MAT344 Week 2

2019/Sep/16

1 This week

This week, we are talking about

1. Binomial coefficients
2. Lattice paths and Catalan numbers
3. The binomial theorem

2 Recap

Last week we talked about

1. Basic counting principles
2. Permutations and combinations
3. Combinatorial proofs

3 Binomial coefficients (Chapter 2.5 in [KT17])

We continue looking at counting problems that we can solve with binomial coefficients.

Exercise 3.1 ([KT17], Example 2.18)**.** *The office assistant is distributing supplies. In how many ways can he distribute 18 identical folders among four office employees: Audrey, Bart, Cecilia and Darren, with the additional restriction that each will receive at least one folder?*

Solution: Imagine the folders placed in a row. Then there are 17 gaps between them. Of these gaps, choose three and place a divider in each. Then this choice divides the folders into four non-empty sets. The first goes to Audrey, the second to Bart, etc. Thus the answer is $\binom{17}{3}$. Figure 1 illustrates the distribution where Audrey gets 6, Bart 1, Cecilia 4 and Darren 7 folders.

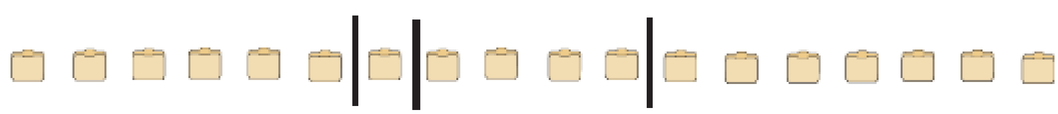

Figure 1: One way of distributing the folders

This sort argument is commonly referred to as a **stars and bars** computation. We have a row of n indistinguishable stars (in the folders example, $n = 18$, but let us consider the case $n = 7$ here)

$$* * * * * * *$$

that we want to separate into k piles. We do this by inserting $k-1$ bars into the ($n-1$ many) spaces between the stars (in the folder example and the stars and bars example, $k=4$)

$$* | * * | * * | * *$$

so we have reduced the counting problem to a previous one that we solved already, with answer $\binom{n-1}{k-1}$.

Exercise 3.2. *Suppose we redo the preceding problem but drop the restriction that each of the four employees gets at least one folder. Now in how many ways can we distribute the folders?*

Solution: How could we reduce this to the previous problem? If we had 22 instead of 18 folders, we could distribute them using the previous method (in $\binom{21}{3}$ many ways) so everyone would have at least one folder. Then we could take one folder from everyone. Let us go back to the bars and stars problem above, with $n=7$ and $k=4$, but this time allowing empty piles. How could we think of this? Eventually we want to have all 7 stars and 3 bars placed in a row, so there are 10 "places" where either a star or a bar can go. If we know where the bars go, the stars will fill all the remaining places, so there are $\binom{10}{3}$ many ways of doing this.

So there are $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$ many ways of separating n indistinguishable stars in to k piles, some of which may be empty.

The following problem does not seem to be immediately related to the bars and stars problem, but we will soon find out that it is:

Exercise 3.3 ([Mor17], Example 5.2)**.** *Chris has promised to bring back donuts for three friends he's studying with. He wants to buy 8 donuts, and the donut shop sells five varieties. How many ways are there for Chris to fill this order?*

Solution: Let us number the flavors as 1, 2, 3, 4, and 5. Chris wants to order 8 donuts, which we can represent as a [5]-string of length 8. We only care about how many times a letter appears in the string, and not *where* it appears, so we may assume that the numbers appear in increasing order, for example

$$111334444$$

is an order where Chris buys 3 bagels with flavor 1, no bagels with flavor 2, 2 bagels with flavor 3, 4 with flavor 4 and none with flavor 5. If we suggestively put separating lines between the different flavors, we get a picture that looks something like this

$$111||33|4444|$$

(notice the two separators between the 1s and the 3s, signifying that there are no 2s). If we know where the separators are, we can recover how many of each number we have, so we don't really need to write all the numbers out, and we end up with a picture

$$* * * || * * | * * * * |$$

and this is a stars and bars computation, whose solution we already know to be $\binom{8+5-1}{5-1} = \binom{12}{4}$.

This counting problem where we are choosing k objects from n, but we are allowed to select a single object more than once is called a **combination with repetition**.

Theorem 3.4 ([Mor17], Theorem 5.3)**.** *The number of ways of choosing k objects from n objects, with repetition allowed is*

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

Proof. We use the idea from the previous problem. We assume that we have an inexhaustible supply of each letter in $[n]$. Since there are n different types of objects, we need $n-1$ bars and k stars. So we need to select k positions from the $n+k-1$ places.

Q.E.D.

We can also reformulate these computations in terms of the number of integer solutions for inequalities.

Exercise 3.5 ([KT17], Example 2.21.). *We count the number of integer solutions to the inequality*

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 538$$

subject to various conditions on the values of $x_1, x_2, \dots, x_6$. For each of the following cases, find the number of solutions in terms of a binomial coefficient.

1. *When all $x_i > 0$ and equality holds.*
2. *When all $x_i \geq 0$ and equality holds.*
3. *When $x_1, x_2, x_4, x_6 > 0, x_3 = 52, x_5 = 194$ and equality holds.*
4. *When all $x_i > 0$ and the inequality is strict. **Hint:** Introduce a new variable x_7 which is the balance. Note that x_7 must be positive.*
5. *When all $x_i \geq 0$ and the inequality is strict. **Hint:** Add a new variable x_7 as above. Now it is the only one which is required to be positive.*
6. *When all $x_i \geq 0$.*

Let us recast all the typical counting problems we have seen so far in terms of strings. In each case, we are going to be counting $[n]$-strings of length k. We have two decisions to make. We should decide if the order in which the letters appear in the string matter and we have to decide whether we are allowed to repeat letters from the alphabet. We can summarize our findings in Table 1 ([Mor17], Table 5.1)

	repetition allowed	repetition not allowed
order matters	n^k	$\frac{n!}{(n-k)!}$
order doesn't matter	$\binom{n+k-1}{k}$	$\binom{n}{k}$

Table 1: number of $[n]$-strings of length k

4 Lattice paths (Chapter 2.5 in [KT17])

Exercise 4.1 ([Bog04], Chapter 1.3.1 Problem 47). *In a part of a city, all streets run either noth-south or east-west and there are no dead ends. Suppose we are standing on a street corner. In how many ways may we walk to a corner that is four blocks north and six blocks east, using as few block as possible?*

Solution: We have to walk ten blocks, and we choose four out of these ten where we walk north. We can do this in $\binom{10}{4}$ many ways.

Example 4.2 ([Bog04], Chapter 1.3.1 Problem 48). *Problem 4.1 has a geometric interpretation in a coordinate plane. A **lattice path** in the plane is a curve made up of line segments that either go from a point (i, j) to the point $(i+1, j)$ or from a point (i, j) to the point $(i, j+1)$ where i and j are integers. (Thus lattice paths always move either up or to the right.) The length of the path is the number of such line segments.*

1. *What is the length of a lattice path from $(0, 0)$ to (m, n)?*
2. *How many such lattice paths of that length are there?*
3. *How many lattice paths are there from (i, j) to (m, n), assuming i, j, m, n are integers?*

Solution:

1. We have to take m steps right and n steps up, for a total of $m + n$ steps.
2. Similarly to problem 4.1, we have to choose m of the $m + n$ steps to be toward the right, so we have $\binom{m+n}{m}$ choices.

3. If $i > m$ or $j > n$ then we can not go from (i, j) to (m, n), since we can only travel up and right. Otherwise we have to make $m - i$ right and $n - j$ up steps, and there are $\binom{m-i+n-j}{m-i}$ many ways of doing so.

Remark 4.3. *There is another kind of lattice path, sometimes called a* ***diagonal lattice path*** *that is common in the literature. These consist of steps from (i, j) to either $(i + 1, j + 1)$ or $(i + 1, j - 1)$. From the enumerative perspective, there is not much difference between these and the ones we will consider. See [Bog04], Chapter 1.3.1 Problem 49. for a comparison.*

Exercise 4.4 ([Bog04], Chapter 1.3.1 Problem 50)**.** *A school play requires a ten dollar donation per person; the donation goes into the student activity fund. Assume that each person who comes to the play pays with a ten dollar bill or a twenty dollar bill. The teacher who is collecting the money forgot to get change before the event. If there are always at least as many people who have paid with a ten as a twenty as they arrive the teacher won't have to give anyone an IOU for change. Suppose $2n$ people come to the play, and exactly half of them pay with ten dollar bills.*

1. *Describe a bijection between the set of sequences of tens and twenties people give the teacher and the set of lattice paths from $(0, 0)$ to (n, n).*

2. *What is the geometric interpretation of a sequence that does not require the teacher to give any IOUs?*

Solution:

1. We take a right step when someone pays with a \$10 bill and an up step when someone pays with a \$20 bill. This defines a lattice path. Since half of all people pay with \$10s, the lattice path will end at (n, n). This is easily seen to be a bijection.

2. With our chosen conventions, a lattice path taht corresponds to a sequence with no IOUs is one that never goes above the diagonal $y = x$.

Definition 4.5. *A* ***Dyck path*** *is a lattice path from $(0, 0)$ to (n, n) that does not go above the diagonal $y = x$.*

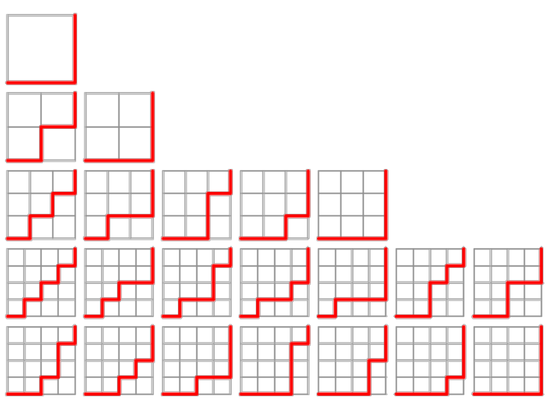

Figure 2: all Dyck paths up to $n = 4$

Proposition 4.6 ([KT17], Example 2.23)**.** *The number of Dyck paths from $(0, 0)$ to (n, n) is the* ***Catalan number***

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Before giving the proof, let's take a look at Figure 2. We see that $C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14$, which agrees with the formula that Proposition 4.6 predicts.

Proof of Proposition 4.6. The formula includes a factor $\binom{2n}{n}$ which we immediately recognize as the number of all lattice paths from $(0,0)$ to (n,n). We will call the ones that go above the diagonal **bad paths**, count them and subtract this number from $\binom{2n}{n}$ to get a formula for C_n. We can equivalently represent a path as a sequence of Us and Rs (up and right steps). Altogether there must be n many Us and n many Rs in the sequence. Let s be a bad path. Since s is bad, it crosses the diagonal after some step. Let i be the first time it happens. Notice that i must be odd, so we let $i = 2j+1$. In the first i steps of s there are exactly j Rs and exactly $j+1$ Us. Now we are going to reflect the tail of the path (starting at step $i+1$) across the line $y = x+1$. Call this path s'. This has the effect of exchanging Rs and Us. Since the tail of the path contained exactly $n-j$ Rs and $n-j-1$ Us, after reflecting s' has $n-j$ Us and $n-j-1$ Rs in its tail, whereas the initial segments up to i of s and s' are identical. So in total s' has $j+(n-j-1) = n-1$ Rs and $(j+1)+(n-j) = n+1$ Us. So s' is a lattice path from $(0,0)$ to $(n-1,n+1)$. By Problem 4.2, part 2, there are $\binom{2n}{n-1}$ many of these paths. What did we just do? For any bad path s, we constructed a path s' from $(0,0)$ to $(n-1,n+1)$.

We claim that this map from bad paths to paths from $(0,0)$ to $(n-1,n+1)$ is a bijection. To see this, notice that any path s' from $(0,0)$ to $(n-1,n+1)$ must cross the diagonal $y=x$. Let i be the first time this happens, and again $i = 2j+1$. Similarly to the original case, we can reflect the tail (starting at step $i+1$) of the path across the line $y = x+1$ to obtain a bad path from $(0,0)$ to (n,n). Also notice that if we start with a bad path s, then obtain the path s', then reflecting the tail of s' (starting from the first time it crosses the diagonal) recovers s, so this is a bijection.

Therefore the number of bad paths is $\binom{2n}{n-1}$, whereas the total number of paths is $\binom{2n}{n}$ so the number of Dyck paths is

$$\binom{2n}{n} - \binom{2n}{n-1} = \binom{2n}{n}\left(1 - \frac{n}{n+1}\right) = \frac{1}{n+1}\binom{2n}{n}.$$

Q.E.D.

The above proof is not very difficult and quite pretty, but it does not really *explain* why we are dividing by $n+1$ in the formula (but it does give a clear interpretation for the formula $\binom{2n}{n} - \binom{2n}{n-1}$). To see a more combinatorial proof, look at Chapter 1.3.1 Problem 52 of [Bog04].

Counting Dyck paths does not immediately seem like a very useful or applicable endeavor. It turns out that there is a dazzling number of counting problems whose answers are given naturally in terms of Catalan numbers. In [Sta99], Stanley gives 66 counting problems that are answered by Catalan numbers (these are available online at `http://www-math.mit.edu/~rstan/ec/catalan.pdf`), and he maintains an addendum of more problems that is currently almost a hundred pages long at `http://www-math.mit.edu/~rstan/ec/catadd.pdf` for a total of 207 counting problems as of now. We will return to some of these interpretations later when we discuss recursion.

5 The Binomial Theorem (Chapter 2.6 in [KT17])

Theorem 5.1 ([KT17], Theorem 2.24)**.** *For every non-negative integer n,*

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

Remark 5.2. *For some reason in the book there is a requirement that x, y and $x+y$ are non-negative real numbers. This is unnecessary. The formula is an equality between two polynomials.*

Proof of Theorem 5.1. View $(x+y)^n$ as the product $\prod_{i=1}^{n}(x+y)$. If we expand this product, every term of the resulting sum contains exactly one term of each factor $(x+y)$. The coefficient of $x^{n-i}y^i$ in $(x+y)^n$ is therefore number of ways of choosing i many ys and $n-i$ many xs. There are $\binom{n}{i}$ many ways of doing this.

Q.E.D.

Some results that we have already seen are easy consequences of the Binomial Theorem, for example

Corollary 5.3 ([Mor17], Corollary 3.19)**.** *For any natural number n, we have*

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

Proof. Substitute $x = y = 1$ into Theorem 5.1.

Q.E.D.

It can also lead us to discover new statements

Exercise 5.4. *For any natural number n, we have*

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i} = 0.$$

Solution: Substitute $x = 1, y = -1$ into Theorem 5.1.

Exercise 5.5. *Find a combinatorial interpretation for this statement.*

Exercise 5.6. *For n odd, find a combinatorial proof. (**Hint:** think of complementation)*

Exercise 5.7. *For n even, find a combinatorial proof.*

Let us think a little more deeply about Theorem 5.1. We can understand it as an equality between two functions in two variables, or, if we set $y = 1$, as an equality between two single-variable functions. Once we realize this, we can do calculus with these functions, and it may lead us to new results about binomial coefficients.

Example 5.8. *By the Binomial Theorem, we have*

$$(1 + x)^n = \sum_{i=0}^{n}\binom{n}{i}x^i.$$

Differentiating both sides yields

$$n(1 + x)^{n-1} = \sum_{i=1}^{n} i\binom{n}{i}x^{i-1}.$$

Note that the $i = 0$ term (the constant term) has zero derivative. This is still an equality between functions*, so we can specify x to be any value. In particular, if we let $x = 1$, we get*

$$n2^{n-1} = \sum_{i=1}^{n} i\binom{n}{i} = \sum_{i=0}^{n} i\binom{n}{i}.$$

Exercise 5.9. *For any natural number, show that*

$$\sum_{i=0}^{n} i\binom{n}{i}(-1)^{i-1} = 0.$$

Exercise 5.10. *Explain how figure 3 visualizes the binomial theorem.*

We can also use the Binomial Theorem to evaluate complicated-looking expressions

Exercise 5.11 ([Mor17], Exercise 3.21.1)**.** *Use the Binomial Theorem to evaluate (i.e. give a formula in terms of n)*

$$\sum_{i=1}^{n}\binom{n}{i}2^i.$$

6 Multinomial Coefficients (Chapter 2.7 in [KT17])

Binomial coefficients are answers to counting problems about *choosing* a certain subset of a set. Let X denote the set of students enrolled in a class with $|X| = n$. In how many ways can exactly k of them pass the class? Let's recall how we interpreted the question in terms of strings. We were looking for k-permutations (strings with no repeats) of X, but we did not care what order the entries appeared in the permutation. For this question, it does not matter if you passed with 100% or 51%. So we divide the number of k-permutations of X (which is $P(n, k)$) by $k!$ to get $\binom{n}{k}$.

However, most of the time you care about what grade you get, not just passing.

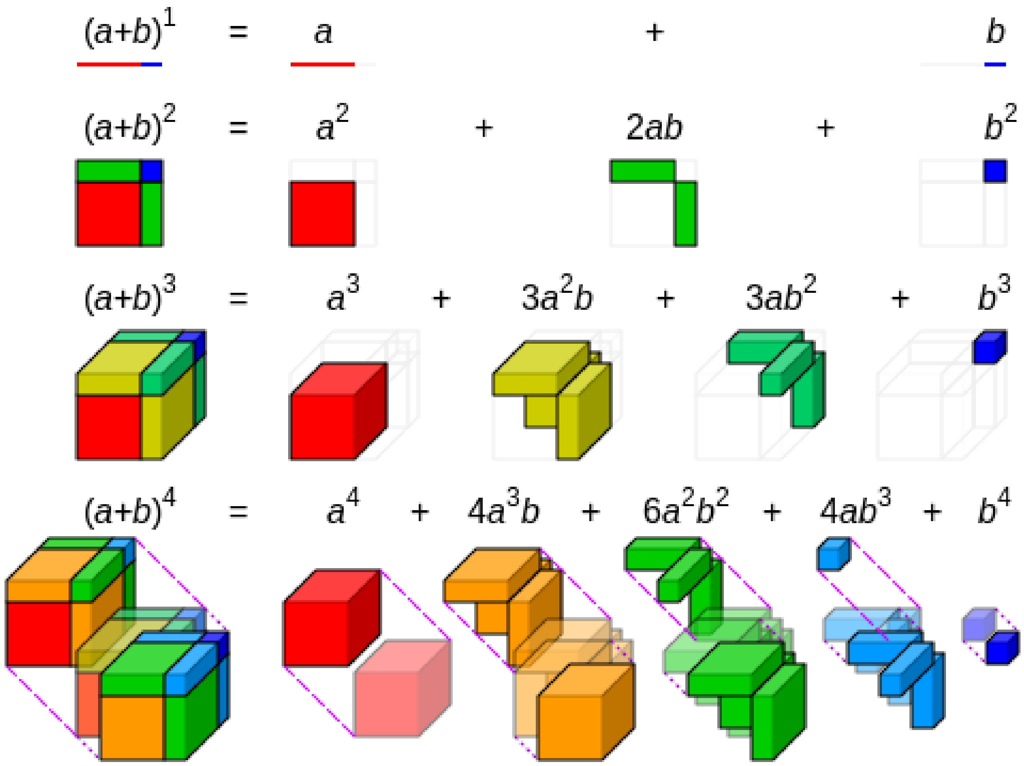

Figure 3: Visualization of the binomial theorem

Example 6.1. *In how many ways can exactly k_G many of them can get a letter grade g for $g \in G$, where $G = \{F, D, C, B, A\}$?*

Let us try to generalize from the pass/fail version. Let us imagine that we write down the students (elements of X) in the order of the increasing total score in their class. Then we decide the cutoffs in a way that exactly k_F many of them will fail, k_D many will get a D and so on. Notice that $\sum_{g \in G} k_G = n$. There are $n!$ many ways of ordering all the students, but this is obviously an overcount. How much did we overcount by? We only care about which grade band each student ended up in, so, for example, if we reorder all the failing students' scores, it makes no difference for our final answer. And similarly for all the other grade bands. So we find that the answer should be

$$\frac{n!}{k_F! k_D! k_C! k_B! k_A!}.$$

Definition 6.2. *Let $k_1, k_2, \ldots, k_r$ be positive integers and $n = \sum_{i=1}^{r} k_i$. Then the number*

$$\binom{n}{k_1, k_2, k_3, \ldots, k_r} = \frac{n!}{k_1! k_2! k_3! \ldots k_r!}$$

is called a ***multinomial coefficient***

Notice that this is slightly different from our notation of binomial coefficients, for example

$$\binom{n}{k} = \binom{n}{k, n-k},$$

but this should not lead to confusion.

The Binomial Theorem (Theorem 5.1) also has a straightforward generalization

Theorem 6.3 (Theorem 2.27 in [KT17])**.** *Let $x_1, x_2, \ldots, x_r$ be real numbers. Then for every nonnegative integer n,*

$$(x_1 + x_2 + \ldots + x_r)^n = \sum_{k_1 + k_2 + \ldots + k_r = n} \binom{n}{k_1, k_2, \ldots, k_r} x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}.$$

In case you have not seen this sort of notation before, the indexing of the summation $\sum_{k_1+k_2+\ldots+k_r=n}$ means that we are summing over all r-tuples $(k_1, k_2, \ldots, k_r)$ of nonnegative integers such that $k_1 + k_2 + \ldots + k_r = n$.

What sort of counting problems are answered by multinomial coefficients?

Exercise 6.4 (Problem 2 (i) in [Sta97]). *In how many different ways can the letters of the word MISSISSIPPI be arranged if the four S's cannot appear consecutively?*

References

[Bog04] Kenneth P. Bogart. *Combinatorics Through Guided Discovery.* Open access, 2004. Available at `http://bogart.openmathbooks.org/`. 3, 4, 5

[KT17] Mitchel T. Keller and William T. Trotter. *Applied Combinatorics.* Open access, 2017. Available at `http://www.rellek.net/appcomb/`. 1, 3, 4, 5, 6, 7

[Mor17] Joy Morris. *Combinatorics.* Open access, 2017. Available at `http://www.cs.uleth.ca/~morris/Combinatorics/Combinatorics.html`. 2, 3, 5, 6

[Sta97] Richard P. Stanley. *Enumerative Combinatorics*, volume 1. Cambridge University Press, 1997. Available at `http://www-math.mit.edu/~rstan/ec/ec1.pdf`. 8

[Sta99] Richard P. Stanley. *Enumerative Combinatorics*, volume 2. Cambridge University Press, 1999. 5