
Linear fractional transformations

0.1. **Definitions.** We study a special class of maps $T : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}$.

A linear fractional transformation of $\mathbb{C} \cup \{\infty\}$ is a map of the form

$$w = T(z) = \frac{az+b}{cz+d}, \qquad a, b, c, d \in \mathbb{C}. \tag{0.1.1}$$

We think of the transformation as depending on the 2×2 matrix

$$g := \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{0.1.2}$$

and write T_g for the transformation. We assume that $g \neq 0$, or else we would be dividing $\frac{0}{0}$ for all z. In fact, if $c = d = 0$, T would be undefined for all z. Note that if we multiply the coefficients by the same $\lambda \neq 0$, we get the same transformation. In particular, if $ad - bc = 0$, then there is cancellation, and the transformation becomes $T(z) = \mathit{constant}$ except for maybe one value of z where the transformation is undefined.

We **assume** that $\mathbf{ad} - \mathbf{bc} \neq \mathbf{0}$.

The following properties are verified by direct calculation.

1: $T_g = T_{g'}$ if and only if there is $\lambda \neq 0$ such that

$$a' = \lambda a,\ b' = \lambda b,\ c' = \lambda c,\ d' = \lambda d.$$

This can be written as

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

2: $T_{g_1 \cdot g_2} = T_{g_1} \circ T_{g_2}$, where $\cdot$ is usual matrix multiplication, and $\circ$ is composition of transformations/functions.

3: If $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $T_g(z) = z$.

In view of this, we define dividing any nonzero number by 0 to be ∞. With this convention, we define

$$T_g(\infty) = \frac{a}{c}, \qquad T_g(-\frac{d}{c}) = \infty. \tag{0.1.3}$$

If $c \neq 0$, $\frac{a}{c}$ is a finite number. If $c = 0$, then $a \neq 0$, and $\frac{a}{c} = \infty$. Similarly for $\frac{d}{c}$.

But we still have to assume that $\frac{0}{0}$ and $0 \cdot \infty$ are undefined.

0.2. **Relation to elementary transformations.** Recall the elementary transformations:

Translation: $T_a(z) = z + a$
Dilation: $T_a(z) = az$ for $a \neq 0$.
Inversion: $R(z) = \frac{1}{z}$.

These are linear fractional transformations, so any composition of simple transformations is a linear fractional transformations. Conversely any linear fractional transformation is a composition of simple transformations. If $c = 0$, this is clear. If $c \neq 0$, we can write

$$\frac{az+b}{cz+d} = \frac{a}{c} - \frac{ad-bc}{c(cz+d)} \tag{0.2.1}$$

The claim follows from this equation.

0.3. **Circles and lines.** The equation of a circle is given by

$$Az\overline{z} + Bz + \overline{B}\overline{z} + C = 0, \qquad A, C \in \mathbb{R}, \quad AC < |B|^2. \tag{0.3.1}$$

The equation of a line is

$$Bz + \overline{B}\overline{z} + C = 0, \qquad C \in \mathbb{R}. \tag{0.3.2}$$

It is easy to verify that translations and dilations send circles to circles, and lines to lines. Inversion also sends a circle to a circle, except if it passes through 0. In this case, $C = 0$. A point $z \neq 0$ on such a circle (0.3.1) with $C =)$ gives the equation

$$B\overline{w} + \overline{B}w + A = 0, \qquad w = \frac{1}{z}. \tag{0.3.3}$$

It makes sense to make the definition that every line contains ∞. Then 0 is mapped by inversion to ∞, and every circle going through 0 is mapped to a line not going through 0. Lines going through 0 have equation (0.3.2) with $C = 0$. Using our conventions, it follows that inversion takes lines through 0, to lines through 0.

Thus elementary transformations take circles and lines to circles or lines. The complement of a line or circle in the extended plane is a union of two connected regions. being continuous, a fractional transformation takes a connected region to a connected region.

Exercise 1. These conventions are compatible with stereographic projection. Prove this fact. Precisely, show that

1. Any circle through the north pole is mapped to a line,
2. Any circle on the sphere not passign through the north pole is mapped to a circle.

Hint: Any circle on the sphere is obtained by intersecting the sphere with a plane.

0.4. **Uniqueness.** The set of 2×2 matrices with complex entries, and of determinant 1 forms a group under matrix multiplication. It is called $SL(2,\mathbb{C})$.

Proposition. *The map*

$$g \mapsto T_g$$

is a group isomorphism between $SL(2,\mathbb{C})/\{\pm Id\}$, *and linear fractional transformations.*

Proof. Every every fractional transformation is of the form T_g with g satisfying $\det g \neq 0$. But g and $\frac{1}{\det g} g$ give the same transformation, and the latter is in $SL(2,\mathbb{C})$. To see that the map is 1-1, note that if $T_{g'} = T_g$, implies $T_{g'} \circ T_{g^{-1}} = Id$, so $T_{g' \cdot g^{-1}} = Id$. Thus it is enough to see that if

$$T_g(z) = z, \qquad \textit{then} \qquad g = Id. \tag{0.4.1}$$

This means

$$\frac{az+b}{cz+d} = z, \qquad az + b = cz^2 + dz. \tag{0.4.2}$$

Plugging in $z = 0$, we get $b = 0$. Plugging in $z = \infty$, we get $c = 0$. Then plugging in $z = 1$, we get $\frac{a}{d} = 1$. Since we also assumed $\det g = ad = 1$, we find $a = d = \pm 1$. ☐

Theorem. *Let* $\{z_0, z_1, z_\infty\}$ *and* $\{z'_0, z'_1, z'_\infty\}$ *be two sets of triplets of distinct points in* $\mathbb{C} \cup \{\infty\}$. *Then there is a unique transformation* T *such that*

$$T(z_0) = z'_0, \qquad T(z_1) = z'_1, \qquad T(z_\infty) = z_\infty.$$

Proof. We show existence first. Assume that $\{z'_0, z'_1, z'_\infty\} = \{0, 1, \infty\}$. If we can prove the assertion for this choice, then the general existence statement follows. This is because for arbitrary $\{z_0, z_1, z_\infty\}$ we can construct

$$\begin{aligned} &T(z_0) = 0, \qquad &&T(z_1) = 1, \qquad &&T(z_\infty) = \infty, \\ &T'(z'_0) = 0, \qquad &&T'(z'_1) = 1, \qquad &&T'(z'_\infty) = \infty, \end{aligned} \tag{0.4.3}$$

and then $T'^{-1} \circ T$ is the desired transformation.
If none of $\{z_0, z_1, z_\infty\}$ are ∞, then

$$T(z) = \frac{z - z_0}{z_1 - z_\infty} \cdot \frac{z_1 - z_\infty}{z_1 - z_0} \tag{0.4.4}$$

is the transformation T. If say $z_0 = \infty$, then

$$T(z) = \frac{z_1 - z_0}{z - z_\infty}. \tag{0.4.5}$$

The other cases are left as an exercise.
Now we show uniqueness. Suppose that T and T' satisfy the claim of the theorem. Then $T'^{-1} \circ T$ fixes $\{z_0, z_1, z_\infty\}$. Suppose T_0 takes $\{z_0, z_1, z_\infty$ to $\{0, 1, \infty\}$. Then $S = T_0 \circ T'^{-1} \circ T \circ T_0^{-1}$ fixes $\{0, 1, \infty\}$. But then we can check easily that $S = T_g$ must satisfy

$$\frac{b}{d} = 0, \qquad \frac{a}{c} = \infty. \tag{0.4.6}$$

Thus $b = c = 0$, and thereofore $\frac{a}{d} = 1$ as well. Thus $S = Id$. It follows that $T'^{-1} \circ T = Id$, or equivalently $T' = T$. ☐

Exercise 2. Complete the proof of the theorem, *i.e.* , find the transformations when $z_1 = \infty$ and when $z_\infty = \infty$. ☐

0.5. **Transformations preserving the real line.**

Exercise 3.

1. Show that any linear fractional transformation that maps the real line to itself can be written as T_g where $a, b, c, d \in \mathbb{R}$.
2. The complement of the real line is formed of two connected regions, the **upper half plane** $\{z \in bC \ : \ Imz > 0\}$, and the **lower half plane** $\{z \in \mathbb{C} \ : \ Imz < 0\}$. Show that a transformation presrving the real line preserves the two half planes if $\det g > 0$, and interchanges them if $\det g < 0$. In particular, show that the group $SL(2, \mathbb{R})$ is precisely the subgroup of $SL(2, \mathbb{C})$ that preserves the upper half plane.

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0.6. **The upper half plane.** Denote the upper half plane by $\mathcal{H}$. Then $SL(2, \mathbb{R})$ acts on it by

$$T_g(z) = \frac{az + b}{cz + d}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{R},\ ad - bc = 1. \tag{0.6.1}$$

The stabilizer of i is the subgroup satisfying
(0.6.2)

$$i = \frac{ai + b}{ci + d} = \frac{(ai + b)(-ci + d)}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + i\frac{ad - bc}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \frac{i}{c^2 + d^2}.$$

Thus $ac = -bd$, and $c^2 + d^2 = 1$. Setting $c = \sin\theta$ and $d = \cos\theta$, we find $a = \cos\theta$ and $b = -\sin\theta$. Thus the stabilizer of i is the subgroup called $SO(2)$,

$$SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \ : \ \theta \in \mathbb{R} \right\}. \tag{0.6.3}$$

On the other hand, let $z = x + iy \in \mathcal{H}$, *i.e.* $y > 0$. Then

$$\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} \tag{0.6.4}$$

has the property that $T_g(i) = x + iy$. Thus for any $z \in \mathcal{H}$, there is $g \in SL(2, \mathbb{R})$ such that $T_g(i) = z$.

Exercise 4. Show that any $g \in SL(2, \mathbb{R})$ can be written uniquely as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{0.6.5}$$

Exercise 5*. Find all fractional linear transformations coming from $SL(2, \mathbb{C})$ which preserve the region

$$\left\{ z \in \mathbb{C} \ : \ |z| < 1 \right\}. \quad \square$$

0.7. **Modular forms.** The group $SL(2, \mathbb{R})$ contains a discrete subgroup

$$SL(2, \mathbb{Z}) = \left\{ g \in SL(2, \mathbb{R}) \ : \ a, b, c, d \in \mathbb{Z} \right\}. \tag{0.7.1}$$

Functions which are analytic (to be defined later in the course) on $\underline{\mathrm{H}}$ satisfying

$$f(T_\gamma(z)) = (cz + d)^k f(z) \qquad \gamma \in SL(2, \mathbb{Z}), \tag{0.7.2}$$

are called modular forms of weight k. Their properties are closely connected to number theory.

Some examples.

- Eisenstein series: $G_{2k}(z) = \sum_{(m,n) \neq (0,0)} \frac{1}{(mz+n)^{2k}}$.
- Theta series: $\theta(z) = \sum_{n \in \mathbb{Z}} e^{in^2 \pi z}$. This is not really a modular form for $SL(2, \mathbb{Z})$, but rather satisfies a more complicated invariance property for a subgroup of $SL(2, \mathbb{Z})$. You may have seen this function in connection with the heat kernel.