November 18, 2004

RELEVANT AND PETITE K-TYPES FOR SPLIT GROUPS

DAN BARBASCH

1. Introduction

This paper discusses the role of the relevant and petite K-types in the classification of the spherical unitary dual for split real groups.

Let G be the rational points of a connected linear reductive group over $\mathbb{F} = \mathbb{R}$ or a p-adic field. Assume G is split, and denote a maximal abelian torus by $\mathbb{A} \cong \mathbb{F}^{\times}$. In both the real and p-adic case, any irreducible spherical representation is realized in a canonical way as a subquotient, denoted $L(\chi)$, of a principal series $X(\chi)$ for a character $\chi \in \widehat{\mathbb{A}}$. Two such representations $L(\chi)$ and $L(\chi')$ are equivalent if and only if there is an element w of the Weyl group W such that $w\chi = \chi'$. The module $L(\chi)$ admits a nondegenerate hermitian form if and only if there is $w \in W$ such that $w\chi = \overline{\chi}^{-1}$. The hermitian form is obtained from a hermitian form on $X(\chi)$; an $L(\chi)$ is unitary if and only if the form on $X(\chi)$ is positive semidefinite. Let K be a maximal compact subgroup as in section 2. For each K-type (E, μ), there is a hermitian form $A(\mu, \chi)$ which depends meromorphically on χ. Thus $L(\chi)$ is unitary if and only if $A(\mu, \chi)$ is positive semidefinite for all μ. In the p-adic case, the results in [BM1] and [BM2] reduce the problem to a finite one, namely to show that certain $a(\sigma, \chi)$ are positive semidefinite for $\sigma \in \widehat{W}$. A more detailed description of this background is in section 2.

Let M be the centralizer of $\mathbb{A}$ in K in the case of a real group. Then $(E^*)^M$ is a representation of the Weyl group. We denote it by σ. Then $A(\mu, \chi)$ induces, by Frobenius reciprocity, a hermitian form $a(\mu, \chi)$ on $(E^*)^M$. In sections 3 and 4 we define a set of K-types, which we call *petite*. They have the property that $a(\mu, \chi)$ coincides with the $a(\sigma, \chi)$ defined in the context of p-adic groups for $\sigma \in \widehat{W}$.

For p-adic groups, the full unitary dual for $GL(n)$ was obtained by Tadic, and for G2 by Muic. For the spherical case, in the classical groups it was obtained in [BM3]. In [B6], I defined a set of W representations called *relevant*, with the following properties:

(a) A representation $L(\chi)$ is unitary if and only if $a(\sigma, \chi)$ is positive semidefinite for σ relevant,

(b) For every relevant σ there is a petite representation E such that $(E^*)^M \cong \sigma$.

This implies the following result

Theorem. *A parameter χ for the real group is unitary* only if *it is unitary for the p-adic group.*

In fact in [B6] I prove th the stronger result that $L(\chi)$ for the real group is unitary *if and only if* $L(\chi)$ is unitary for the p-adic group.

For F4, the Iwahori spherical unitary dual is obtained in [C], and the spherical unitary dual in [BC]. In these cases we find a set of relevant W-representations analogous to the classical cases. The remaining sections of this paper are devoted to identifying a petite representation E for each relevant σ such that $(E^*)^M \cong \sigma$. This implies that theorem 1 holds in all cases.

The proof of the matchup between petite K-types and relevant W-types is different from the one in [B6]. Instead of identifying the groups M and W explicitly and using restrictions to Levi components, I use properties of fine K-types and tensor products. To decompose tensor products in the exceptional cases, I used the packages *GAP* and *LiE*. The results for E6-E8 contain matchups for a larger class of petite K-types than needed for theorem 1.

For the real group of type G2, the full unitary dual is contained in the work of [V]. So the *if and only if* analogue of theorem 1 is known to hold for spherical representations.

I would like to thank David Vogan for generously sharing his ideas about the relation between K-types, Weyl group representations and signatures. They were the catalyst for this paper. The paper is an outgrowth of a series of talks given at the conference in Dubrovnik in the summer of 2003. I would like to thank the organizers for giving me the opportunity of participating, and express my gratitude for their work to make the conference a success.

This research was supported by NSF grants DMS-0070561 and DMS0300172.

2. Notation and background

2.1. Let $\mathbb{F}$ be a local field, either $\mathbb{R}$ or a p-adic field. In the latter case, let $\mathcal{R} \supset \mathcal{P}$ be the ring of integers and its maximal prime ideal respectively. Let G be a split connected reductive real or p-adic group, and fix a rational Borel subgroup $B = AN$. In the real case, fix a Cartan involution θ, and let K be the maximal compact group which is its fixed points. In the p-adic case, let $K := G(\mathcal{R})$. A character $\chi \in \widehat{A}$ is called *unramified*, if $\chi\big|_{A\cap K}$ is trivial. Since G is split, $A \cong (\mathbb{F}^\times)^n$ where n is the rank of G. Thus an unramified character is of the form

$$\chi = (\chi_1, \dots, \chi_n) \tag{2.1.1}$$

where $\chi_i(t) = |t|_i^{\nu}$. Let $\mathcal{L} := (0)$ if $\mathbb{F} = \mathbb{R}$, $\mathcal{L} := (\frac{2i\pi}{\log q}\mathbb{Z})$ in the p-adic case. We can therefore identify χ with an element in $(\mathbb{C}/\mathcal{L})^n$. As is well known, any irreducible spherical admissible representation of G is the unique spherical subquotient of a *standard* induced module

$$X_B(\chi) := Ind_B^G(\chi \otimes \delta_B^{-1/2}) \tag{2.1.2}$$

where χ is an unramified character, and δ_B is the modulus function of B. Denote this subquotient by $L(\chi)$. Two such quotients $L(\chi)$ and $L(\chi')$ are isomorphic if and only if there is an element $w \in W := N_K(A)/C_K(A)$ such that $w\chi = \chi'$. More precisely, if $(Re\chi, \alpha) \geq 0$ for all roots $\alpha \in \Delta(\mathfrak{n}, \mathfrak{a})$, then $X(\chi)$ has a unique irreducible quotient, namely $L(\chi)$. We call such a χ *dominant*. If on the other hand $(Re\chi, \alpha) \leq 0$ for all $\alpha \in \Delta(\mathfrak{n}, \mathfrak{a})$, then $X(\nu)$ has a unique irreducible submodule which is again $L(\chi)$. We call such a χ *antidominant*.

Let $B' = AN'$ be another Borel subgroup. Then there is an intertwining operator

$$A(B, B', \chi) : X_B(\chi) \longleftrightarrow X_{B'}(\chi) \tag{2.1.3}$$

given by

$$A(B, B', \chi)f(g) := \int_{N'/N \cap N'} f(gn')\, dn'. \tag{2.1.4}$$

The integral is convergent for χ very dominant, and has a meromorphic extension to all χ. It has no poles if χ is dominant. Let $w \in W$, and write $B' := w(B)$. There is an isomorphism $R_w : X_{B'}(\chi) \longrightarrow X_B(w\chi)$ given by

$$R_w f(g) := f(gw^{-1}). \tag{2.1.5}$$

We write

$$A(w, \chi) := R_w \circ A(B, B', \chi) : \; X_B(\chi) \longrightarrow X_B(w\chi). \tag{2.1.6}$$

Recall that $G = KB$, and let (E, μ) be a K-type. Then the intertwining operator (2.1.3) gives rise to a map

$$A(\mu, w, \chi) : \mathrm{Hom}_K[E, X_B(\chi)] \longrightarrow \mathrm{Hom}_K[E, X_B(w\chi)]. \tag{2.1.7}$$

By Frobenius reciprocity, we get a map

$$a(\mu, w, \chi) : (E^*)^M \longrightarrow (E^*)^M. \tag{2.1.8}$$

We normalize the intertwining operator (2.1.3) so that $a(triv, w, \chi) = 1$ by multiplying it with the appropriate meromorphic function in χ.

The hermitian dual of $X_B(\chi)$ is $X_B(\chi^h)$, where $\chi^h := (\overline{\chi})^{-1}$ Indeed, if $v \in X_B(\chi)$ and $w \in X_B(\chi^h)$, then vw transforms according to δ_B under the right action of B. The hermitian pairing is

$$\big(v, w\big) := \int_{G/B} v(g)\overline{w(g)}\, dg. \tag{2.1.9}$$

Thus the hermitian dual of $L(\chi)$ is $L(\chi^h)$, and $L(\chi)$ is hermitian if and only if there is $w \in W$ such that $w\chi = \chi^h$. Suppose this is the case and

$B' := w(B)$. If χ is dominant, then χ^h is antidominant. So the image of $A(w,\chi)$ is exactly $L(\chi)$ The hermitian form is given by

$$\langle v_1, v_2\rangle := \big(v_1, A(w,\chi)(v_2)\big), \tag{2.1.10}$$

where $(\ ,\ )$ is the pairing in (2.1.9). Fix a positive definite K-invariant hermitian form for each K-type (E,μ). Then as in (2.1.5-2.1.6), we get hermitian symmetric maps $a(\mu,\chi)$. Then $L(\chi)$ is unitary if and only if $a(\mu,\chi)$ is positive semidefinite for all μ.

2.2. Suppose $Im\chi \neq 0$. Then $Im\chi$ defines a proper standard parabolic subgroup $P = LU$, where

$$\Delta(L) = \{\alpha \ :\ (\alpha, Im\chi) = 0\}, \qquad \Delta(U) = \{\alpha \ :\ (\alpha, Im\chi) > 0\}. \tag{2.2.1}$$

Let $L_M(\chi)$ be the spherical irreducible module of M corresponding to χ. Then $L_M(\chi) = L_M(\chi_r) \otimes \chi_i$ where χ_r is such that $Im\chi_r = 0$ and χ_i is unitary.

Theorem.

$$L(\chi) = Ind_M^G[L_M(\chi_r)\otimes \chi_i],$$

and $L(\chi)$ is unitary if and only if $L_M(\chi_r)$ is unitary.

Because of this theorem, we assume that χ is real, *i.e.* $Im\chi = 0$.

3. Fine K-types

3.1. Let α be a simple root and $P_\alpha = M_\alpha N$ be the standard parabolic subgroup so that M_α is θ-stable, and the Lie algebra of M_α is isomorphic to the $sl(2,\mathbb{R})$ generated by the root vectors $E_{\pm\alpha}$. We assume that $\theta E_\alpha = -E_{-\alpha}$. Let $D_\alpha = \sqrt{-1}(E_\alpha - E_{-\alpha})$ and $s_\alpha = e^{\sqrt{-1}\pi D_\alpha/2}$. Then $s_\alpha^2 = m_\alpha$ is in $M \cap M_\alpha$, while s_α itself gives rise to the nontrivial element in the Weyl group.

3.2. Let (μ_a, V_a) and (μ_b, V_b) be representations of K. Then $\mathrm{Hom}_M[V_a, V_b]$ is endowed with a representation of $N_K(M)$ via

$$n \cdot f(v) := \mu_a(n^{-1}) f(\mu_b(n)v).$$

Under this action, $M \subset N_K(M)$ acts trivially, so we get a representation of W. Note that

$$\mathrm{Hom}_M[V_a, V_b] \cong \mathrm{Hom}_M[V_a \otimes V_b^*, Triv].$$

In particular, if $V_a = E_\mu$, and $V_b = Triv$, we get the more familiar action of W on $(E_\mu^*)^M$.

Definition. *A K-type is called **fine** (Vogan), if $\mu(iD_\alpha) = 0, \pm 1$.*

More generally, a K-type is said to have level less than or equal to r, if $|\mu(iD_\alpha)| \le r$.

The fine K-types are the lowest K-types of principal series.
In the case of a linear group, M is abelian, so $\widehat{M}$ is formed of characters. Fix a representative δ for each W-orbit, and a fine K-type μ_δ. Then

$$\mu_\delta \otimes \mu_\delta^* \text{ is formed of level} \leq 2 \text{ K-types only.}$$

3.3. We will use the previous formula to determine the Weyl group representation on $\mu_\delta \otimes \mu_\delta^*$. The following properties hold.

- ${}^\vee\Delta^\delta := \{\check{\alpha} \mid \delta(m_\alpha) = 1\}$ is a roots system.
- The Weyl group generated by the roots in ${}^\vee\Delta^\delta$ is a normal subgroup of W_δ.
- The quotient $R_\delta := W_\delta/W_\delta^0$ is is a product of $\mathbb{Z}_2$'s.
- $\widehat{R_\delta}$ acts simply transitively on the fine K-types containing δ.

Inflate τ to W_δ. Having fixed a μ_δ, there is a 1-1 correspondence

$$\{\tau \in \widehat{W_\delta} \mid \tau|_{W_\delta^0} = triv\} \longleftrightarrow \{\mu_{\delta,\tau}\}, \qquad triv \longleftrightarrow \mu_\delta.$$

Theorem. *As a W-module,*

$$\mathrm{Hom}_M[\mu_{\delta,1}, \mu_{\delta,\tau}] \cong Ind_{W_\delta}^W[\tau].$$

Proof. A fine K-type has the property that its restriction to M is multiplicity free, and forms a single orbit under the action of M. The proof follows from the properties listed before the proof, and standard properties of representations of finite groups. □

4. Relevant and Petite K-types

4.1. Recall the notation from section 3.1. Since the square of any element in M is in the center and M normalizes the root vectors, $\mathrm{Ad}\, m(D_\alpha) = \pm D_\alpha$. Let (E, μ) be a K-type, and grade $E^* = \oplus E_i^*$ according to the absolute values of the eigenvalues of D_α (which are integers). Then M preserves this grading, and

$$(E^*)^M = \bigoplus_{i\ even} (E_i^*)^M.$$

The map $\psi_\alpha : sl(2, \mathbb{R}) \longrightarrow \mathfrak{g}$ determined by

$$\psi_\alpha \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = E_\alpha, \qquad \psi_\alpha \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = E_{-\alpha}$$

determines a map

$$\Psi_\alpha \ : \ SL(2, \mathbb{R}) \longrightarrow G \tag{4.1.1}$$

with image G_α, a connected group with Lie algebra isomorphic to $sl(2, \mathbb{R})$. Let A_α be the maps (2.1.6) for M_α. Recall that for $SL(2, \mathbb{R})$, K-types are parametrized by integers, and only K-types parametrized by even integers occur in the spherical principal series.

Proposition. *On* $(E^*_{2m})^M$,

$$a(2m, s_\alpha, \nu) = \begin{cases} Id & \textit{if } m = 0, \\ \prod_{0<j\le m} \frac{2j-1-<\nu,\check\alpha>}{2j-1+<\nu,\check\alpha>}\ Id & \textit{if } m \ne 0. \end{cases}$$

Proof. The formula is well known for $SL(2,\mathbb{R})$. The second assertion follows from this and the listed properties of intertwining operators. □

Let $w \in W$. Then any reduced decomposition $w = s_1 \cdots s_k$ gives rise to a factoring

$$A(w,\chi) = \prod A(s_i, s_{i+1}\cdot\ \dots\ \cdot s_k\chi). \tag{4.1.2}$$

In turn each $A(s_i, s_{i+1}\cdot \dots \cdot s_k\chi)$ is induced from the corresponding A_{α_i}.

Corollary. *In particular,* $A(w,\chi)$ *is an isomorphism unless* $\langle\nu,\check\alpha\rangle \in \mathbb{N}$ *for some root* α.

Proof. This follows from the properties of the intertwining operator listed above. □

Definition. *A K-type* μ *is called petite if each* A_{α_i} *satisfies*

$$a(\mu, s_i, \nu) = \begin{cases} Id & \textit{on the +1 eigenspace of } s_\alpha, \\ \frac{1-<\nu,\check\alpha>}{1+<\nu,\check\alpha>}\ Id & \textit{on the -1 eigenspace of } s_\alpha. \end{cases}$$

In other words, the restriction of μ to $K_\alpha := M_\alpha \cap K$ consists of K-types with $2m = 0, \pm 2$ only. The hermitian symmetric matrix $a(\mu,\chi)$ depends only on the W-structure of $(E^*_\mu)^M$.

Theorem. *Level* ≤ 3 *K-types are petite.*

4.2. **Affine Hecke algebra.** We now show that the formulas in the previous section coincide with corresponding ones in the p-adic case. Recall from [BM3] that the induced module is $X(\chi) := \mathbb{H} \otimes_{\mathbb{A}} \mathbb{C}_\chi$ where $\mathbb{H} = \mathbb{C}[W] \times \mathbb{A}$ is the graded affine Hecke algebra. The abelian subalgebra $\mathbb{A}$ is generated by $\omega \in S(\mathfrak{a})$ ($\mathfrak{a} = Lie(A)_c$) and $\mathbb{C}[W]$ is generated by $\{t_\alpha\}_{\alpha\ simple}$ satisfying $t_\alpha^2 = 1$. They are subject to the relations

$$\omega t_\alpha = s_\alpha(\omega) t_\alpha + c_\alpha < \omega, \alpha >, \qquad \omega \in \mathfrak{a}. \tag{4.2.1}$$

Because the group is assumed to be split, $c_\alpha = 1$. The intertwining operator $A(w,\nu)$ is a product of operators A_{α_i} according to a reduced decomposition of $w = s_{\alpha_1} \cdot \dots \cdot s_{\alpha_k}$. If α is a simple root, let

$$r_\alpha := (t_\alpha\check\alpha - c_\alpha)\frac{1}{\check\alpha - c_\alpha}, \qquad A_\alpha\ :\ x \otimes 1\!\!1_\nu \mapsto x r_\alpha \otimes 1\!\!1_{s_\alpha\nu}. \tag{4.2.2}$$

The $A(w,\chi)$ have the same properties as their counterparts in the real case. The r_α are being multiplied on the right, so we can replace $\check\alpha$ with $w\langle\chi,\check\alpha\rangle$ in the formulas. Furthermore,

$$\mathbb{C}[W] = \sum_{\sigma\in\widehat{W}} V_\sigma \otimes V_\sigma^*.$$

Since r_α acts as multiplication on the right, it gives rise to an operator

$$r_\sigma(s_\alpha,\nu) : V_\sigma^* \longrightarrow V_\sigma^*.$$

Theorem. *Let (E,μ) be a petite K-type such that $V_\sigma^* \cong (E_\mu^*)^M$. Then $a(\mu,w,\chi)$ on $(E_\mu^*)^M$ coincide with $r_\sigma(w,\chi)$ on V_σ^*.*

Proof. The assertion is clear from definition (4.1) and formula (4.2.2). □

Since $\mathbb{H}$ has a $*$ operation given by

$$\omega^* = -\overline{\omega} + \sum_{\alpha\in\Delta(\mathfrak{n})} (\omega,\alpha)t_\alpha, \tag{4.2.3}$$

it makes sense to talk about its unitary dual.

We remark that in the work of Lusztig and Kazhdan-Lusztig, the Hecke algebra is defined in terms of the dual root system so that there is no discrepancy between α and $\check{\alpha}$ in the formulas. For p-adic groups the determination of the Iwahori spherical unitary dual is equivalent to the determination of the unitary dual of $\mathbb{H}$. As explained in [BM1] and [BM2], an infinitesimal character is given by the conjugacy class of a semisimple element $s \in \check{G}$. In turn this element has a decomposition $s = s_e s_h$, where s_e is elliptic and s_h is hyperbolic. The Iwahori spherical dual consisting of representations with infinitesimal character s is in 1-1 correspondence with the unitary dual of an affine graded Hecke algebra $\mathbb{H}(s_e)$ with real infinitesimal character. A representation is called *spherical* if it contains the trivial W-type. The classification of spherical representations with real infinitesimal character is the same as for the real and p-adic cases. The same holds for hermitian modules.

4.3. **Relevant K-types.** *Relevant* K-types are a subset of petite K-types which are necessary and sufficient for detecting unitarity. We list them for the simple root systeems. For the classical groups they are already in [B6].

The representations of $W(A_{n-1}) = S_n$ are parametrized by partitions of n, written as $(a_1,\dots,a_k),\ a_i \le a_{i+1}$. The representations of $W(B_n) \cong W(C_n)$ are parametrized as in [L1] by pairs of partitions

$$(a_1,\dots,a_k)\times(b_1,\dots,b_l),$$
$$a_i \le a_{i+1},\quad b_j \le b_{j+1},\quad \sum a_i + \sum b_j = n. \tag{4.3.1}$$

Precisely, the representation σ parametrized by (4.3.1) is as follows. Let $k = \sum a_i,\ l = \sum b_j$. Recall that $W \cong S_n \rtimes \mathbb{Z}_2^n$. Let χ be the character of $\mathbb{Z}_2^n$ which is trivial on the first k $\mathbb{Z}_2$'s, sign on the rest. Its centralizer in S_n is $S_k\times S_l$. Let σ_1 and σ_2 be the representations of $S_k,\ S_l$ corresponding to the partitions (a) and (b). Then σ, is

$$Ind_{(S_k\times S_l)\times\mathbb{Z}_2^n}^W\ [(\sigma_1\otimes\sigma_2)\otimes\chi].$$

For $W(D_n)$, the representations are parametrized as in (4.3.1) except that $(a)\times(b)$ and $(b)\times(a)$ parametrize the same representation and when $(a)=(b)$, there are two of them $(a)\times(a)_{I,\ II}$. This is because the restriction of $(a)\times(b)$ to $W(D_n)$ is irreducible when $(a)\neq(b)$ and equal to the restriction of $(b)\times(a)$, while the restriction of $(a)\times(a)$ consists of two nonisomorphic irreducible representations labelled $(a)\times(a)_{I,\ II}$. These are usually easy to deal with.

4.4. **Orthogonal groups.** Because we are dealing with the spherical case, we can use the orthogonal group instead of its connected component. We follow Weyl's conventions to parametrize the representations of $O(n)$. Embed $O(a)\subset U(a)$ in the standard way. An irreducible representation of $O(n)$ is parametrized by

$$(a_1,\dots,a_k,0,\dots,0;\epsilon),\qquad a_i\geq a_{i+1},\ \epsilon=\pm 1. \tag{4.4.1}$$

The ϵ is (sometimes) abbreviated as $\pm$. The parameter in (4.4.1) is the irreducible representation generated by the highest weight vector of the irreducible representation of $U(a)$ with highest weight

$$(a_1,\dots,a_k,\underbrace{1,\dots,1}_{n-(1-\epsilon)k},0,\dots,0). \tag{4.4.2}$$

For $O(n,n)$ we have $K=O(n)\times O(n)$. The fine K-types (in the sense of [Vo]), and their restrictions to M are

$$\begin{array}{ll} K-type & M-type \\ (\underbrace{1,\dots,1}_{k},0,\dots,0)\otimes(0,\dots,0) & \\ & \binom{n}{k}\ characters\ \delta_k \\ (0,\dots,0)\otimes(\underbrace{1,\dots,1}_{k},0,\dots,0), & \end{array} \tag{4.4.3}$$

and we suppress the $\pm$. The meaning of (4.4.3) is that both representations of K in the left column restrict to an orbit of characters of M. The size of the orbit is $\binom{n}{k}$, and we fix a representative denoted δ_k.

The K-types

$$(0,\dots,0;+)\otimes(\underbrace{2,\dots,2}_{r},0,\dots,0;+) \tag{4.4.4}$$

$$(\underbrace{1,\dots,1}_{k},0,\dots,0;\epsilon)\otimes(\underbrace{1,\dots,1}_{k},0,\dots,0;\epsilon)\qquad \epsilon=\pm \tag{4.4.5}$$

are **relevant**. The representation of W on V^M is

$$(r,n-r)\times(0) \longleftrightarrow (4.4.4) \tag{4.4.6}$$

$$(n-k)\otimes(k), \quad k\le[n/2] \longleftrightarrow (4.4.5)\text{ with }+, \tag{4.4.7}$$

$$(n-k)\otimes(k), \quad k>[n/2] \longleftrightarrow (4.4.5)\text{ with }-. \tag{4.4.8}$$

For $O(n+1,n)$, we have $K=O(n+1)\times O(n)$. The fine K-types ([Vo]), and their restriction to M are

$$K-type \qquad\qquad M-type$$

$$(0,\dots,0)\otimes(\underbrace{1,\dots,1}_{k},0,\dots,0) \quad \binom{n}{k}\ characters\ \delta_k \tag{4.4.9}$$

The K-types

$$(0,\dots,0;+)\otimes(\underbrace{2,\dots,2}_{r},0,\dots,0;+) \tag{4.4.10}$$

$$(\underbrace{1,\dots,1}_{k},0,\dots,0;+)\otimes(\underbrace{1,\dots,1}_{k},0,\dots,0;+) \qquad \text{for } k\le[n/2] \tag{4.4.11}$$

$$(\underbrace{1,\dots,1}_{n+1-k},0,\dots,0;-)\otimes(\underbrace{1,\dots,1}_{n-k},0,\dots,0;-) \qquad \text{for } n\ge k>[n/2] \tag{4.4.12}$$

are **relevant,** and the corresponding representations of $W(B_n)$ on V^M are

$$(r,n-r)\times(0), \quad r\le n/2 \longleftrightarrow (4.4.11) \tag{4.4.13}$$

$$(n-k)\otimes(k), \quad k\le[n/2] \longleftrightarrow (4.4.12)\text{ with }+, \tag{4.4.14}$$

$$(n-k)\otimes(k), \quad k>[n/2] \longleftrightarrow (4.4.12)\text{ with }-. \tag{4.4.15}$$

Theorem ([B6])**.** *A spherical irreducible representation for $\mathbb{H}$ is unitary if and only if the hermitian form is positive definite on the W-types*

$$(r,n-r)\times(0), \qquad (n-k)\times(k).$$

Corollary ([B6])**.** *A spherical representation $L(\chi)$ for a real split orthogonal group is unitary only if the corresponding $L(\chi)$ for $\mathbb{H}$ is unitary.*

In other words, the set of spherical unitary parameters for the real split group is contained in the set of spherical unitary parameters for $\mathbb{H}$. In fact in [B6] the stronger result is proved that the unitary duals of G and $\mathbb{H}$ coincide.

4.5. **Symplectic groups.** The maximal compact subgroup of $Sp(n,\mathbb{R})$ is $U(n)$. Its irreducible representations are parametrized by

$$(a_1,\dots,a_n), \qquad a_i\ge a_{i+1},\ a_i\in\mathbb{Z}. \tag{4.5.1}$$

The fine K-types and their restrictions to M are

$$\begin{array}{ll} K-type & M-type \\ \mu_+(k) := (\underbrace{1,\dots,1}_{k},0,\dots,0) & \\ & \binom{n}{k}\ characters\ \delta_k \\ \mu_-(k) := (0,\dots,0,\underbrace{-1,\dots,-1}_{k}). & \end{array} \tag{4.5.2}$$

The relevant K-types are

$$(\underbrace{2,\dots,2}_{k},0,\dots,0) \longleftrightarrow (n-k)\times(k), \tag{4.5.3}$$

$$(0,\dots,0,\underbrace{-2,\dots,-2}_{k}) \longleftrightarrow (k)\times(n-k), \tag{4.5.4}$$

$$(\underbrace{1,\dots,1}_{r},0,\dots,0,\underbrace{-1,\dots,-1}_{r}) \longleftrightarrow (r,n-r), \tag{4.5.5}$$

Theorem ([B6])**.** *A spherical representation for $\mathbb{H}$ is unitary if and only if the hermitian form is positive definite on the W-types*

$$(r,n-r)\times(0), \qquad (n-k)\times(k).$$

Corollary. *A spherical representation $L(\chi)$ for a real split orthogonal group is unitary only if the corresponding $L(\chi)$ for $\mathbb{H}$ is unitary.*

In fact in [B6] the stronger result is proved that the unitary duals of G and $\mathbb{H}$ coincide.

4.6. **General linear group.** The maximal compact subgroup of $GL(n,\mathbb{R})$ is $O(n)$. The fine K-types and their restrictions to M are

$$\begin{array}{ll} K-type & M-type \\ (\underbrace{1,\dots,1}_{k},0,\dots,0) & \binom{n}{k}\ characters\ \delta_k \end{array} \tag{4.6.1}$$

According to [V1], a set of relevant K-types is given by

$$(\underbrace{2,\dots,2}_{k},0,\dots,0). \tag{4.6.2}$$

The corresponding representation of S_n on V^M is, according to [B6],

$$(\underbrace{2,\dots,2}_{k},0,\dots,0) \longleftrightarrow (n-k,k) \tag{4.6.3}$$

In the case n even, we have

$$(2,\dots,2,\pm 2) \longleftrightarrow (n/2,n/2)_{I,II}. \tag{4.6.4}$$

The theorems and corollaries analogous to the ones in sections 4.4 and 4.5 follow from [V1] and [T]. For a construction of the relevant K-types for

$SL(n)$ starting from the Weyl gorup module on E^M, the interested reader may consult [P].

4.7. **G2.** We refer to [V] for more details about the structure of the split group of type $G2$. The split real form which consists of the real points of the complex simply connected linear group of type $G2$ has

$$K := [SU(2)_s \times SU(2)_l]/\{\pm Id\}. \tag{4.7.1}$$

It is more convenient to work with the (nonlinear) double cover for which

$$\widetilde{K} = SU(2)_s \times SU(2)_l. \tag{4.7.2}$$

The Borel subgroup is $\widetilde{B} = \widetilde{M}AN$ where $\widetilde{M}$ is the group of order 8 which contains the central subgroup $\{\pm Id\}$ in (4.7.1) as a central sugroup, and quotient $\mathbb{Z}_2 \times \mathbb{Z}_2$. Let T be the maximal compact subgroup. We identify the Lie algebra $\mathfrak{t}$ with $\{(a_1, a_2, a_3) \mid a_1 + a_2 + a_3 = 0\}$, and use the standard positive system of roots

$$\{(1,-1,0),\ (-1,2,-1),\ (0,1,-1),\ (1,0,-1),\ (1,1,-2),\ (2,-1,-1)\}. \tag{4.7.3}$$

A K-type is determined by its highest weight which we denote

$$(p \mid q) \longleftrightarrow (q, \frac{p-q}{2}, \frac{-p-q}{2}), \qquad p,\ q \in \mathbb{N}. \tag{4.7.4}$$

The positive roots in $\mathfrak{k}$ are $(0,1,-1)$ and $(2,-1,-1)$. The fine K-types in the sense of [Vo] are

$$\begin{array}{lr} \widetilde{K}-type & \widetilde{M}-type \\ (0 \mid 0) & \delta_0, \quad \text{the trivial character}, \\ (1 \mid 0) & \delta_2, \quad \text{a representation of dimension 2}, \\ (2 \mid 0) & \delta_3, \quad \text{three characters}. \end{array} \tag{4.7.5}$$

The one dimensional represntations all factor to M, while the two dimensional one is a genuine representation of $\widetilde{M}$. The Weyl group is generated by s_s, the reflection about the simple short root, and s_l, the reflection about the simple long root. Its representations are parametrized as in [L1],

$$\begin{array}{lr} 1_1 & trivial \\ 1_2 & s_s = 1,\ s_l = -1, \\ 1_3 & s_s = -1,\ s_l = +1, \\ 1_4 & s_s = -1,\ s_l = -1, \\ 2_1 & reflection, \\ 2_2 & 2_1 \otimes 1_2. \end{array} \tag{4.7.6}$$

Theorem ([C]). *A spherical representation of* $\mathbb{H}$ *is unitary if and only if the* r_σ *are positive definite on the* W*-types*

$$1_1,\ 2_1,\ 2_2.$$

Proposition. *The following list consists of petite K-types.*

$K-type$	*W-type on* $(V^*)^M$
$(0 \mid 0)$	1_1,
$(3 \mid 1)$	2_1,
$(4 \mid 0)$	2_2.

More detailed results along these lines can be found in [C].

4.8. **F4.** The split real form which is the rational points of the complex linear group of type F4 has maximal compact subgroup

$$K := [Sp(1) \times Sp(3)]/\{\pm Id\}. \tag{4.8.1}$$

It is more convenient to use the double cover which has

$$\widetilde{K} = [Sp(1) \times Sp(3)] \tag{4.8.2}$$

The Borel subgroup is $\widetilde{B} = \widetilde{M}AN$, where $\widetilde{M}$ is a finite nonabelian group of order 2^5. It has a central subgroup of order 2 (the $\pm Id$ in (4.8.1)) such that the quotient is $\mathbb{Z}_2^4$.

Let T be a maximal compact Cartan subgroup. We use the standard positive system and roots,

$$\{2\epsilon_1,\ 2\epsilon_k,\ \epsilon_k \pm \epsilon_\ell\}_{2\le k\le \ell\le 4}. \tag{4.8.3}$$

The highest weight of a $\widetilde{K}$-type will be denoted

$$(a_1 \bigm| a_2, a_3, a_4, a_5), \qquad a_i \in \mathbb{N}, \quad a_2 \ge a_3 \ge a_4 \ge 0. \tag{4.8.4}$$

The *fine* K-types (in the sense of [Vo]), and their restrictions to $\widetilde{M}$ are

$\widetilde{K}-type$	$\widetilde{M}-type$	W_δ	
$(0 \mid 0,0,0)$	δ_0, one character	F_4,	
$(1 \mid 0,0,0)$	δ_2, a representation of dimension 2	F_4	
$(2 \mid 0,0,0)$	δ_3, three characters	C_4,	(4.8.5)
$(0 \mid 1,0,0)$	δ_6, three representations of dimension 2	B_4,	
$(1 \mid 1,0,0)$	δ_{12}, twelve characters	B_3A_1	

The 1-dimensional representations all factor to M. The remaining ones are genuine for $\widetilde{M}$.

The representations of the Weyl group of type F4 are parametrized as in [L1].

Theorem ([C])**.** *A spherical representation of* $\mathbb{H}$ *of type F4 is unitary if and only if the* r_σ *are positive semidefinite on the W-types*

$$1_1,\ 2_3,\ 8_1,\ 4_2,\ 9_1.$$

The next result was obtained joint with D. Vogan.

Proposition. *The following list consists of petite K-types.*

$K-type$	W-type on $(V^*)^M$
$(0 \mid 0,0,0)$	$1_1,$
$(0 \mid 1,1,0)$	$2_1,$
$(4 \mid 0,0,0)$	$2_3,$
$(1 \mid 2,1,0)$	$8_1,$
$(1 \mid 1,1,1)$	$4_2,$
$(2 \mid 2,0,0)$	$9_1.$

Corollary. *A spherical representation for the real split group G of type F4 is unitary only if the corresponding $L(\chi)$ for $\mathbb{H}$ is unitary.*

The results in [C] provide an explicit list of the unitary dual of $\mathbb{H}$. The unitarity of the unipotent representations and the relevant irreducibility results are not (yet) available.

4.9. **E6.** The maximal compact subgroup of the rational points of the simply connected complex group of type E6 is

$$K = Sp(4)/\{\pm Id\}. \tag{4.9.1}$$

Again it is more convenient to use the double cover which has

$$\widetilde{K} = Sp(4). \tag{4.9.2}$$

We use the same conventions as in section 4.8 for the positive roots and highest weights of representations of $\widetilde{K}$. The finite group $\widetilde{M}$ has size 2^7, and its quotient by the central group $\{\pm Id\}$ is a $\mathbb{Z}_2^6$. The fine K-types and their restrictions to $\widetilde{M}$ are

$\widetilde{K}-type$	$\widetilde{M}-type$	W_δ^0
$(0,0,0,0)$	δ_1, the trivial character,	E_6,
$(1,0,0,0)$	δ_8, a representation of dimension 8,	E_6,
$(1,1,0,0)$	δ_{27}, twenty seven characters,	D_5,
$(2,0,0,0)$	δ_{36}, thirty six characters,	A_5A_1.

(4.9.3)

The second representation is genuine for $\widetilde{M}$, the others all factor to M.

The W-types are parametrized as in [L1].

Theorem ([BC])**.** *A spherical representation of $\mathbb{H}$ of type E6 is unitary if and only if the r_σ are positive semidefinite on the W-types*

$$1_p,\ 6_p,\ 20_p,\ 30_p,\ 15_q.$$

We denote by ω_i the fundamental weights of $sp(4)$. In coordinates they are $\omega_i = \sum_{j\le i}\epsilon_j$.

Proposition. *The following list consists of the petite K-types which have no nontrivial M-fixed vectors, and the underlying Weyl group representations.*

$K-type$	*W-type on* $(V^*)^M$
$(0)=(0,0,0,0)$	$1_p,$
$\omega_4=(1,1,1,1)$	6_p
$2\omega_2=(2,2,0,0)$	$20_p,$
$4\omega_1=(4,0,0,0)$	$15_q,$
$2\omega_1+\omega_4=(3,1,1,1)$	$30_p,$
$\omega_1+\omega_2+\omega_3=(3,2,1,0)$	$64_p,$
$3\omega_1+\omega_3=(4,1,1,0)$	$60_p,$
$2\omega_3=(2,2,2,0)$	$15_p,$
$2\omega_1+2\omega_2=(4,2,0,0)$	$81_p,$
$3\omega_2=(3,3,0,0)$	$24_p,$
$6\omega_1=(6,0,0,0)$	$24'_p.$

Corollary. *A spherical representation* $L(\chi)$ *for the real group* G *of type E6 is unitary only if the corresponding representation for* $\mathbb{H}$ *is unitary.*

4.10. **E7.** The maximal compact subgroup of the split real form of the simply connected group of type E7 is

$$K=SU(8)/\{\pm Id\}. \tag{4.10.1}$$

We work with the double cover for which

$$\widetilde{K}=SU(8). \tag{4.10.2}$$

The finite group $\widetilde{M}$ has size 2^8, and its quotient by the center in (4.10.1) is a $\mathbb{Z}_2^7$. The fine K-types are

(4.10.3)

$\widetilde{K}-type$	$\widetilde{M}-type$	W^0_δ
(0)	δ_1, trivial representation,	E_7
ω_1	δ_8, eight dimensional representation,	E_7
ω_7	δ_8^*, eight dimensional representation,	E_7
$\omega_2,\ \omega_6$	δ_{28}, twenty eight characters,	E_6
$2\omega_1,\ 2\omega_7$	δ_{36}, thirty six characters,	A_7
$\omega_1+\omega_7$	δ_{63}, sixty three characters,	D_5

In (4.10.3), the ω refer to the fundamental weights of $\widetilde{K}$, usual labelling.

The second and third are genuine representations of $\widetilde{M}$, the others are single orbits under the action of W.

The Weyl group representations are parametrized as in [L1].

Theorem ([BC])**.** *A spherical representation of* $\mathbb{H}$ *of type E7 is unitary if and only if* r_σ *is positive semidefinite for*

$$1_a,\ 7'_a,\ 27_a,\ 56'_a,\ 21'_b,\ 35_b,\ 105_b.$$

Proposition. *The following list gives the petite K-types with M-spherical vectors, and the corresponding Weyl group representations:*

$K-type$	*W-type on* $(V^*)^M$
(0)	$1_a,$
ω_4	$7'_a$
$2\omega_2,\ 2\omega_6$	$21'_b,$
$\omega_2+\omega_6$	$27_a,$
$2\omega_1+2\omega_7$	$35_b,$
$4\omega_1,\ 4\omega_7$	$15'_a,$
$\omega_2+\omega_3+\omega_7,\ \omega_1+\omega_5+\omega_6$	$105'_a,$
$\omega_1+\omega_4+\omega_7$	$56'_a,$
$2\omega_1+\omega_3+\omega_7,\ \omega_1+\omega_5+2\omega_7$	$189'_b,$
$\omega_1+\omega_3+\omega_6,\ \omega_2+\omega_5+\omega_7$	$168_a,$
$3\omega_1+\omega_5,\ \omega_3+3\omega_7$	$105_b,$
$\omega_3+\omega_5$	$21_a,$
$\omega_1+\omega_2+\omega_5,\ \omega_3+\omega_6+\omega_7$	$120_a,$
$\omega_1+\omega_2+\omega_6+\omega_7,$	$168_a+210_a,$
$3\omega_1+3\omega_7,$	$84_a+105_c,$
$\omega_1+2\omega_2+\omega_7, \omega_1+\omega_6+\omega_7,$	$189'_c$
$3\omega_1+\omega_2+\omega_7, \omega_1+\omega_6+3\omega_7,$	$216'_a,$
$3\omega_1+\omega_6+\omega_7, \omega_1+\omega_2+3\omega_7,$	$280_b,$
$5\omega_1+\omega_7, \omega_1+5\omega_7,$	$84'_b.$

Corollary. *A spherical representation* $L(\chi)$ *for the real group G of type E7 is unitary only if the corresponding representation for* $\mathbb{H}$ *is unitary.*

4.11. **E8.** The maximal compact subgroup of the split real form of the simply connected complex group of type E8 is

$$Spin(16)/\{Id, \omega\}, \tag{4.11.1}$$

for ω the appropriate element of order two in the center (the quotient is **not** SO(16)). As before we work with the double cover which has

$$\widetilde{K} = Spin(16). \tag{4.11.2}$$

The group $\widetilde{M}$ has size 2^9, and its quotient by the center in (4.11.1) is $\mathbb{Z}_2^8$. The fine K-types are

$$\begin{array}{ll} \widetilde{K}-type & \widetilde{M}-type \\ (0) & \delta_1, \text{ trivial representation,} \\ \omega_1 & \delta_{16}, \text{ sixteen dimensional representation,} \\ \omega_2 & \delta_{120}, \text{ one hundred and twenty characters,} \\ 2\omega_1 & \delta_{135}, \text{ one hundred thirty five characters,} \end{array} \tag{4.11.3}$$

Only the second representation is genuine.

The representations of the Weyl group are parametrized as in [L1].

Theorem. *A spherical representation of $\mathbb{H}$ of type E8 is unitary if and only if r_σ is positive semidefinite for*

$$1_x,\ 8_z,\ 35_x,\ 50_x,\ 84_x,\ 112_z,\ 400_z,\ 300_x,\ 210_x.$$

Proposition. *The following list gives petite K-types and the corresponding Weyl group representations on $(V^*)^M$:*

$K-type$	W-type on $(V^*)^M$
(0)	$1_x,$
ω_8	8_z
ω_4	$35_x,$
$2\omega_2$	$84_x,$
$\omega_2+\omega_8$	$112_z,$
$4\omega_1$	$50_x,$
$3\omega_1+\omega_7$	$400_z,$
$\omega_3+\omega_7$	$160_z,$
ω_6	$28_x,$
$\omega_1+\omega_5$	$210_x,$
$\omega_1+\omega_2+\omega_7$	$560_x,$
$\omega_2+\omega_4$	$567_x,$
$2\omega_3$	$300_x,$
$2\omega_1+\omega_4$	$700_x,$
$3\omega_1+\omega_3$	$1050_x,$
$\omega_1+\omega_2+\omega_3$	$1344_x,$
$3\omega_2$	$525_x,$
$2\omega_1+2\omega_2$	$972_x,$
$4\omega_1+\omega_2$	$700_{xx},$
$6\omega_1$	$168_y.$

Corollary. *A spherical representation $L(\chi)$ for the real group G of type E7 is unitary only if the corresponding representation for $\mathbb{H}$ is unitary.*

5. Some proofs

5.1. Let $\widetilde{K}$ be a compact group, $\widetilde{M} \subset \widetilde{N}$ finite groups such that $\widetilde{M}$ is normal in $\widetilde{N}$. Denote by W the quotient $\widetilde{N}/\widetilde{M}$. In the applications, $\widetilde{N}$ is the normalizer of $\mathfrak{a}$ in $\widetilde{K}$, which is simply connected.

Let $V_a,\ V_b$ be representations of $\widetilde{K}$. In the nicer cases, the restrictions of $V_a,\ V_b$ to M are multiples of the same representation V_δ, which in turn extends to a representation of $\widetilde{K}$. This is the case for δ_{16} and genuine representations of $\widetilde{K}$.

Proposition. *There is a natural action of W on $\mathrm{Hom}_M[V_a, V_b]$.*

Proof. The action on $f \in \mathrm{Hom}[V_a, V_b]$ is given by

$$(n \cdot f)(v) = \mu_a(n^{-1})f(\mu_b(n)v). \tag{5.1.1}$$

This is clearly an action of $\widetilde{N}$. Since $nm = (nmn^{-1})n$, and $\widetilde{M}$ is normal, $nmn^{-1} \in \widetilde{M}$. Then the fact that the action does not depend on the right M-coset, follows from the fact that f is an M-homomorphism. □

This action is compatible with the canonical isomorphism

$$\mathrm{Hom}_M[V_a, V_b] \cong [V_a^* \otimes V_b]^M. \tag{5.1.2}$$

It is also compatible with the isomorphism

$$\mathrm{Hom}_M[V_a, V_b]^* \cong \mathrm{Hom}[V_b^*, V_a^*]. \tag{5.1.3}$$

This action is a generalization of the usual one on V^M.

5.2. Now consider two genuine modules $V_a,\ V_b$. Then there is a homomorphism

$$\mathrm{Hom}_M[V_a, V_\delta] \otimes \mathrm{Hom}_M[V_\delta, V_a] \longrightarrow \mathrm{Hom}_M[V_a, V_b]. \tag{5.2.1}$$

This is compatible with the action of W. When $V_a,\ V_b$ are multiples of V_δ, and V_δ extends irreducibly to $\widetilde{K}$, this is an isomorphism.

5.3. **Orthogonal groups.** In the case of $SO(n+1, n)$, the Weyl group is $W(B_n)$. The centralizer $W_\delta^0 = W_\delta$ corresponding to $(0) \otimes (\underbrace{1, \dots, 1}_{k}, 0, \dots, 0)$ is isomorphic to $W(C_k) \times C_{n-k})$. Then

$$Ind_{W(C_k \times C_{n-k})}^{W(C_n}[triv] = \sum (n-k+\ell, k-\ell) \times (0). \tag{5.3.1}$$

The corresponding tensor product is

$$\sum (0) \otimes (\underbrace{2, \dots, 2}_{a}, 1, \dots, 1, \underbrace{0, \dots, 0}_{b}). \tag{5.3.2}$$

These K-types are automatically level ≤ 2. The formulas imply that these relevant W-types are realized in the M-types of petite K-types. This is sufficient for the purpose of corollary 4.4 which says that a parameter for the real group is unitary only if it is unitary for the p-adic group. For the more precise results in (4.4.13-4.4.15) one needs to show first that the factors occuring in (5.3.2 with $a > 0$ do not have any M fixed vectors. This can be done by showing that such factors occur in tensor products of distinct fine K-types which cannot have any M fixed vectors. To sort out the matchup of the remaining K-types with the W-represntations, one needs to compare the formulas (5.3.1) and (5.4.2) for various values of k.

Tensor products of fine K-types of $SO9n{+}1, n)$ are not sufficient to realize the reamining relevant Weyl group representations as M-fixed vectors of petite K-types. To achieve this, we observe that $Spin(n+1, n)$, the rational points of $Spin(2n+1, \mathbb{C})$ has another pair of fine K-types

$$spin \otimes spin^{\pm}. \tag{5.3.3}$$

In this case $W_\delta^0 = S_n$, while $W_\delta = S_n \times \mathbb{Z}_2$. Then

$$\begin{aligned} Ind_{S_n \times \mathbb{Z}_2}[triv \otimes triv] &= \sum (n-k) \times (k), \qquad k \ even, \\ Ind_{S_n \times \mathbb{Z}_2}[triv \otimes sgn] &= \sum (n-k) \times (k), \qquad k \ odd. \end{aligned} \tag{5.3.4}$$

We combine these formulas with

$$\begin{aligned} spin \otimes spin &= \sum (\underbrace{1, \dots, 1}_{k}, 0, \dots, 0), \qquad \text{for Spin(2n+1)}, \\ spin^{+} \otimes spin^{-} &= \sum_{k \ even} (\underbrace{1, \dots, 1}_{n-k-1}, 0, \dots, 0), \\ spin^{+} \otimes spin^{+} &= (1, \dots, 1) + \sum_{k>0 \ even} (\underbrace{1, \dots, 1}_{n-k}, 0, \dots, 0), \\ spin^{-} \otimes spin^{-} &= (1, \dots 1, -1) + \sum_{k>0 \ even} (\underbrace{1, \dots, 1}_{n-k}, 0, \dots, 0). \end{aligned} \tag{5.3.5}$$

This proves that the sum of petite representations in formulas (4.4.9) contains the sum of all the relevant W-representations. This is sufficient for corollary 4.4.

In the case $O(n, n)$, we can in fact derive the matchup of the petite K-types with the relevant W-representations as in (4.4.6-eq:4.4.7) in a manner similar to the symplectic group case.

5.4. **Symplectic groups.** The basic tool is the tensor product formulas

$$\begin{aligned} \mu_{+}(k) \otimes \mu_{-}(k) &= \sum (\underbrace{1, \dots, 1}_{a}, \underbrace{0, \dots, 0}_{b}, -1, \dots, -1), \\ \mu_{+}(k) \otimes \mu_{+}(k) &= \sum (\underbrace{2, \dots, 2}_{a}, \underbrace{1, \dots, 1}_{b}, 0, \dots, 0), \qquad 2k \leq n \end{aligned} \tag{5.4.1}$$

with $2a + b = 2k$. These K-types are automatically level ≤ 2, so petite. The stabilizer W^0_δ for $\mu_\pm(k)$ is $W(D_k) \times W(C_{n-k})$, while $W_\delta = W(C_k) \times W(C_{n-k})$. Then

$$Ind^{W(C_k)}_{W(D_k)}[triv] = (k) \times (0) + (0) \times (k). \tag{5.4.2}$$

The induced modules corresponding to (5.4.1) are

$$\begin{aligned} &Ind^{W(C_n)}_{W(C_k)\times W(C_{n-k})}[(k) \times (0) \otimes (n-k) \times (0)] = \sum (n-k+\ell, k-\ell) \times (0),\\ &Ind^{W(C_n)}_{W(C_k)\times W(C_{n-k})}[(0) \times (k) \otimes (n-k) \times (0)] = (n-k) \times (k). \end{aligned} \tag{5.4.3}$$

These results imply that any relevant Weyl group representation is realized in $(E^+_\mu)^M$ for some petite K-type. To get the more precise result of (4.5.3-4.5.4), one needs to decompose $\mu_+(r) \otimes \mu_+(s)$ for all $r,\ s$. For $r \neq s$, none of the factors have M fixed vectors. Then one makes a comparison of the remaining representations occuring in the tensor products for $r = s$ with the induced representations of the corresponding Weyl groups.

5.5. **G2.** The stabilizer of δ_3 is the subgroup $W(A_{1,s} \times A_{1,l})$. The induced module

$$Ind^{W(G2)}_{W(A_{1,s}\times A_{1,l}}[triv] = 1_1 + 2_2 \tag{5.5.1}$$

corresponds to the tensor product

$$(2 \mid 0) \otimes (2 \mid 0) = (4 \mid 0) + (2 \mid 0) + (0 \mid 0). \tag{5.5.2}$$

On the other hand,

$$(1 \mid 0) \otimes (1 \mid 0) = (2 \mid 0) + (0 \mid 0). \tag{5.5.3}$$

Since the M-fixed vectors in the tensor product have dimension 1, it follows that $(2 \mid 0)$ has no M fixed vectors. Combining this with the fact that the M fixed vectors of $\mathfrak{s} = (3 \mid 1)$ equal the reflection representation, we conclude

$$\begin{aligned} (0 \mid 0) &\longleftrightarrow\ 1_1,\\ (4 \mid 0) &\longleftrightarrow\ 2_2,\\ (3 \mid 1) &\longleftrightarrow\ 2_1. \end{aligned} \tag{5.5.4}$$

5.6. **F4.** We first compute some tensor products of the δ's. In the process we get the restrictions of the $\widetilde{K}$-types in theorem 4.8 to $\widetilde{M}$.

- $\delta_2 \otimes \delta_2 = \delta_0 + \delta_3$ because it equals
$$(1 \mid 0,0,0) \otimes (1 \mid 0,0,0) = (2 \mid 0,0,0) + (0 \mid 0,0,0).$$
- $\delta_2 \otimes \delta_3$ equals
$$(1 \mid 0,0,0) \otimes (2 \mid 0,0,0) = (3 \mid 0,0,0) + (1 \mid 0,0,0) = 2\delta_2 + \delta_2,$$
because there is no room for a δ_6. So we also get $(3 \mid 0,0,0) = 2\delta_2$.
- $\delta_2 \otimes \delta_6$ equals
$$(1 \mid 0,0,0) \otimes (0 \mid 1,0,0) = (1 \mid 1,0,0) = \delta_{12}.$$

- $\delta_2\otimes\delta_{12}=\delta_2\otimes\delta_2\otimes\delta_6$ equals
$$(\delta_1+\delta_3)\otimes\delta_6=\delta_6+\delta_3\otimes\delta_6.$$
It also equals
$$(1\mid 0,0,0)\otimes(1\mid 1,0,0)=(2\mid 1,0,0)+(0\mid 1,0,0)=\delta_3\otimes\delta_6+\delta_6=4\delta_6.$$
Then $\delta_3\otimes\delta_6=a\delta_2+b\delta_6$. But
$$\delta_2\otimes\delta_3\otimes\delta_6=3\delta_2\otimes\delta_6=3\delta_{12},$$
so $a=0$. So also $(2\mid 1,0,0)=\delta_3\otimes\delta_6=3\delta_6$.
- $\delta_6\otimes\delta_6=3\delta_1+a\delta_3+b\delta_{12}$. Because
$$\delta_3\otimes\delta_6\otimes\delta_6=3\delta_6\otimes\delta_6,$$
$a=3$ and $b=2$. Since also
$$(0\mid 1,0,0)\otimes(0\mid 1,0,0)=(0\mid 2,0,0)+(0\mid 1,1,0)+(0\mid 0,0,0),$$
we conclude
$$\begin{aligned}(0\mid 2,0,0)&=3\delta_3+\delta_{12},\\(0\mid 1,1,0)&=2\delta_1+\delta_{12}.\end{aligned}$$

We now compute the induced modules
$$\begin{aligned}&Ind_{W(B_4)}^{W(F_4)}[triv]=1_1+2_1,\\&Ind_{W(C_4)}^{W(F_4)}[triv]=1_1+2_3,\\&Ind_{W(B_3A_1)}^{W(F_4)}[triv]=1_1+2_1+9_1,\end{aligned}\tag{5.6.1}$$
corresponding to the tensor products
$$\begin{aligned}&(0\mid 1,0,0)\otimes(0\mid 1,0,0)=(0\mid 2,0,0)+(0\mid 1,1,0)+(0\mid 0,0,0),\\&(2\mid 0,0,0)\otimes(4\mid 0,0,0)=(2\mid 0,0,0)+(0\mid 0,0,0),\\&(1\mid 1,0,0)\otimes(1\mid 1,0,0)=(2\mid 2,0,0)+(2\mid 1,1,0)+(2\mid 0,0,0)+\\&+(0\mid 2,0,0)+(0\mid 1,1,0)+(0\mid 0,0,0).\end{aligned}\tag{5.6.2}$$
Noting also that
$$(2\mid 2,0,0)=(2\mid 0,0,0)\otimes(0\mid 2,0,0)=\delta_3\otimes(3\delta_3+\delta_{12},\tag{5.6.3}$$
we conclude that the M-fixed vectors of $(2\mid 0,0,0)$ are nine dimensional, and therefore
$$\begin{aligned}(0\mid 1,1,0)&\longleftrightarrow 2_1,\\(4\mid 0,0,0)&\longleftrightarrow 2_3,\\(2\mid 2,0,0)&\longleftrightarrow 9_1.\end{aligned}\tag{5.6.4}$$

Because $(1\mid 1,1,1)$ is the representation of K on $\mathfrak{s}$, the Weyl group representation on the fixed vectors is the reflection representation 4_2.

Now consider
$$\mathrm{Hom}_M[(1\mid 1,0,0),(0\mid 1,1,0)]=\mathrm{Hom}_M[\delta_{12},2\delta_1+\delta_{12}].\tag{5.6.5}$$

It is therefore 12-dimensional, so induced from a character of $W(B_3A_1)$. The tensor product

$$(1 \mid 1,0,0) \otimes (0 \mid 1,1,0) = (1 \mid 2,1,0) + (1 \mid 1,1,1) + (1 \mid 1,0,0) \tag{5.6.6}$$

has the property that the W-representation on the M-fixed vectors is induced from this 1-dimensional character. It follows that the M-fixed vectors of $(1 \mid 2,1,0)$ are 8-dimensional. To determine what represntation it is, we note that the restriction of 4_2 is

$$4_2 \longrightarrow [(2\times 1)\otimes(2)] + [(3\times 0)\otimes(1^2)], \tag{5.6.7}$$

and the induced module from the character is

$$[(3\times 0)\otimes(1^2)] \longrightarrow 4_2 + 8_1. \tag{5.6.8}$$

The proof of theorem 4.8 follows.

5.7. **E6.** The proof will occupy several sections. We will exploit the fact that genuine representations are all multiples of δ_8.

The following induced modules,

$$\begin{aligned} Ind_{W(D5)}^{W(E6)}[triv] &= 1_p + 6_p + 20_p,\\ Ind_{W(A5A1)}^{W(E6)}[triv] &= 1_p + 15_q + 20_p, \end{aligned} \tag{5.7.1}$$

correspond to the tensor products

$$\begin{aligned} \omega_2\otimes\omega_2 &= (2\omega_1) + (\omega_1+\omega_3) + (2\omega_2) + (\omega_4) + (\omega_2) + (0),\\ 2\omega_1\otimes 2\omega_1 &= (4\omega_1) + (2\omega_1+\omega_2) + (2\omega_1) + (2\omega_2) + (\omega_2) + (0). \end{aligned} \tag{5.7.2}$$

We know that $\omega_4 \longleftrightarrow 6_p$ because this is the representation of $\widetilde{K}$ on $\mathfrak{s}$ of the Cartan decomposition. We conclude that

$$\begin{aligned} (2\omega_1+\omega_2) + (2\omega_2) &\longleftrightarrow 20_p,\\ (\omega_1+\omega_3) + (2\omega_2) &\longleftrightarrow 20_p. \end{aligned} \tag{5.7.3}$$

But $(2\omega_1)\otimes(\omega_2)$ does not have any M-fixed vectors, so $2\omega_1+\omega_2$ which is a factor does not have any either. We conclude that

$$\begin{aligned} (0) &\longleftrightarrow 1_p,\\ (\omega_4) &\longleftrightarrow 6_p,\\ (2\omega_2) &\longleftrightarrow 20_p,\\ (4\omega_1) &\longleftrightarrow 15_q, \end{aligned} \tag{5.7.4}$$

and the rest of the representations considered so far do not have any M-fixed vectors. Using tensoring with ω_1 we also find that

$$\begin{aligned} (\omega_1) &\longleftrightarrow 1_p,\\ (\omega_3) &\longleftrightarrow 6_p,\\ (\omega_1+\omega_2) &\longleftrightarrow 20_p,\\ (3\omega_1) &\longleftrightarrow 15_q, \end{aligned} \tag{5.7.5}$$

5.8. The tensor product

$$\omega_3 \otimes \omega_3 = (2\omega_1) + (\omega_1 + \omega_3) + (2\omega_2) + (\omega_2 + \omega_4) + (\omega_2) + (2\omega_3) + (0) \quad (5.8.1)$$

corresponds to

$$6_p \otimes 6_p = 1_p + 20_p + 15_p. \quad (5.8.2)$$

Since $\omega_4 = 6\delta_1 + \delta_{36}$, $\omega_2 \otimes \omega_4$ does not have any M-fixed vectors. Thus $\omega_2 + \omega_4$ doesn't either. We conclude

$$(2\omega_3) \longleftrightarrow 15_p. \quad (5.8.3)$$

5.9. The representations $(3\omega_1)$ and ω_3 are genuine, and

$$3\omega_1 \otimes \omega_3 = (3\omega_1 + \omega_3) + (2\omega_1 + \omega_2) + (2\omega_1 + \omega_4) + (\omega_1 + \omega_3) \quad (5.9.1)$$

corresponds to

$$6_p \otimes 15_q = 30_p + 60_p. \quad (5.9.2)$$

Since

$$\omega_1 \otimes (\omega_1 + \omega_4) = (2\omega_1 + \omega_4) + (\omega_1 + \omega_3) + (\omega_2 + \omega_4) + (\omega_4), \quad (5.9.3)$$

and the dimension of $(\omega_1 + \omega_4)$ is $8 \cdot 36$, we conclude that

$$\begin{aligned} (2\omega_1 + \omega_4) &\longleftrightarrow 30_p, \\ (3\omega_1 + \omega_3) &\longleftrightarrow 60_p. \end{aligned} \quad (5.9.4)$$

This completes the proof of corollary 4.9.

5.10. The tensor product

$$\begin{aligned} &(\omega_1 + \omega_2) \otimes \omega_3 = (\omega_1 + \omega_2 + \omega_3) + (2\omega_1 + \omega_2) + (2\omega_1 + \omega_4) + \\ &(2\omega_1) + 2(\omega_1 + \omega_3) + (2\omega_2) + (\omega_2 + \omega_4) + (\omega_2) + (\omega_4) \end{aligned} \quad (5.10.1)$$

corresponds to

$$6_p \otimes 20_p = 6_p + 30_p + 20_p + 64_p. \quad (5.10.2)$$

We conclude that

$$(\omega_1 + \omega_2 + \omega_3) \longleftrightarrow 64_p. \quad (5.10.3)$$

5.11. The tensor product

$$\begin{aligned} &3\omega_1 \otimes (\omega_1 + \omega_2) = (4\omega_1 + \omega_2) + (4\omega_1) + (3\omega_1 + \omega_3) + \\ &(2\omega_1 + 2\omega_2) + 2(2\omega_1 + \omega_2) + (2\omega_1) + \\ &(\omega_1 + \omega_2 + \omega_3) + (\omega_1 + \omega_3) + (2\omega_2) + (\omega_2) \end{aligned} \quad (5.11.1)$$

corresponds to

$$15_q \otimes 20_p = 15_q + 20_p + 60_p + 60_s + 64_p + 81_p. \quad (5.11.2)$$

We conclude that

$$(4\omega_1 + \omega_2) + (2\omega_1 + 2\omega_2) \longleftrightarrow 81_p + 60_s. \quad (5.11.3)$$

5.12. The tensor product

$$\begin{aligned}&(\omega_1+\omega_2)\otimes(\omega_1+\omega_2)=\\&(4\omega_1)+(3\omega_1+\omega_3)+(2\omega_1+2\omega_2)+3(2\omega_1+\omega_2)+\\&(2\omega_1+\omega_4)+2(2\omega_1)+2(\omega_1+\omega_2+\omega_3)+3(\omega_1+\omega_3)+\\&(3\omega_2)+2(2\omega_2)+(\omega_2+\omega_4)+(2\omega_2)+(2\omega_3)+(\omega_4)+(0)\end{aligned}\tag{5.12.1}$$

corresponds to

$$20_p\otimes 20_p=1_p+6_p+15_q+15_p+2\cdot 20_p+24_p+30_p+60_p+2\cdot 64_p+81_p.\tag{5.12.2}$$

We conclude that

$$(2\omega_1+2\omega_2)+(3\omega_2)\longleftrightarrow 81_p+24_p.\tag{5.12.3}$$

5.13. The tensor product

$$\begin{aligned}&3\omega_1\otimes 3\omega_1=(6\omega_1)+(4\omega_1+\omega_2)+(4\omega_1)+\\&(2\omega_1+2\omega_2)+2(2\omega_1+\omega_2)+(2\omega_1)+\\&(3\omega_2)+(2\omega_2)+(\omega_1)+(0)\end{aligned}\tag{5.13.1}$$

corresponds to

$$15_q\otimes 15_q=1_p+15_q+20_p+24_p+24'_p+60_s+81_p.\tag{5.13.2}$$

We conclude that

$$(6\omega_1)+(4\omega_1+\omega_2)+(2\omega_1+2\omega_2)+(3\omega_2)\longleftrightarrow 24_p+81_p+60_s+24'_p.\tag{5.13.3}$$

Combining this with (5.12.3), and (5.11.3), we get

$$\begin{aligned}&(3\omega_2)\leftrightarrow 24_p,\\&(4\omega_1+\omega_2)\leftrightarrow 60_s,\\&(2\omega_1+2\omega_2)\leftrightarrow 81_p,\\&(6\omega_1)\leftrightarrow 24'_p.\end{aligned}\tag{5.13.4}$$

5.14. **E7.** In this case, genuine representations restrict to combinations of δ_8 and δ_8^*. The proof will occupy several sections. We first note that

$$(\omega_4)\longleftrightarrow 7'_a,\tag{5.14.1}$$

because it is the representation of K on $\mathfrak{s}$ in the Cartan decomposition.

We record the following induced representations:

$$\begin{aligned}&Ind_{A7}^{E7}[triv]=1_a+35_b+21'_b+15'_a,\\&Ind_{E6}^{E7}[triv]=1_a+27_a+7'_a+21'_b,\\&Ind_{D6}^{E7}[triv]=1_a+27_a+35_b.\end{aligned}\tag{5.14.2}$$

For the first one, $W_\delta\neq W_\delta^0$. It corresponds to the sum of

$$\begin{aligned}&2\omega_1\otimes 2\omega_1=(4\omega_1)+(2\omega_1+\omega_2)+(2\omega_2),\\&2\omega_1\otimes 2\omega_7=(2\omega_1+2\omega_7)+(\omega_1+\omega_7)+(0).\end{aligned}\tag{5.14.3}$$

The first line is $21'_b + 15_a$, the second one $1_a + 35_b$. For the second equation, $W_\delta \neq W_\delta^0$ as well. It corresponds to the sum

$$\begin{aligned}(\omega_2) \otimes (\omega_2) &= (2\omega_2) + (\omega_1 + \omega_3) + (\omega_4),\\ \omega_2 \otimes \omega_6 &= (\omega_2 + \omega_6) + (\omega_1 + \omega_7) + (0).\end{aligned} \tag{5.14.4}$$

The first line is $7'_a + 21'_b$, the second one $1_a + 27_a$. For the third equation, $W_\delta = W_\delta^0$. It corresponds to

$$\begin{aligned}&(\omega_1 + \omega_7) \otimes (\omega_1 + \omega_7) =\\ &(2\omega_1 + 2\omega_7) + (2\omega_1 + \omega_6) + 2(\omega_1 + \omega_7) + (\omega_2 + \omega_6) + (\omega_2 + \omega_7) + (0).\end{aligned} \tag{5.14.5}$$

We conclude that

$$\begin{aligned}(2\omega_1 + 2\omega_7) &\longleftrightarrow 35_b,\\ (\omega_2 + \omega_6) &\longleftrightarrow 27_a,\end{aligned} \tag{5.14.6}$$

Furthermore, because of (5.14.1),

$$(2\omega_2) + (\omega_1 + \omega_3) \longleftrightarrow 21'_b. \tag{5.14.7}$$

But

$$(\omega_1) \otimes (\omega_3) = (\omega_1 + \omega_3) + (\omega_4), \tag{5.14.8}$$

and the dimension of ω_3 is $8 \cdot 7$, So $\omega_1 + \omega_3$ cannot have a 21-dimnsional space of M-fixed vectors. In fact neither repreesntation in the right hand side of 5.14.8 can contain the trivial M-type. Thus

$$(2\omega_2) \longleftrightarrow 21'_b. \tag{5.14.9}$$

It also follows that

$$(4\omega_1) + (2\omega_1 + \omega_2) \longleftrightarrow 15'_a. \tag{5.14.10}$$

Since

$$(\omega_1 + \omega_2) \otimes \omega_1 = (2\omega_1 + \omega_2) + (\omega_1 + \omega_3) + (2\omega_2), \tag{5.14.11}$$

it follows that neither $(2\omega_1 + \omega_2)$ nor $(\omega_1 + \omega_3)$ can contain the trivial M-type, so

$$(4\omega_1) \longleftrightarrow 15'_a. \tag{5.14.12}$$

5.15. The following tensor products,

$$\begin{aligned}(\omega_1 + \omega_6) \otimes \omega_1 &= (2\omega_1 + \omega_6) + (\omega_1 + \omega_7) + (\omega_2 + \omega_6),\\ (\omega_1 + 2\omega_7) \otimes \omega_1 &= (2\omega_1 + 2\omega_7) + (\omega_1 + \omega_7) + (\omega_2 + 2\omega_7),\end{aligned} \tag{5.15.1}$$

and the results in section 5.14 allow us to conclude that

$$\begin{aligned}(\omega_1) &\longleftrightarrow 1_a,\\ (\omega_3) &\longleftrightarrow 7'_a\\ (3\omega_1) &\longleftrightarrow 15'_a\\ (\omega_1 + \omega_2) &\longleftrightarrow 21'_b,\\ (\omega_1 + \omega_6) &\longleftrightarrow 27_a,\\ (\omega_1 + 2\omega_7) &\longleftrightarrow 35_b.\end{aligned} \tag{5.15.2}$$

From

$$\begin{aligned} &(\omega_1+\omega_6)\otimes\omega_1=(2\omega_1+\omega_6)+(\omega_1+\omega_7)+(\omega_2+\omega_6)\\ &(3\omega_1)\otimes\omega_7=(3\omega_1+\omega_7)+(2\omega_1)\end{aligned} \tag{5.15.3}$$

we find that $(3\omega_1+\omega_7)$ and $2\omega_1+\omega_6)$ are not M-spherical.

5.16. Now consider

$$7'_a\otimes 7'_a=1_a+21_a+27_a, \tag{5.16.1}$$

and

$$\begin{aligned} &\omega_3\otimes\omega_3=(2\omega_3)+(\omega_1+\omega_5)+(\omega_2+\omega_4)+(\omega_6),\\ &\omega_3+\omega_5=(\omega_3+\omega_5)+(\omega_1+\omega_7)+(\omega_2+\omega_6)+(0).\end{aligned} \tag{5.16.2}$$

It follows that

$$(\omega_3+\omega_5)\longleftrightarrow 21_a, \tag{5.16.3}$$

and aside from the already known cases, none of the representations occuring are M-spherical.

Now observe that

$$(\omega_5)\otimes(\omega_1+\omega_2)=(\omega_1+\omega_2+\omega_5)+(2\omega_1+\omega_6)+(\omega_1+\omega_7)+(\omega_2+\omega_6) \tag{5.16.4}$$

corresponds to

$$7'_a\otimes 21'_b=27_a+120_a. \tag{5.16.5}$$

Because $(2\omega_1+\omega_6)$ does not have M-fixed vectors, we conclude that

$$(\omega_1+\omega_2+\omega_5)\longleftrightarrow 120_a, \tag{5.16.6}$$

The same argument using

$$\omega_3\otimes(\omega_1+\omega_2)=(\omega_1+\omega_2+\omega_3)+(2\omega_1+\omega_4)+(\omega_1+\omega_5)+(\omega_2+\omega_4) \tag{5.16.7}$$

shows that none of the representations in the RHS of 5.16.7 can have M-fixed vectors.

5.17. Note that

$$\begin{aligned} &\omega_3\otimes(\omega_1+\omega_6)=(\omega_1+\omega_3+\omega_6)+\\ &(2\omega_1)+(\omega_1+\omega_2+\omega_7)+(\omega_2)+(\omega_3+\omega_7)+(\omega_4+\omega_6)\\ &\omega_5\otimes(\omega_1+\omega_6)=(\omega_1+\omega_5+\omega_6)+\\ &(\omega_1+\omega_4+\omega_7)+(\omega_1+\omega_3)+(\omega_4)+(\omega_5+\omega_7)+(2\omega_6).\end{aligned} \tag{5.17.1}$$

The representations in the first equation do not have M-fixed vectors, while the ones in the second equation correspond to

$$7'_a\otimes 27_a=7'_a+21'_b+56'_a+105'_a. \tag{5.17.2}$$

Since

$$\omega_7\otimes(\omega_1+\omega_4)=(\omega_1+\omega_4+\omega_7)+(\omega_1+\omega_3)+(\omega_4), \tag{5.17.3}$$

and the dimension of $\omega_1+\omega_4$ is $8\cdot 63$, we conclude that the dimension of the M-fixed vectors in $(\omega_1+\omega_4+\omega_7)$ is 56, so it follows that

$$\begin{aligned}(\omega_1+\omega_5+\omega_6) &\longleftrightarrow 105'_a,\\ (\omega_1+\omega_4+\omega_7) &\longleftrightarrow 56'_a.\end{aligned} \tag{5.17.4}$$

We also conclude that

$$(\omega_1+\omega_4) \longleftrightarrow 7'_a+56'_a. \tag{5.17.5}$$

5.18. Note that

$$\begin{aligned}&(\omega_3)\otimes(\omega_1+2\omega_7)=(\omega_1+\omega_3+2\omega_7)+(\omega_1+\omega_2+\omega_7)+\\ &(\omega_3+\omega_7)+(\omega_4+2\omega_7),\\ &(\omega_3)\otimes(\omega_1+2\omega_7)=(\omega_1+\omega_3+2\omega_7)+(\omega_1+\omega_2+\omega_7)+\\ &(\omega_3+\omega_7)+(\omega_4+2\omega_7).\end{aligned} \tag{5.18.1}$$

The first equation consists of represntations without M-fixed vectors, the second one corresponds to

$$7'_a\otimes 35_b=56'_a+189'_b. \tag{5.18.2}$$

It follows that

$$(\omega_1+\omega_5+2\omega_7)\leftrightarrow 189'_b. \tag{5.18.3}$$

5.19. Note that

$$\begin{aligned}(\omega_3)\otimes(3\omega_1)&=(3\omega_1+\omega_3)+(2\omega_1+\omega_4),\\ (\omega_5)\otimes(3\omega_1)&=(3\omega_1+\omega_5)+(2\omega_1+\omega_6).\end{aligned} \tag{5.19.1}$$

The first equation does not contain any represntations with M-fixed vectors, while the second one corresponds to

$$7'_a\otimes 15'_a=105_b. \tag{5.19.2}$$

So

$$(3\omega_1+\omega_5)\longleftrightarrow 105_b. \tag{5.19.3}$$

The proof of corollary 4.10 is complete.

5.20. The tensor products decompose according to

$$\begin{aligned}&(\omega_1+\omega_2)\otimes(\omega_1+\omega_2)=(2\omega_1+2\omega_2)+(3\omega_1+\omega_3)+(2\omega_1+\omega_4)+\\ &2(\omega_1+\omega_2+\omega_3)+(3\omega_2)+(\omega_2+\omega_4)+(2\omega_3),\\ &(\omega_1+\omega_2)\otimes(\omega_6+\omega_7)=(\omega_1+\omega_2+\omega_6+\omega_7)+(2\omega_1+\omega_6)+\\ &(2\omega_1+2\omega_7)+2(\omega_1+\omega_7)+(\omega_2+\omega_6)+(\omega_2+2\omega_7)+(0).\end{aligned} \tag{5.20.1}$$

The first equation does not have any representations with M-fixed vectors, while the second one corersponds to

$$21'_b\otimes 21'_b=1_a+27_a+35_b+168_a+210_a. \tag{5.20.2}$$

Thus

$$(\omega_1+\omega_2+\omega_6+\omega_7)\longleftrightarrow 168_a+210_a. \tag{5.20.3}$$

5.21. We decompose the tensor products

$$(\omega_1+\omega_2)\otimes(\omega_1+\omega_6)=(3\omega_1+\omega_7)+(2\omega_1+\omega_2+\omega_6)+(2\omega_1)+$$
$$2(\omega_1+\omega_2+\omega_7)+(\omega_1+\omega_3+\omega_6)+(2\omega_2+\omega_6)+(\omega_2)+(\omega_3+\omega_7),$$

$$(\omega_1+\omega_2)\otimes(\omega_2+\omega_7)=(\omega_1+2\omega_2+\omega_7)+(2\omega_1+\omega_3+\omega_7)+(2\omega_1+\omega_2)+$$
$$2(\omega_1+\omega_3)+(\omega_1+\omega_4+\omega_7)+(\omega_1+\omega_4+\omega_7)+(2\omega_2)+(\omega_2+\omega_3+\omega_7). \tag{5.21.1}$$

Since also

$$21'_b\otimes 27_a=7'_a+21'_b+56'_a+105'_a+189'_b+189'_c, \tag{5.21.2}$$

and the first equation does not contain any representations with M-fixed vectors, we conclude

$$\begin{aligned}(2\omega_1+\omega_3+\omega_7)&\longleftrightarrow 189'_{b,c},\\ (\omega_1+2\omega_2+\omega_7)&\longleftrightarrow 189'_{b,c}.\end{aligned} \tag{5.21.3}$$

The ambiguity is resolved in section 5.26.

5.22. We decompose the tensor products

$$\begin{aligned}(3\omega_1)\otimes(3\omega_1)&=(6\omega_1)+(4\omega_1+\omega_2)+(2\omega_1+2\omega_2)+(3\omega_2),\\ (3\omega_1)\otimes(3\omega_7)&=(3\omega_1+3\omega_7)+(2\omega_1+2\omega_7)+(\omega_1+\omega_7)+(0).\end{aligned} \tag{5.22.1}$$

The first equation does not contain any representations with M-fixed vectors, the second one corresponds to

$$15'_a\otimes 15'_a=1_a+35_b+84_a+105_c. \tag{5.22.2}$$

It follows that

$$(3\omega_1+3\omega_2)\longleftrightarrow 84_a+105_c. \tag{5.22.3}$$

5.23. We decompose the tensor products

$$\begin{aligned}&(\omega_1+\omega_6)\otimes(\omega_1+\omega_6)=(2\omega_1+\omega_4)+(2\omega_1+\omega_5+\omega_7)+\\ &(2\omega_1+2\omega_6)+2(\omega_1+\omega_5)+2(\omega_1+\omega_6+\omega_7)+(\omega_2+\omega_4)+\\ &(\omega_2+\omega_5+\omega_7)+(\omega_2+2\omega_6)+(\omega_6)+(2\omega_7),\\ &(\omega_1+\omega_6)\otimes(\omega_2+\omega_7)=(\omega_1+\omega_2+\omega_6+\omega_7)+(2\omega_1+2\omega_7)+\\ &(\omega_1+\omega_2+\omega_5)+2(\omega_1+\omega_7)+2(\omega_2+\omega_6)+(\omega_1+\omega_2+\omega_5)+\\ &(\omega_2+2\omega_7)+(\omega_3+\omega_5)+(\omega_3+\omega_6+\omega_7)+(0).\end{aligned} \tag{5.23.1}$$

The first equation does not contain any M-spherical represntations, while the second on corresponds to

$$27_a\otimes 27_a=1_a+21_a+2\cdot 27_a+35_b+2\cdot 120_a+168_a+210_a. \tag{5.23.2}$$

The same conclusion as (5.20.3) follows.

5.24. We decompose the tensor products

$$\begin{aligned}&(3\omega_1)\otimes(\omega_1+\omega_6)=(4\omega_1+\omega_6)+(3\omega_1+\omega_7)+\\&(2\omega_1+\omega_2+\omega_6)+(\omega_1+\omega_2+\omega_7),\\&(3\omega_1)\otimes(\omega_2+\omega_7)=(3\omega_1+\omega_2+\omega_7)+(2\omega_1+\omega_2)+\\&(2\omega_1+\omega_3+\omega_7)+(\omega_1+\omega_3).\end{aligned}\tag{5.24.1}$$

The first equation does not contain any M-spherical represntations, while the second on corresponds to

$$15'_a\otimes 27_a=189'_b+216'_a.\tag{5.24.2}$$

We conclude that

$$(3\omega_1+\omega_2+\omega_7)\longleftrightarrow 216'_a.\tag{5.24.3}$$

5.25. We decompose the tensor products

$$\begin{aligned}&(3\omega_1)\otimes(\omega_1+\omega_2)=(4\omega_1+\omega_2)+(3\omega_1+\omega_3)+(2\omega_1+2\omega_2)+\\&(\omega_1+\omega_2+\omega_3),\\&(3\omega_1)\otimes(\omega_6+\omega_7)=(3\omega_1+\omega_6+\omega_7)+(2\omega_1+\omega_6)+\\&(2\omega_1+2\omega_7)+(\omega_1+\omega_7).\end{aligned}\tag{5.25.1}$$

The first equation does not contain any M-spherical represntations, while the second on corresponds to

$$15'_a\otimes 21'_b=35_b+280_b\tag{5.25.2}$$

We conclude that

$$(3\omega_1+\omega_6+\omega_7)\longleftrightarrow 280_b.\tag{5.25.3}$$

5.26. We decompose the tensor products

$$\begin{aligned}&(3\omega_1)\otimes(\omega_1+2\omega_7)=(4\omega_1+2\omega_2)+(3\omega_1+\omega_7)+\\&(2\omega_1+\omega_2+2\omega_7)+(2\omega_1)+(\omega_1+\omega_2+\omega_7)+(\omega_2),\\&(3\omega_1)\otimes(2\omega_1+\omega_7)=(5\omega_1+\omega_7)+(4\omega_1)+\\&(3\omega_1+\omega_2+\omega_7)+(2\omega_1+\omega_2)+(\omega_1+2\omega_2+\omega_7)+(2\omega_2).\end{aligned}\tag{5.26.1}$$

The first equation does not contain any M-spherical representations, while the second on corresponds to

$$15'_a\otimes 35_b=15'_a+21'_b+84'_a+189'_c+216'_a.\tag{5.26.2}$$

Combining this with section 5.21, we conclude that

$$\begin{aligned}&(5\omega_1+\omega_7)\longleftrightarrow 84'_a,\\&(\omega_1+2\omega_2+\omega_7)\longleftrightarrow 189'_c,\\&(2\omega_1+\omega_3+\omega_7)\longleftrightarrow 189'_b\end{aligned}\tag{5.26.3}$$

Remark. Here's another way to match the 21-dimensional Weyl group representations. The restriction of $2\omega_2$ to the maximal compact subgroup of the Levi component of type D_6 contains

$$(1,1,1 \mid 1,1,1) + (1,0,0 \mid 1,0,0) + (2,0,0 \mid 0,0,0), \tag{5.26.4}$$

and representations which do not occur in the spherical principal series. Meanwhile the restriction of $\omega_3+\omega_5$ does not contain the second two factors. The restrictions of the Weyl group representations are

$$\begin{aligned} 21_a &: \; 5\times 1 + 51 \times 0 + 3 \times 3_-, \\ 21'_b &: 5 \times 1 + 4 \times 11. \end{aligned}$$

Next we observe that

$$\begin{aligned} \omega_2 \otimes \omega_2 &= (2\omega_2) + (\omega_1+\omega_3) + (\omega_4), \\ \omega_3 \otimes \omega_5 &= (\omega_3+\omega_5) + (\omega_2+\omega_6) + (0). \end{aligned}$$

Recalling that $\omega_2 = \delta_{28}$, $\omega_3 = 7\delta_8$, $\omega_5 = 7\delta_8^*$, and the fact that $\omega_4 = 8\delta_1 + \delta_{63}$ implies that the multiplicity of δ_1 in $2\omega_2$ and $\omega_3+\omega_5$ is 21. The representations of $SO(6)\times SO(6)$ in (5.26.4) match the restriction of 21_a. The restriction of $\omega_3+\omega_5$ contains

$$(1,0,0 \mid 1,0,0) \tag{5.26.5}$$

exactly once, and none of the other ones in (5.26.4). Thus it cannot contain 1_a, $7'_a$ or 21_a, so it must be $21'_b$.

5.27. **E8.** Let (μ, V) be the fine K-type ω_2. Then the W-representation is

$$\mathrm{Hom}_M[V,V] \cong Ind_{W(E7A1)}^{W(E8)}[triv] = 1_x + 35_x + 84_x. \tag{5.27.1}$$

This is because the stabilizer of δ_{120} in $W(E8)$ is $W(E7A1)$, and V is a sum of the W-orbit of a single character.

Similarly, if (μ, V) is the fine K-type $2\omega_1$,

$$\mathrm{Hom}_M[V,V] \cong Ind_{W(D8)}^{W(E8)}[triv] = 1_x + 84_x + 50_x. \tag{5.27.2}$$

These representations are self-dual, and their tensor products are

$$\begin{aligned} \omega_2\otimes\omega_2 &= (2\omega_2) + (\omega_1+\omega_3) + (2\omega_1) + (\omega_2) + (\omega_4) + (0), \\ (2\omega_1)\otimes(2\omega_1) &= (4\omega_1) + (2\omega_1+\omega_2) + (2\omega_1) + (2\omega_2) + (\omega_2) + (0). \end{aligned} \tag{5.27.3}$$

Similarly ω_3 has dimension $35 \cdot 16$, and

$$\omega_1 \otimes \omega_3 = (\omega_1+\omega_3) + (\omega_2) + (\omega_4) \tag{5.27.4}$$

Thus the multiplicity of δ_1 in $(\omega_1+\omega_3)+(\omega_4)$ is 35. On the other hand, $\dim \omega_4 = 8020$, so its multiplicity in ω_4 is nonzero. From (5.27.3) it follows that the multiplicity is exactly 35, and in fact

$$\omega_4 \longleftrightarrow 35_x, \quad 2\omega_2 \longleftrightarrow 84_x. \tag{5.27.5}$$

We also conclude that the multiplicity of δ_1 in $\omega_1+\omega_3$ is zero.

Consider $\omega_1+\omega_2$ which has dimension $84\cdot 16$, so δ_1 occurs 84 times. Then

$$(\omega_1+\omega_2)\otimes\omega_1=(2\omega_1+\omega_2)+(2\omega_1)+(\omega_1+\omega_3)+(2\omega_2)+(\omega_2). \quad (5.27.6)$$

Thus only $2\omega_2$ contains δ_1. This also implies

$$\omega_3 \longleftrightarrow 35_x. \quad (5.27.7)$$

Combined with (5.27.3) we get

$$4\omega_1 \longleftrightarrow 50_x. \quad (5.27.8)$$

5.28. The representation ω_7 has dimension $8\cdot 16$ while $\omega_8=8\delta_1+\delta_{120}$, and the W-representation on $(\omega_8)^M$ is 8_x, because this is the representation on $\mathfrak{s}$ from the Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{s}$. Then

$$\omega_1\otimes\omega_7=(\omega_1+\omega_7)+(\omega_8). \quad (5.28.1)$$

So δ_1 does not occur in $\omega_1+\omega_7$. This implies that $\mathrm{Hom}_M[\omega_1,\omega_7]$ is 8_x. We combine this with

$$\begin{aligned} &35_x\otimes 8_x=160_z+112_x+8_x,\\ &\omega_3\otimes\omega_7=(\omega_3+\omega_7)+(\omega_2+\omega_8)+(\omega_1+\omega_7)+(\omega_8). \end{aligned} \quad (5.28.2)$$

Then $(\omega_1+\omega_7)$ does not contain δ_1 and ω_8 contains it 8 times. We have to distribute the other two representations among $\omega_3+\omega_7$ and $\omega_2+\omega_8$. We have

$$\omega_2\otimes\omega_8=(\omega_2+\omega_8)+(\omega_2)+(\omega_8), \quad (5.28.3)$$

and it contains δ_1 120 times. Thus $\omega_2+\omega_8$ contains δ_1 112 times, and combined with (5.28.2) we get

$$\omega_2+\omega_8\longleftrightarrow 112_x, \qquad \omega_3+\omega_7\longleftrightarrow 160_z. \quad (5.28.4)$$

5.29. We apply the previous reasoning to

$$\begin{aligned} 35_x\otimes 35_x=&210_x+84_x+567_x+300_x+35_x+28_x+1_x,\\ (\omega_3)\otimes(\omega_3)=&(2\omega_3)+(\omega_1+\omega_3)+(\omega_1+\omega_5)+(2\omega_1)+\\ &+(2\omega_2)+(\omega_2+\omega_4)+(\omega_2)+(\omega_4)+(\omega_6)+(0). \end{aligned} \quad (5.29.1)$$

Then

$$\omega_1\otimes\omega_5=(\omega_1+\omega_5)+(\omega_6)+(\omega_4), \quad (5.29.2)$$

and the dimension of ω_5 being $273\cdot 16$, we get that the multiplicity of δ_1 in $(\omega_6)+(\omega_1+\omega_5)$ is 238. From

$$\omega_2\otimes\omega_4=(\omega_2+\omega_4)+(\omega_1+\omega_3)+(\omega_1+\omega_5)+(\omega_2)+(\omega_4)+(\omega_6), \quad (5.29.3)$$

and $\omega_4=35\delta_1+7\delta_{120}+7\delta_{135}$, we get that δ_1 must occur 567 times in $\omega_2+\omega_4$. The formula for ω_4 follows from the equation (5.6.7) which yields the system

$$\begin{aligned} &1820=35+120a_1+135b_1,\\ &7020=120a_2+135b_2,\\ &a_1+a_2=34,\\ &b_1+b_2=35. \end{aligned} \quad (5.29.4)$$

The numbers 1820, 7020 are the dimensions of ω_4 and $\omega_1+\omega_3$, and the two last equations come from

$$\begin{aligned} \omega_1 \otimes \omega_1 &= (2\omega_1) + (\omega_2) + (0) \\ \delta_{16} \otimes \delta_{16} &= \delta_{135} + \delta_{120} + \delta_1. \end{aligned} \tag{5.29.5}$$

Furthermore,

$$\omega_8 \otimes \omega_8 = (2\omega_8) + (\omega_1+\omega_7) + (\omega_6) + (\omega_4) + (\omega_4) + (\omega_2) + (0) \tag{5.29.6}$$

and δ_1 occurs 184 times. Thus if ω_6 contains δ_1, it must correspond to 28_x. Since the dimension is $8008 = a + 120b + 135c$, it follows that $a > 0$. We conclude that

$$\begin{aligned} &(\omega_1+\omega_5) \longleftrightarrow 28_x, \\ &(\omega_6) \longleftrightarrow 210_x, \\ &(\omega_2+\omega_4) \longleftrightarrow 567_x, \\ &(2\omega_3) \longleftrightarrow 300x, \\ &\omega_5 \longleftrightarrow 28_x + 35_x + 210_x. \end{aligned} \tag{5.29.7}$$

5.30. The previous arguments also show that

$$3\omega_1 \longleftrightarrow 50_x. \tag{5.30.1}$$

Then

$$\begin{aligned} &50_x \otimes 8_x = 400_x, \\ &(3\omega_1) \otimes (\omega_7) = (3\omega_1+\omega_7) + (2\omega_1+\omega_8). \end{aligned} \tag{5.30.2}$$

But

$$(\omega_1+\omega_8) \otimes \omega_1 = (2\omega_1+\omega_8) + (\omega_1+\omega_7) + (\omega_2+\omega_8) + (\omega_8). \tag{5.30.3}$$

Since $\dim(\omega_1+\omega_8) = 120\cdot 16$, the previous results show that δ_1 cannot occur in $2\omega_1+\omega_8$, and we conclude

$$(3\omega_1+\omega_7) \longleftrightarrow 400_x. \tag{5.30.4}$$

5.31. The previous sections show

$$3\omega_1 \longleftrightarrow 50_x, \qquad \omega_3 \longleftrightarrow 35_x. \tag{5.31.1}$$

The tensor product decomposes

$$50_x \otimes 35_x = 1050_x + 700_x. \tag{5.31.2}$$

On the other hand,

$$\begin{aligned} &(3\omega_1) \otimes (\omega_3) = (3\omega_1+\omega_3) + (2\omega_1+\omega_2) + (2\omega_1+\omega_4) + (\omega_1+\omega_3), \\ &(\omega_1) \otimes (\omega_1+\omega_4) = \\ &(2\omega_1+\omega_4) + (\omega_2+\omega_4) + (\omega_1+\omega_5) + (\omega_1+\omega_3) + (\omega_4). \end{aligned} \tag{5.31.3}$$

In the first equation only $2(\omega_1+\omega_4)$ and $(3\omega_1+\omega_3)$ can have fixed vectors, and their dimensions add up to 1750. Counting dimensions of M-fixed vectors in the second equation, we find that the dimension of the M-fixed vectors in

$(2\omega_1+\omega_4)$ is 700. Thus $2\omega_1+\omega_4$ must contain 700_x and $(3\omega_1+\omega_3)$ must contain 1050_x.

This completes the proof of corollary 4.11.

5.32. Recall that $50_x \longleftrightarrow (3\omega_1)$. The tensor product decomposes

$$50_x \otimes 50_x = 1_x + 50_x + 84_x + 168_y + 525_x + 700_{xx} + 972_x. \tag{5.32.1}$$

It corresponds to

$$\begin{aligned}&(3\omega_1)\otimes(3\omega_1) =\\ &(6\omega_1)+(4\omega_1+\omega_2)+(4\omega_1)+(2\omega_1+2\omega_2)+(2\omega_1+\omega_2)+\\ &(2\omega_1)+(3\omega_2)+(2\omega_2)+(\omega_2)+(0).\end{aligned} \tag{5.32.2}$$

Since

$$\begin{aligned}&(\omega_1+\omega_2)\otimes\omega_1 = (2\omega_1+\omega_2)+(2\omega_2)+(\omega_1+\omega_3)+(2\omega_1)+(\omega_2),\\ &\omega_1\otimes\omega_3 = (\omega_4)+(\omega_2)+(\omega_1+\omega_3),\end{aligned} \tag{5.32.3}$$

and taking the earlier results into account, we find that $(\omega_1+\omega_3)$ and $(2\omega_1+\omega_2)$ cannot have M-fixed vectors. This shows that the sum of Weyl group representations in (5.32.1) corresponds to the M-fixed vectors in (5.32.2). We omit the details of how they match the irreducible modules. Finally, recall that $(\omega_1+\omega_2)$ corresponds to 84_x, while ω_3 corresponds to 35_x. Then

$$84_x \otimes 35_x = 1344_x + 700_x + 210_x + 567_x + 84_x + 35_x, \tag{5.32.4}$$

while

$$\begin{aligned}&(\omega_1+\omega_2)\otimes(\omega_3) =\\ &(2\omega_1+\omega_2)+(2\omega_1+\omega_4)+(2\omega_1)+(\omega_1+\omega_2+\omega_3)+2(\omega_1+\omega_3)+\\ &+(\omega_1+\omega_5)+(2\omega_2)+(\omega_2+\omega_4)+(2\omega_2)+(\omega_2)+(\omega_4).\end{aligned} \tag{5.32.5}$$

The fact that the representation of $W(E_8)$ on the M-fixed vectors of $(\omega_1+\omega_2+\omega_3)$ is 1344_x follows from these two equations.

References

[ABV] J. Adams, D. Barbasch, D. Vogan, *The Langlands classification and irreducible characters of real reductive groups*, Progress in Mathematics, Birkhäuser, Boston-Basel-Berlin, (1992), vol. 104.

[BB] D. Barbasch, M. Bozicevic *The associated variety of an induced representation*proceedings of the AMS **127 no. 1** (1999), 279-288

[B1] D. Barbasch, *The unitary dual of complex classical groups*, Inv. Math. **96** (1989), 103–176.

[B2] ———, *Unipotent representations for real reductive groups*, Proceedings of ICM, Kyoto 1990, Springer-Verlag, The Mathematical Society of Japan, 1990, pp. 769–777.

[B3] ———, *The spherical unitary dual for split classical p-adic groups*, Geometry and representation theory of real and p-adic groups (J. Tirao, D. Vogan, and J. Wolf, eds.), Birkhauser-Boston, Boston-Basel-Berlin, 1996, pp. 1–2.

[B4] ———, *Orbital integrals of nilpotent orbits* , Proceedings of Symposia in Pure Mathematics, vol. 68, (2000) 97-110.

[B5] ———, *The associated variety of a unipotent representation* preprint in preparation

[B6] ______, *The spherical unitary dual of split real and p-adic groups* preprint in preparation, http://www.math.cornell.edu/~barbasch

[BC] D. Barbasch, D. Ciubotaru, *The spherical unitary dual of split p-adic groups of type E*

[BM1] D. Barbasch and A. Moy *A unitarity criterion for p-adic groups*, Inv. Math. **98** (1989), 19–38.

[BM2] ______, *Reduction to real infinitesimal character in affine Hecke algebras*, Journal of the AMS **6 no. 3** (1993), 611-635.

[BM3] ______, *Unitary spherical spectrum for p-adic classical groups*, Acta Applicandae Math **5 no. 1** (1996), 3-37.

[BS] D. Barbasch, M. Sepanski *Closure ordering and the Kostant-Sekiguchi correspondence*, Proceedings of the AMS **126 no. 1** (1998), 311-317.

[BV1] D. Barbasch, D. Vogan *The local structure of characters* J. of Funct. Anal. **37 no. 1** (1980) 27-55

[BV2] D. Barbasch, D. Vogan *Unipotent representations of complex semisimple groups* Ann. of Math., 121, (1985), 41-110

[BV3] D. Barbasch, D. Vogan *Weyl group representation and nilpotent orbits* Representation theory of reductive groups (Park City, Utah, 1982), **Progr. Math., 40**, Birkhuser Boston, Boston, MA, (1983), 21-33.

[C] D. Ciubotaru *The Iwahori spherical unitary dual of the split group of type F4* preprint

[CM] D. Collingwood, M. McGovern *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold Co., New York, (1993).

[D] D. Djokovic *Closures of conjugacy classes in classical real linear Lie groups II* Trans. Amer. Math. Soc. **270 no. 1**, (1982), 217-252.

[KnV] A. Knapp, D. Vogan *Cohomological induction and unitary representations* Princeton University Press, Princeton Mathematical Series vol. 45, 1995.

[L1] G. Lusztig *Characters of reductive groups over a finite field* Annals of Math. Studies, Princeton University Press vol. 107.

[McG] W. McGovern *Cells of Harish-Chandra modules for real classical groups* Amer. Jour. of Math., 120, (1998), 211-228.

[P] A. Pantano, Ph.D. thesis, Princeton Uinversity, 2004

[SV] W. Schmid, K. Vilonen *Characteristic cycles and wave front cycles of representations of reductive groups*, Ann. of Math., 151 (2000), 1071-1118.

[Stein] E. Stein *A*nalysis in matrix space and some new representations of $SL(n, \mathbb{C})$ Ann. of Math. **86** (1967) 461-490

[T] M. Tadic *Classification of unitary representations in irreducible representations of general linear groups*, Ann. Sci. École Norm. Sup. (4) 19, (1986) no. 3, 335-382.

[V] D. Vogan *The unitary dual for real G2*

[V1] D. Vogan *The unitary dual of GL(n) over an archimedean field*, Inv. Math., 83 (1986), 449-505.

[V2] ______ *Irreducible characters of semisimple groups IV* Duke Math. J. 49, (1982), 943-1073

[Vo] ______ *Representations of real reductive Lie groups*, Birkhäuser, Boston-Basel-Stuttgart, 1981

[ZE] A. Zelevinsky *Induced representations of reductive* $\mathfrak{p}$*-adic groups II. On irreducible representations of* $\mathrm{GL}(n)$, Ann. Sci. cole Norm. Sup. (4) **13 no. 2** 165-210

(D. Barbasch) Dept. of Mathematics, Cornell University, Ithaca, NY 14850
E-mail address: `barbasch@math.cornell.edu`