Semisimple Lie Algebras
Math 649, 2013
Root Systems

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REFERENCES: Bourbaki chapter 6, Humphreys chapter III

Simple Roots

Definition 1

A subset $\Pi \subset R$ is called *a base* if

(I) Π is a basis of V

(II) Any $\beta \in R$ can be written $\beta = \sum n_\alpha \alpha,\ n_\alpha \in \mathbb{Z}$, either all positive or all negative. The roots in Π are called simple.

Properties:

1. A root is called positive if all the $n_\alpha \geq 0$, negative if all the $n_\alpha \leq 0$. Then $R = R^+ \cup R^-$, $\quad R^+ \cap R^- = \emptyset$. where $R^\pm$ are the positive (negative) roots.
2. If $\alpha, \beta \in R^+$, then $\alpha + \beta \in R^+$, or it is not a root.
3. We say $\alpha \leq \beta$ if $\beta - \alpha \in R^+$.
4. If $\alpha, \beta \in \Pi$, $\alpha \neq \beta$, then $(\alpha, \beta) \leq 0$ and $\alpha - \beta$ is not a root.

Proof of (4)

Proof.

If $\gamma = \alpha - \beta$ is a root, either $\gamma \in R^+$, or $-\gamma \in R^+$. Either way, it violates (II). □

Theorem 2

Every root system has a base.

Proof.

Let H_α be the hyperplane where $\check{\alpha}$ is zero. Let $0 \neq h_0 \in V \setminus \cup H_\alpha$.
Then

$$\check{\alpha}(h_0) \neq 0 \quad \text{for all} \quad \alpha \in R. \tag{1}$$

Let $R^+ := \{\alpha \mid \check{\alpha}(h_0) > 0\}$. Then $R = R^+ \coprod R^-$ where $R^- = -R^+$. We say $\alpha \in R^+$ is indecomposable if α cannot be written as

$$\alpha = \beta + \gamma \qquad \beta, \gamma \in R^+. \tag{2}$$

□

Proof of theorem 2, continued

Claim: The set of indecomposable elements is a base.

Proof.

Call this set Π.

(1) Each $\alpha \in R^+$ is an $\mathbb{N}$-linear combination of elements in Π.

(2) $\alpha, \beta \in \Pi$ distinct, then $\alpha - \beta$ is not a root and $(\alpha, \beta) \leq 0$. Otherwise $\beta = \alpha + \gamma$ or $\alpha = \beta + \gamma$ with $\gamma \in R^+$.

(3) Π forms a linear independent set. Suppose $\sum_{\alpha \in \Pi} n_\alpha \alpha = 0$.

Then we get a relation $r = \sum n_\alpha \alpha = \sum m_\beta \beta,\ \ n_\alpha, m_\beta > 0$, and the two sets are disjoint. But then $(r, r) = \sum n_\alpha m_\beta (\alpha, \beta) \leq 0$. So $r = 0$. But then $0 = (r, h_0) = \sum n_\alpha \langle \alpha, h_0 \rangle = \sum m_\beta \langle \beta, h_0 \rangle$, so all $n_\alpha, m_\beta = 0$. □

Proposition 1

Each base is obtained in this fashion.

Proof.

Choose h_0 such that $\langle \alpha, h_0 \rangle > 0$ for all $\alpha \in \Pi$. Let $R^{\pm}$ be the positive and negative systems corresponding to h_0. Clearly

$$R^{+} = \left\{ \beta \in R \mid \beta = \sum_{\alpha \in \Pi} n_{\alpha} \alpha, \;\; n_{\alpha} \in \mathbb{N} \right\}. \tag{3}$$

□

Weyl Chambers

A Weyl chamber is a connected components $\mathcal{C}$ of $V - \bigcup_{\alpha \in R} H_\alpha$. Recall that a regular element $h_0 \subset V$ is an element so that

$$\langle \alpha, h_0 \rangle \neq 0 \quad \text{for any} \quad \alpha \in R. \tag{4}$$

Each h_0 regular determines a Weyl chamber $\mathcal{C}(h_0)$

$$\{\text{Weyl chambers}\} \longleftrightarrow \{\text{Bases}\}. \tag{5}$$

Lemma 3

The Weyl group $W(R)$ permutes the Weyl chambers.

Proof.

Clear. □

Lemma 4

If $\gamma \in R^+$ is not simple, there is $\alpha \in \Pi$ such that $\gamma - \alpha \in R^+$.

Proof.

There is $\alpha \in R^+$ such that $(\gamma, \alpha) > 0$. This implies the claim of the lemma. If $(\gamma, \alpha) \leq 0$ for all $\alpha \in \Pi$, then since

$$\gamma = \sum_{\beta \in \Pi} n_\beta \beta, \;\; n_\beta \geq 0, \tag{6}$$

we get

$$(\gamma, \gamma) = \sum n_\beta (\gamma, \beta) \leq 0 \quad \text{so} \quad \gamma = 0. \tag{7}$$

□

Corollary 5

Every root $\gamma \in R^+$ *can be written as*

$$\gamma = \alpha_1 + \cdots + \alpha_k \qquad \alpha_j \in \Pi \tag{8}$$

such that $\alpha_1 + \cdots + \alpha_i$ *is a root for each* $i \leq k$*.*

Proof.

Exercise. □

Lemma 6

Let $\alpha \in \Pi$. *Then*

$$s_\alpha(R^+) = (R^+ \setminus \{\alpha\}) \cup \{-\alpha\}. \tag{9}$$

Proof.

Clearly $s_\alpha(\alpha) = -\alpha$. Suppose $\gamma \neq \alpha$ is in R^+. Then

$$\gamma = \sum_{\beta \neq \alpha} n_\beta \cdot \beta + n_\alpha \cdot \alpha \tag{10}$$

$s_\alpha(\beta) = \beta + m \cdot \alpha$ with $m \geq 0$. There is at least one $n_\beta \neq 0$ in the first sum. It follows that the coefficient of β in $s_\alpha(\gamma)$ is > 0. So all coefficients are ≥ 0. Thus $s_\alpha(\gamma) \in R^+$. □

Corollary 7

If $\rho := \frac{1}{2}\sum_{\alpha\in R^+}\alpha$, *then*

$$s_\beta(\rho) = \rho - \beta \quad \text{for any} \quad \beta\in\Pi. \tag{11}$$

Proof.

Exercise. □

Theorem 8 (Main Theorem)

Let Π be a base of R.

1. *If $h_0 \in V$ is regular, there exists $w \in W$ such that*
$$\langle w(h_0), \alpha \rangle > 0 \quad \text{for all} \quad \alpha \in \Pi.$$
2. *If $\Pi' \subset R$ is another base, there is a $w \in W$ such that $w(\Pi') = \Pi$.*
3. *If $\alpha \in R$ is any root, there is $w \in W$ such that*
$$w\alpha \in \Pi. \tag{12}$$
4. *W is generated by s_α, $\alpha \in \Pi$.*
5. *If $w(\Pi) = \Pi$, then $w = 1$.*

Proof of Main Theorem

Proof.

Let W' be the subgroup geneated by s_α, $\alpha \in \Pi$. We prove (1)-(3) for W' and then (4).

(1) Let $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$ where R^+ is defined by Π. Let $w \in W'$ be such that

$$\langle w(h_0), \rho\rangle \quad \text{is maximal.} \tag{13}$$

Let $\alpha \in \Pi$. Look at

$$(s_\alpha w(h_0), \rho) = (w(h_0), \rho - \alpha) \leq (w(h_0), \rho). \tag{14}$$

Thus $(w(h_0), \alpha) \geq 0$ as claimed; in fact > 0 because $(w(h_0), \alpha) \neq 0$ for any α.

(2) Exercise.

□

Proof of Main Theorem, continued

Proof.

(3) Because of (2), it is enough to see that for any $\alpha \in R$ there is at least one base it belongs to. Look at $V \backslash H_\alpha$. Since R is reduced, all other H_β, $\beta \neq \pm\alpha$ are distinct from H_α. Choose h_0 so that

$$(h_0, \alpha) = \varepsilon > 0 \quad |(h_0, \beta)| > \varepsilon \quad \beta \neq \pm\alpha. \tag{15}$$

Then $\alpha \in \Pi(h_0)$.
(4) Let $\alpha \in R$. Then there is $w \in W'$ such that $w(\alpha) = \alpha_1 \in \Pi$. We get

$$s_\alpha = s_{w(\alpha_1)} = w \circ s_{\alpha_1} \circ w^{-1} \tag{16}$$

so s_α is a product of simple root reflections.
$(w = s_{\beta_1} \circ s_{\beta_2} \cdots)$ □

Proof of Main Theorem, continued

Proof.

(5) Suppose $w(\Pi) = \Pi$. Write a minimal expression

$$w = s_1 \circ \cdots \circ s_k, \qquad s_i = s_{\alpha_i} \qquad \alpha_i \in \Pi \tag{17}$$

not necessarily distinct. Then write

$$w_i = s_1 \cdots s_i, \quad \text{(in particular} \quad w_k = w). \tag{18}$$

$$w(\alpha_k) = w_{k-1} \cdot s_k(\alpha_k) = w_{k-1}(-\alpha_k) = -w_{k-1}(\alpha_k). \tag{19}$$

By the assumption, $w(\Pi) = \Pi$, so $w(\alpha_k) > 0$, and therefore w_{k-1} maps α_k to a negative root. In such a case we can show that there is $t < i$ such that

$$w_{k-1} = s_1 \cdots s_{t-1}\ s_{t+1} \cdots s_{k-1} \tag{20}$$

Proof of Main Theorem, continued

continued.

Let t be the smallest so that

$$\gamma = s_{t+1} \cdots s_{k-1}(\alpha_k) > 0. \tag{17}$$

Then since $s_t(\gamma)$ is negative, $\gamma = \alpha_t$. So

$$s_t = (s_{t+1} \cdots s_{k-1}) s_k (s_{k-1} \cdots s_{t+1}). \tag{18}$$

Plug this into the reduced expression for w to find a strictly shorter expression of w in terms of simple reflections. □

Irreducible systems

We assume that the root system R is irreducible. This means that we cannot decompose $R = R_1 \cup R_2$ disjoint union so that each R_i is a root system.

Recall that $\alpha < \beta$ means that either $\beta = \alpha$ or $\beta - \alpha$ is a sum of positive roots.

This implies that if $\alpha_1 \in R_1$ and $\alpha_2 \in R_2$, then $\alpha_1 \pm \alpha_2$ cannot be a root. Furthermore, $V = V_1 \oplus V_2$ where V_i are the spans of the roots in R_i.

A root is called maximal, if for any other root α, $\beta - \alpha$ is a sum of positive roots (or is not a root).

Lemma 9

If R is irreducible, there is a unique maximal root β. Furthermore, $\beta = \sum_{\gamma \in \Pi} m_\gamma \gamma$ with all $m_\gamma > 0$.

Proof of Lemma 9

Proof.

Let β be a maximal root. Then $\beta = \sum_{\alpha \in \Pi} m_\alpha \alpha, \ m_\alpha \geq 0$.

Observe that $(\gamma, \beta) \geq 0$ for all $\gamma \in \Pi$. Otherwise $(\gamma, \beta) < 0$ implies $\gamma + \beta$ is a root, which is strictly bigger. Define
$\Pi_1 := \{\alpha \in \Pi \mid m_\alpha > 0\}, \ \Pi_2 := \{\alpha \in \Pi \mid m_\alpha = 0\}$.
If $\gamma \in \Pi_2$, then $(\gamma, \beta) = \sum_{\alpha \in \Pi_1} m_\alpha (\gamma, \alpha) \leq 0$. So $(\gamma, \beta) = 0$ and therefore also $(\gamma, \alpha) = 0$ for all $\alpha \in \Pi_1$. Then $R^+ = R_1^+ \cup R_2^+$, a contradiction. Thus all $m_\alpha > 0$ and $(\beta, \alpha) \geq 0$ for all $\alpha \in \Pi$, and $(\beta, \alpha) > 0$ for at least one simple root.
If β' is another maximal root, $(\beta, \beta') > 0$, so $\beta - \beta'$ is a root. One of β, β' cannot be maximal. □

Lemma 10

The W orbit of any root spans V.

Proof.

Follows from previous facts about irreducibility of V. See part 2 in proposition 4 in the lecture from April 2-4. □

Lemma 11

R has at most 2 root lengths.

Proof.

Let $\alpha_1,\ \alpha_2$ be two roots. We can conjugate one of the roots by the Weyl group so that they have nonzero inner product. The classification of root systems of rank 2 shows that the ratio of the lengths of such roots is either 2 or 3. If for example $\|\alpha_2\|^2 = 2\|\alpha_1\|^2$ and $\|\alpha_3\|^2 = 3\|\alpha_1\|^2$ then $\|\alpha_3\|^2 = \frac{3}{2}\|\alpha_2\|^2$, a contradiction. □

Lemma 12

The maximal root is long.

Proof.

Suppose β is maximal and α is arbitrary. We want to show $||\alpha||^2 \leq ||\beta||^2$. We can conjugate α by W so that it satisfies $(\alpha, \gamma) \geq 0$ for all $\gamma \in \Pi$. At least one $(\alpha, \gamma) > 0$. Then $(\beta, \alpha) > 0$, and $\beta - \alpha$ must be a root in R^+, so a sum of simple roots with nonnegative coefficients. Therefore $(\alpha, \beta - \alpha) \geq 0$, and

$$(\beta, \beta) = (\beta - \alpha, \beta - \alpha) + 2(\alpha, \beta - \alpha) + (\alpha, \alpha) \geq (\alpha, \alpha). \quad (19)$$

□

Classification of irreducible root systems

The irreducible (or simple) root systems are characterized by their corresponding Cartan matrices.

Definition 13 (Cartan Matrix)

An $n \times n$ matrix with integer entries $A = [a_{ij}]$ is called a generalized Cartan matrix if

1. $a_{ii} = 2$,
2. $a_{ij} \leq 0$ for $i \neq j$,
3. $a_{ij} = 0$ if and only if $a_{ji} = 0$.

It is called a Cartan matrix if in addition $detA \neq 0$.

If R is an irreducible root system, then the matrix $A = [\langle \check{\alpha}_i, \alpha_j \rangle]$ is a Cartan matrix.

Proposition 2

Two root systems with the same Cartan matrix are isomorphic.

A Cartan matrix gives rise to a **Coxeter-Dynkin Graph**. This has one vertex for each simple root $\alpha_1 \cdots \alpha_\ell$. Two of them are joined if $\langle \alpha_i, \alpha_j \rangle < 0$ as many times as the ratio of lengths.
The arrow points to shorter root.
The classification of the Dynkin diagrams is described in Humphreys chapter III.