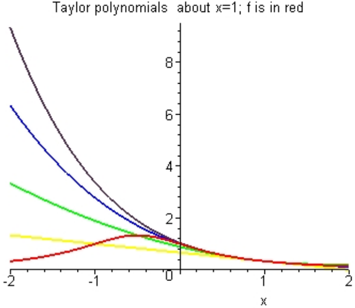
Approximation properties of different Taylor polynomials near $x = 1$.



1 Variable Taylor Series

Taylor polynomials of $f(x) = \frac{1}{1+x+x^2}$ about $x = 1$.

Typically higher order approximate well for a greater distance.



(Bad behavior beyond the radius of convergence.)

1 Variable Taylor Series

Show that the Maclaurin series of $\sin x$ converges to $\sin x$ for all x.

1 Variable Taylor Series

Show that the Maclaurin series of $\sin x$ converges to $\sin x$ for all x.

This is *different* than saying the radius of convergence is ∞. Some functions, e.g.

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-\frac{1}{x}} & \text{if } x > 0 \end{cases}$$

have a Taylor series which always converges, but not to $f(x)$ even near 0. In this case, every derivative is 0 at $x = 0$ and the Maclaurin series is identically 0!

1 Variable Taylor Series

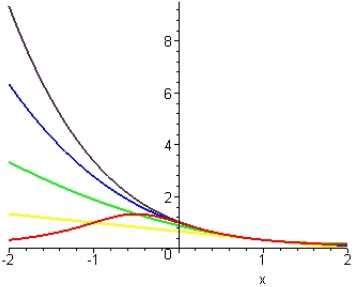
Show that the Maclaurin series of $\sin x$ converges to $\sin x$ for all x.

The point here is that all derivatives are bounded functions taking values between -1 and $+1$. So you can show the remainder $R_n(x)$ goes to 0 as $n \to \infty$.

Taylor polynomials of $f(x) = \frac{1}{1+x+x^2}$ about $x = 1$.

Typically higher order approximate well for a greater distance.

Taylor polynomials about x=1; f is in red



1 Variable Taylor Series

It is more advanced than our course, but there is an amazing connection between when Taylor series converge to a function and differentiability of f(x) for x viewed as a complex variable.

1 Variable Taylor Series

It is more advanced than our course, but there is an amazing connection between when Taylor series converge to a function and differentiability of f(x) for x viewed as a complex variable. For example, consider the MacLaurin series of $f(x) = \frac{1}{1+x^2}$. This fcn. is complex differentiable when $x \neq \pm i = \pm\sqrt{-1}$. And if one identifies the complex number $a + bi$ with the point $(a, b) \in R^2$, the nearest bad point of $\pm i \sim (0, \pm 1)$ is distance 1 from $0 = 0 + 0i \sim (0, 0)$. So from this advanced point of view, the radius of convergence is 1!

1 Variable Taylor Series

Similarly, consider the above picture for $f(x) = \frac{1}{1+x+x^2}$ about $x = 1$. This fcn. is bad when $1 + x + x^2 = 0$, or $x = \frac{-1 \pm i\sqrt{3}}{2}$. The distance from $1 + 0i \sim (1, 0)$ to $-(\frac{1}{2}, \frac{\sqrt{3}}{2})$ is $\sqrt{3}$, so from this advanced point of view, that is the radius of convergence for this Taylor series.

Multivariable Taylor Polynomials

Given $f : U \subset R^n \to R$, to find the formula for the k'th Taylor polynomial P_k of a C^k function about the point $p_0 = (a_1, a_2, \ldots, a_n) \in U$, we consider the 1-variable function $g(t)$ defined by

$$g(t) = f(p_0 + th)$$

where $h = (h_1, h_2, \ldots, h_n) \in R^n$.

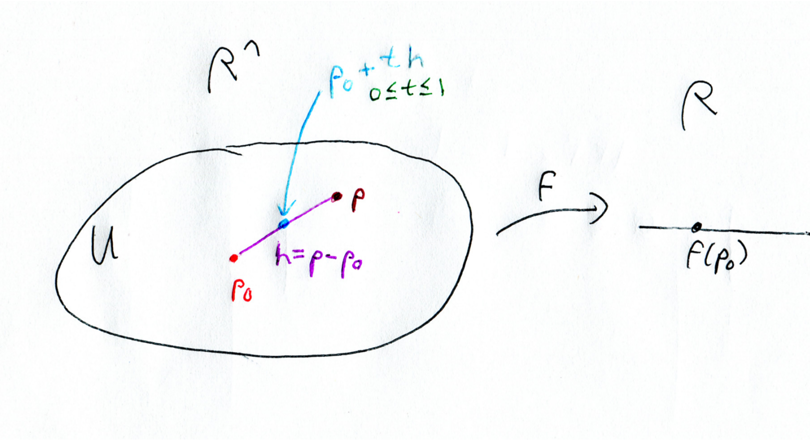
If we are interested in relating the Taylor polynomial P_k to the behavior of f at the point $p = (x_1, x_2, \ldots, x_n) \in U$ we might choose

$$h = (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$$

as long as the entire line segment between p_0 and p lies in U.

Multivariable Taylor Polynomials

$$g(t) = f(p_0 + th)$$



Multivariable Taylor Polynomials

(2 variable case, f a C^3 function)

$$g(t) = \quad f(x_0 + th, y_0 + tk)$$

Multivariable Taylor Polynomials

(2 variable case, f a C^3 function)

$$\begin{aligned} g(t) &= f(x_0 + th, y_0 + tk) \\ g'(t) &= hf_x(x_0 + th, y_0 + tk) \quad + kf_y(x_0 + th, y_0 + tk) \end{aligned}$$

Multivariable Taylor Polynomials

(2 variable case, f a C^3 function)

$$
\begin{aligned}
g(t) = &\ f(x_0 + th, y_0 + tk) \\
g'(t) = &\ hf_x(x_0 + th, y_0 + tk) + kf_y(x_0 + th, y_0 + tk) \\
g''(t) = &\ h(hf_{xx}(x_0 + th, y_0 + tk) + kf_{xy}(x_0 + th, y_0 + tk)) \\
&+ k(hf_{yx}(x_0 + th, y_0 + tk) + kf_{yy}(x_0 + th, y_0 + tk))
\end{aligned}
$$

Multivariable Taylor Polynomials

(2 variable case, f a C^3 function)

$$
\begin{aligned}
g(t) &= f(x_0+th, y_0+tk) \\
g'(t) &= hf_x(x_0+th, y_0+tk) \quad +kf_y(x_0+th, y_0+tk) \\
g''(t) &= h(hf_{xx}(x_0+th, y_0+tk) \quad +kf_{xy}(x_0+th, y_0+tk)) \\
&\quad + k(hf_{yx}(x_0+th, y_0+tk) \quad +kf_{yy}(x_0+th, y_0+tk)) \\
&= \left[h^2 f_{xx}(x_0+th, y_0+tk) \quad +2hkf_{xy}(x_0+th, y_0+tk) \right. \\
&\qquad \left. +k^2 f_{yy}(x_0+th, y_0+tk)\right]
\end{aligned}
$$

Multivariable Taylor Polynomials

(2 variable case, f a C^3 function)

$$
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g(t) =&\ f(x_0+th, y_0+tk) \\
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&+ k(hf_{yx}(x_0+th, y_0+tk) + kf_{yy}(x_0+th, y_0+tk)) \\
=&\ \left[h^2 f_{xx}(x_0+th, y_0+tk) + 2hkf_{xy}(x_0+th, y_0+tk) \right. \\
&\left. + k^2 f_{yy}(x_0+th, y_0+tk)\right] \\
g(0) =&\ f(x_0, y_0)
\end{aligned}
$$

Multivariable Taylor Polynomials

(2 variable case, f a C^3 function)

$$
\begin{aligned}
g'(t) &= hf_x(x_0+th, y_0+tk) && +kf_y(x_0+th, y_0+tk) \\
g''(t) &= h(hf_{xx}(x_0+th, y_0+tk) && +kf_{xy}(x_0+th, y_0+tk)) \\
&\quad + k(hf_{yx}(x_0+th, y_0+tk) && +kf_{yy}(x_0+th, y_0+tk)) \\
&= \big[h^2 f_{xx}(x_0+th, y_0+tk) && +2hkf_{xy}(x_0+th, y_0+tk) \\
& && +k^2 f_{yy}(x_0+th, y_0+tk)\big] \\
g(0) &= f(x_0, y_0) \\
g'(0) &= hf_x(x_0, y_0) && +kf_y(x_0, y_0)
\end{aligned}
$$

Multivariable Taylor Polynomials

(2 variable case, f a C^3 function)

$$
\begin{aligned}
g''(t) = \; & h(hf_{xx}(x_0+th, y_0+tk) \quad +kf_{xy}(x_0+th, y_0+tk)) \\
& + k(hf_{yx}(x_0+th, y_0+tk) \quad +kf_{yy}(x_0+th, y_0+tk)) \\
= \; & \left[h^2 f_{xx}(x_0+th, y_0+tk) \quad +2hkf_{xy}(x_0+th, y_0+tk)\right. \\
& \left. +k^2 f_{yy}(x_0+th, y_0+tk)\right]
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& \left. +k^2 f_{yy}(x_0, y_0)\right]
\end{aligned}
$$

Multivariable Taylor Polynomials

So the 1-variable Taylor (about $t = 0$) with Remainder at $t = 1$

$$\begin{aligned} g(t) &= \quad P_1 + \quad R_1 \\ g(1) &= \quad g(0) + g'(0)(1-0) + \quad \frac{1}{2}g''(c)(1-0)^2 \end{aligned}$$

for some $c \in (0, 1)$ becomes the multivariable

Multivariable Taylor Polynomials

becomes the multivariable

$$\begin{aligned} f(x_0+th, y_0+tk) &= \qquad P_1 \qquad + R_1 \\ f(x_0+h, y_0+k) &= f(x_0,y_0) + hf_x(x_0,y_0) + kf_y(x_0,y_0) \\ (\text{i.e. } f(x,y)) &= f(x_0,y_0) + hf_x(x_0,y_0) + kf_y(x_0,y_0) \\ &\quad + \frac{1}{2}\left[h^2 f_{xx}(x^*,y^*) + 2hkf_{xy}(x^*,y^*)\right. \\ &\quad \left. + k^2 f_{yy}(x^*,y^*)\right] \end{aligned}$$

for some point (x^*, y^*) on the (interior of the) line segment between (x_0, y_0) and (x, y).

Multivariable Taylor Polynomials

Similarly

$$g'''(0) = \quad h^3 f_{xxx}(x_0, y_0) + 3h^2 k f_{xxy}(x_0, y_0) + 3hk^2 f_{xyy}(x_0, y_0) + k^3 f_{yyy}(x_0, y_0)$$

(with $g'''(t)$ similar) and 1-variable Taylor $g(1) = P_2 + R_2$ become for a C^3 function

$$f(x, y) = \quad f(x_0, y_0) + hf_x(x_0, y_0) + kf_y(x_0, y_0) + \frac{1}{2}\left[h^2 f_{xx}(x_0, y_0) + 2hkf_{xy}(x_0, y_0) + k^2 f_{yy}(x_0, y_0)\right] + \frac{1}{3!}\left[h^3 f_{xxx}(x^*, y^*) + 3h^2 k f_{xxy}(x^*, y^*) + 3hk^2 f_{xyy}(x^*, y^*) + k^3 f_{yyy}(x^*, y^*)\right]$$

Multivariable Taylor Polynomials

The coefficients 1 1, 1 2 1 and 1 3 3 1 in the above derivative expressions are in fact the binomial coefficients from Pascal's triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

which describes the expansion of $(x + y)^m$.

Multivariable Taylor Polynomials

Problem: Find the first and second Taylor polynomials of

$$f(x, y) = \cos(x + 3y) + \sin x$$

about $(x, y) = (\frac{\pi}{2}, \pi)$.

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about $(x, y) = (\frac{\pi}{2}, \pi)$.

Estimate the error in approximating f by P_1 for $|x - \frac{\pi}{2}| < .1$ and $|y - \pi| < .1$.

Problem: Find the first and second Taylor polynomials of

$$f(x, y) = \cos(x + 3y) + \sin x$$

about $(x, y) = (\frac{\pi}{2}, \pi)$.

Estimate the error in approximating f by P_1 for $|x - \frac{\pi}{2}| < .01$ and $|y - \pi| < .01$.

Multivariable Taylor Polynomials

Problem: Find the first and second Taylor polynomials of

$$f(x, y) = \cos(x + 3y) + \sin x$$

about $(x, y) = (\frac{\pi}{2}, \pi)$.

Find an ϵ so that the error in approximating f by P_1 for $|x - \frac{\pi}{2}| < \epsilon$ and $|y - \pi| < \epsilon$ is at most .01.

Or at most 10^{-6}.

More than 2 Variables

Now the 1-variable function $g(t) = f(p_0 + th)$ where $p_0 \in R^n$ and $h = (h_1, h_2, \ldots, h_n)$ satisfies

$$g'(t) = \Sigma_{i=1}^{n} h_i (D_i g)(p_0 + th_i)$$

where D_i denotes the i'th partial derivative.

More than 2 Variables

And

$$g''(t) = \Sigma_{i=1}^{n} h_i \left(\Sigma_{j=1}^{n} h_j D_j (D_i g)(p_0 + th_i) \right)$$

or

$$g''(t) = \Sigma_{i=1}^{n} \Sigma_{j=1}^{n} h_i h_j (D_{ij} g)(p_0 + th_i).$$

More than 2 Variables

And

$$g''(t) = \Sigma_{i=1}^{n} h_i \left(\Sigma_{j=1}^{n} h_j D_j (D_i g)(p_0 + th_i)\right)$$

or

$$g''(t) = \Sigma_{i=1}^{n} \Sigma_{j=1}^{n} h_i h_j (D_{ij} g)(p_0 + th_i).$$

Noting that

$$(h_1 + h_2 + \ldots + 2_n)^2 = \Sigma_{i=1}^{n} h_i \Sigma_{j=1}^{n} h_j = \Sigma_{i=1}^{n} \Sigma_{j=1}^{n} h_i h_j$$

we can see how the binomial (actually multinomial) coefficients in $(h_1 + h_2 + \ldots + 2_n)^m$ enter when we collect terms in the Taylor polynomials.

More than 2 Variables

A good way to write the degree m part of the m'th Taylor polynomial in terms of multi-index notation is

$$\Sigma_I \frac{1}{|I|!}\left(\frac{|I|!}{i_1! i_2! \dots i_n!} h^I D_I f\right)$$

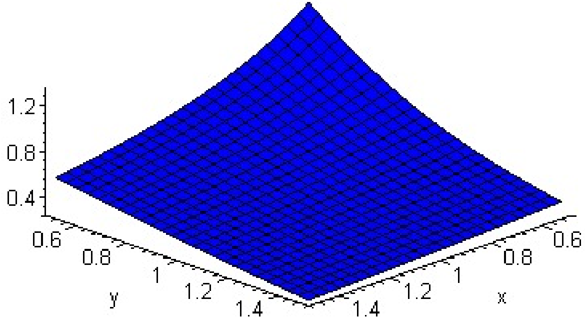
where the n powers are arranged in a vector $I = (i_1, i_2, \dots, i_n)$ and

$$\begin{aligned} |I| &= \quad i_1 + i_2 + \dots i_n \; (= m \text{ for terms contributing to } P_m) \\ h^I &= \quad h_1^{i_1} h_2^{i_2} \dots h_n^{i_n} \\ D_I f &= \quad D_1^{i_1} D_2^{i_2} \dots D_n^{i_n} f \end{aligned}$$

where for example $D_3^5 f$ means partially differentiate f 5 times with respect to the third variable x_3.

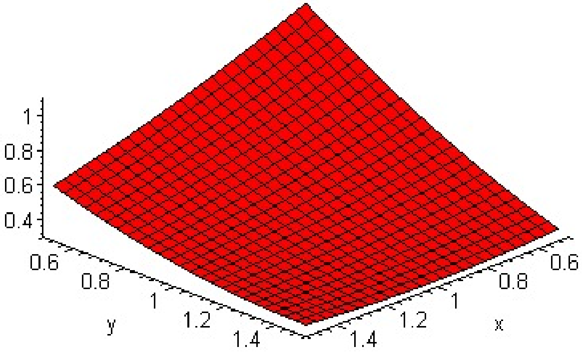
Multivariable Taylor Pictures

Graph of $f(x, y) = \frac{1}{x+y^2}$ for $.5 \leq x, y \leq 1.5$



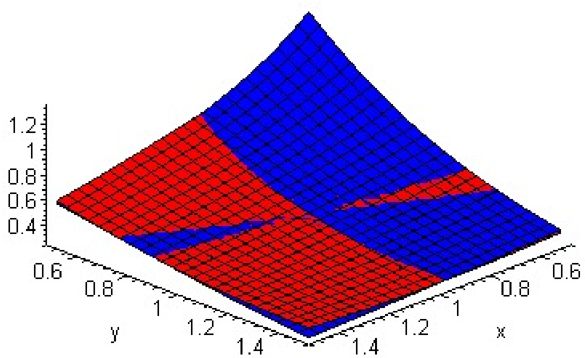
Multivariable Taylor Pictures

P_2 about $(1,1)$ for $f(x,y) = \frac{1}{x+y^2}$ for $.5 \leq x, y \leq 1.5$



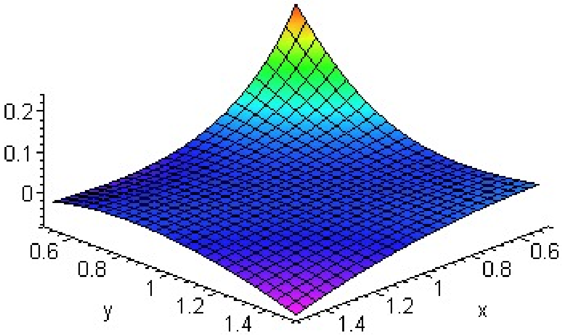
Multivariable Taylor Pictures

Both superimposed for $.5 \leq x, y \leq 1.5$



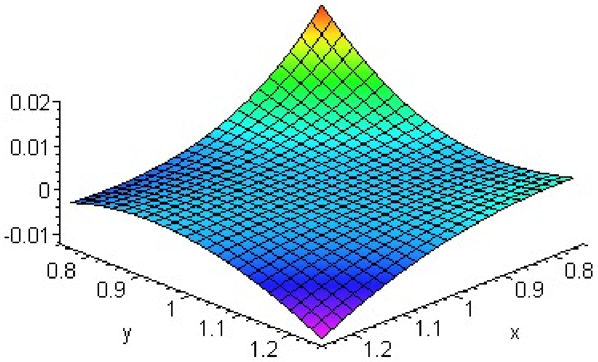
Multivariable Taylor Pictures

Error in using P_2 for f for $.5 \leq x, y \leq 1.5$



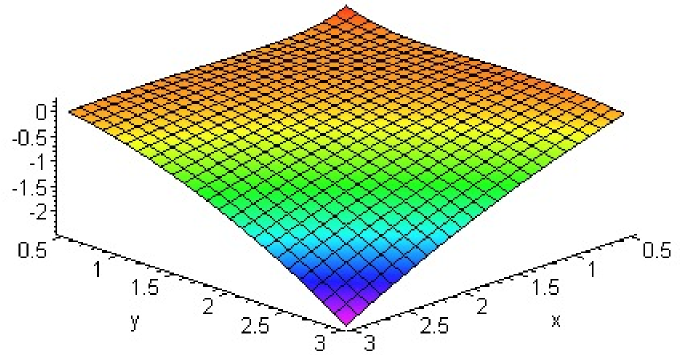
Multivariable Taylor Pictures

Error in using P_2 for f for $.75 \leq x, y \leq 1.25$



Multivariable Taylor Pictures

Error in using P_2 for f for $0 \leq x, y \leq 3$



Higher Partials in Polar Cordinates

Let $x = r\cos\theta, y = r\sin\theta$. Given $f(x, y)$, express

$$\frac{\partial^2 f}{\partial r^2}$$

in terms of partials of f with respect to x and y.

Higher Partials in Polar Cordinates

Really there are two functions here; namely $f(x, y)$ and

$$g(r, \theta) = f(r \cos \theta, r \sin \theta).$$

But many applied fields routinely use the same letter f for both functions.

Higher Partials in Polar Cordinates

The key step is realizing that the task (via the chain rule) of relating $\frac{\partial f}{\partial r}$ to $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ is just like the task of doing the analagous thing with expressions like

$$\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right).$$

(The right hand side "operator notation" means partially differentiate the function $\frac{\partial f}{\partial x}$ of (x, y) with respect to r.)

Higher Partials in Polar Cordinates

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\begin{aligned}
\frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \\
&= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \\
\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right) &= \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\right)\frac{\partial x}{\partial r} + \left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right)\frac{\partial y}{\partial r} \\
&= \left(\frac{\partial^2 f}{\partial x^2}\right)\cos\theta + \left(\frac{\partial^2 f}{\partial y \partial x}\right)\sin\theta
\end{aligned}$$

Higher Partials in Polar Cordinates

A similar computation with $\dfrac{\partial^2 f}{\partial \theta^2}$ would show that the laplacian

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

is given in polar coordinates by

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 f}{\partial \theta^2}\right).$$

Method of Characteristics

Problem 33 on page 146 from your homework asks you to show using the chain rule that if $z = f(\frac{y}{x})$ for a 1-variable differentiable function f, then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

Method of Characteristics

Problem 33 on page 146 from your homework asks you to show using the chain rule that if $z = f(\frac{y}{x})$ for a 1-variable differentiable function f, then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

You don't need it to do this homework problem, but a natural question is

$$\text{Does } x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0 \text{ imply } z = f(\frac{y}{x})$$

for some 1-variable differentiable function f?

Method of Characteristics

The answer is yes, and we can easily show it using what we learned in section 2.6!
(i.e. essentially chain rule ideas applied to curves.)

Method of Characteristics

The beautiful idea for the equation

$$a(x, y)\frac{\partial z}{\partial x} + b(x, y)\frac{\partial z}{\partial y} = c(x, y, z)$$

is to consider curves $(x(t), y(t)$ called characteristics) satisfying

$$\begin{aligned} \frac{dx}{dt} &= \quad a(x(t), y(t)) \\ \frac{dy}{dt} &= \quad b(x(t), y(t)). \end{aligned}$$

Method of Characteristics

Here for

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$$

we have

$$\frac{dx}{dt} = x$$
$$\frac{dy}{dt} = y$$

with solutions $x(t) = x_0 e^t$ and $y(t) = y_0 e^t$.

Method of Characteristics

Here for

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$$

we have

$$\begin{aligned} \frac{dx}{dt} &= x \\ \frac{dy}{dt} &= y \end{aligned}$$

with solutions $x(t) = x_0 e^t$ and $y(t) = y_0 e^t$.

The key observation is that for a solution $z(x, y)$

$$\begin{aligned} \frac{d}{dt}\left[z(x(t), y(t))\right] &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= \frac{\partial z}{\partial x}x + \frac{\partial z}{\partial y}y \\ &= 0\ ! \end{aligned}$$

Method of Characteristics

Here for

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$$

with solutions $x(t) = x_0 e^t$ and $y(t) = y_0 e^t$.
The key observation is that for a solution $z(x, y)$

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So $z(x, y)$ is constant along lines $\frac{y}{x} = k$ for any k and thus

$$z(x, y) = z(1, \frac{y}{x}) = f(\frac{y}{x})$$

as we wanted to show.

Lagrange Multipliers Intro

Problem: Maximize $f(x, y) = x^4 + 4x^2y^2$ among points satisfying the constraint $g(x, y) = x^2 + y^2 = 1$.

Lagrange Multipliers Intro

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Only points on the circle $x^2 + y^2 = 1$ count !

Lagrange Multipliers Intro

Problem: Maximize $f(x, y) = x^4 + 4x^2y^2$ among points satisfying the constraint $g(x, y) = x^2 + y^2 = 1$.

Parameterizing the circle as $x = \cos t$ and $y = \sin t$ is one approach, but even when you can nicely parameterize the constraint set LM is often useful.

Lagrange Multipliers Intro

Problem: Maximize $f(x,y) = x^4 + 4x^2y^2$ among points satisfying the constraint $g(x,y) = x^2 + y^2 = 1$.

The method of LM says to look for constrained local extrema, seek solutions of

$$\nabla f = \lambda \nabla g$$

for some unknown constant λ. (As well as places on the constraint set where $\nabla g = 0$.)

Lagrange Multipliers Intro

The need to check $\nabla g = 0$ too:

Problem: Minimize $f(x, y) = y$ on the constraint set

$$(y - x^2)(y - 4x^2) = 0.$$

Lagrange Multipliers Intro

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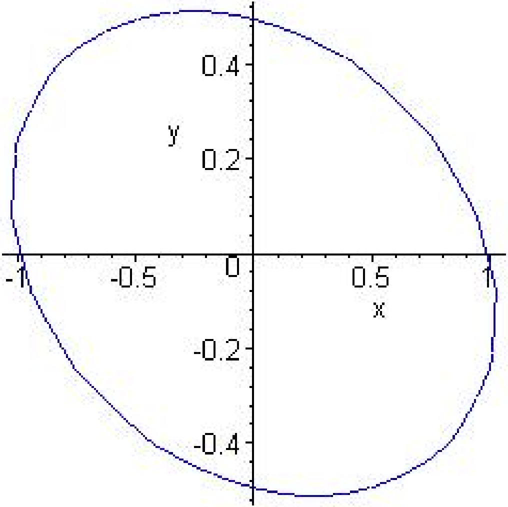
Problem: Minimize $f(x, y) = y$ on the constraint set

$$(y - x^2)(y - 4x^2) = 0.$$

Clearly $(x, y) = (0, 0)$ is the answer, but $\nabla f \neq \lambda \nabla g$ there since ∇g is 0 at $(0, 0)$.

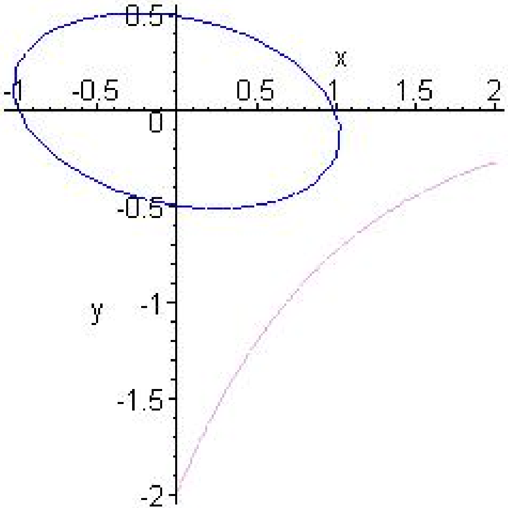
Maximize $f(x, y) = ye^x$.
On the constraint set $x^2 + xy + 4y^2 = 1$.

The constraint set is an ellipse.



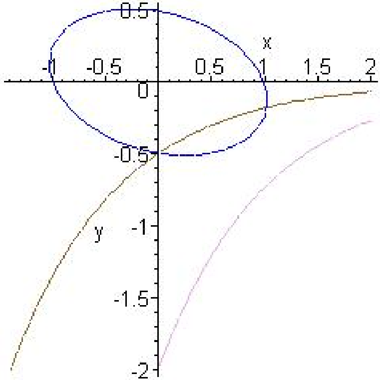
Maximize $f(x,y) = ye^x$. On the constraint set $x^2 + xy + 4y^2 = 1$.

With the level curve $f(x,y) = -2$.



Maximize $f(x,y) = ye^x$. On the constraint set $x^2 + xy + 4y^2 = 1$.

The level curve $f(x,y) = -.5$. meets the ellipse obliquely.



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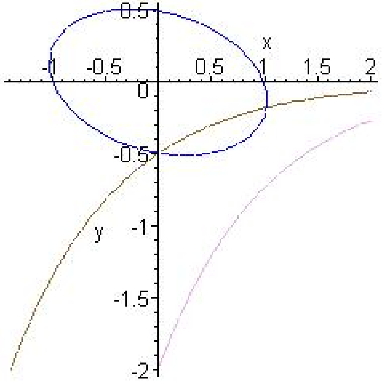
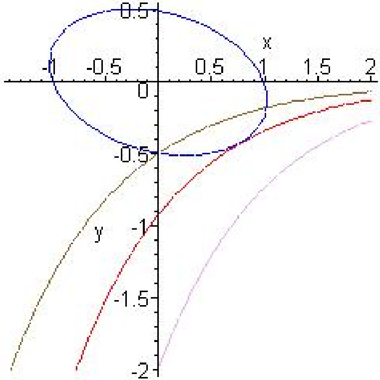


The essence of LM is that at a non-tangent intersection point, b/c $D_{\vec{u}} f = \nabla f \cdot \vec{u}$, f changes to first order as you move along the constraint set. So not a local constr. min or max.

Maximize $f(x, y) = ye^x$.
On the constraint set $x^2 + xy + 4y^2 = 1$.

The level curve $f(x, y) = -.9276551883$ is tangent.



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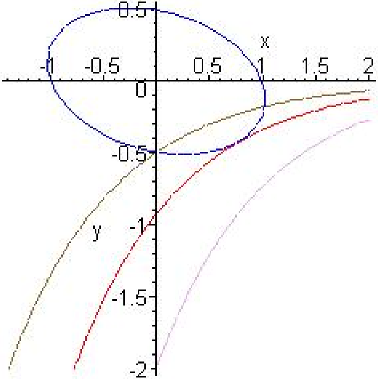

This value was found by solving $\nabla f = \lambda \nabla g$ numerically.
$((x,y) = (.776, -.427).)$

Maximize $f(x, y) = ye^x$.
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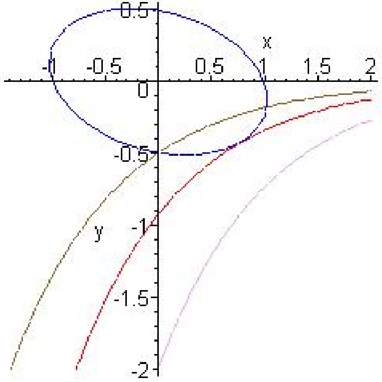
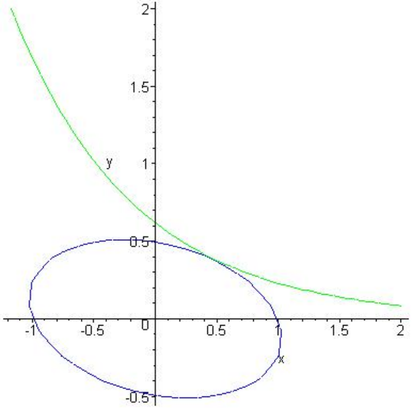
The level curve $f(x, y) = -.9276551883$ is tangent.



This value was found by solving $\nabla f = \lambda \nabla g$ numerically.
$((x, y) = (.776, -.427).)$
It appears to be the minimum value of f on the constraint set.

Maximize $f(x, y) = ye^x$.
On the constraint set $x^2 + xy + 4y^2 = 1$.

The level curve $f(x, y) = .6185121367$ is also tangent.



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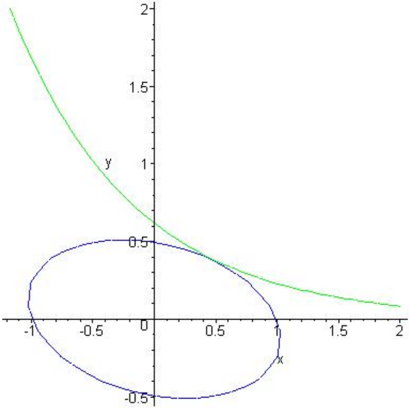
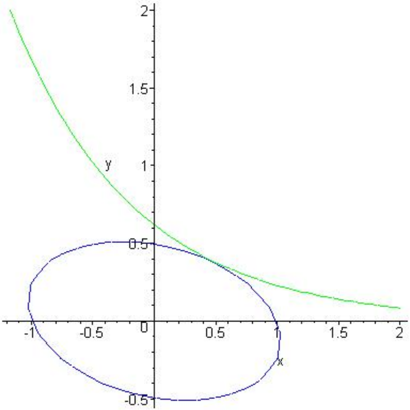


Also found numerically. $((x, y) = (.459, .381).)$

Maximize $f(x, y) = ye^x$. On the constraint set $x^2 + xy + 4y^2 = 1$.

The level curve $f(x, y) = .6185121367$ is also tangent.



Also found numerically. $((x, y) = (.459, .381).)$
It appears to be the maximum value of f on the constraint set.