MAT344 Week 4

2019/Sept/30

1 This week

This week, we are talking about

1. The Pigeonhole Principle
2. Graph Theory

2 Recap

Last week we talked about

1. Induction
2. Recursion

3 The Pigeonhole principle (Chapter 4.1 in [KT17])

If we have $n+1$ pigeons and n holes that we have to place all the pigeons in, there will be at least one hole with at least two pigeons in it. The mathematical way of saying this common sense observation is the following:

Proposition 3.1 (Proposition 4.1. in [KT17])**.** *If $f : X \to Y$ is a function and $|X| > |Y|$, then there exists an element $y \in Y$ and distinct elements $x, x' \in X$ such that $f(x) = f(x') = y$.*

While this seems like an obvious statement, it has many applications.

Exercise 3.2. *Show that at any party there are two people who have the same number of friends at the party assuming that all friendships are mutual.*

Solution: Let n be the number of people at the party. Each person can have $0, 1, \ldots, n-1$ friends present at the party. But if one person has no friends at the party, then there can't be a person who has $n-1$ friends at the party, since then they are friends with everyone at the party and friendships are mutual. So there are n people with $n-1$ possible numbers of friends at the party. By the pigeonhole principle, there are at least two people with the same number of friends present.

Exercise 3.3. *Show that if* 101 *integers are chosen from the set* $\{1, 2, \ldots, 200\}$ *then one of the chosen integers divides another.*

Solution: Let the chosen integers be $a_1, \ldots, a_{101}$. For each k, write $a_k = 2^{p_k} b_k$ for b_k odd. All of the 101 odd numbers b_k have to be from the 100 odd integers in $\{1, 3, 5, \ldots 199\}$, so by the pigeonhole principle there is m, l such that $b_m = b_l$. Either $p_m < p_l$ and therefore a_m divides a_l or $p_m > p_l$ and a_l divides a_m.

The pigeonhole principle also appears in many proofs of theorems that are far from obvious, for example:

Theorem 3.4 (Erdős-Szekeres, Theorem 4.2 in [KT17])**.** *If m and n are non-negative integers, then any sequence of $mn+1$ distinct real numbers either has an increasing subsequence of $m+1$ terms or it has a decreasing subsequence of $n+1$ terms.*

Proof. Let $\sigma = (x_1, x_2, \dots, x_{mn+1})$ be a sequence of $mn+1$ distinct real numbers. For each $i = 1, 2, \dots, mn+1$, let a_i be the maximum number of terms in a increasing subsequence of σ with x_i as the first term. Also, let b_i be the maximum number of terms in a decreasing subsequence of σ with x_i as the last term. If some $a_i \geq m+1$ or $b_i \geq n+1$, we are done, so it remains to consider the case where $a_i \leq m$ and $b_i \leq n$ for all i. Since there are mn different ordered pairs of the form (a, b) with $1 \leq a \leq m, 1 \leq b \leq n$, by the pigeonhole principle that there must be integers $i_1 < i_2$ such that $(a_{i_1}, b_{i_1}) = (a_{i_2}, b_{i_2})$. Since all the numbers are distinct, either $x_{i_1} < x_{i_2}$ or $x_{i_1} > x_{i_2}$.

1. If $x_{i_1} < x_{i_2}$, we can extend any increasing subsequence starting with x_{i_2} by putting x_{i_1} at the start, so $a_{i_1} > a_{i_2}$, a contradiction.

2. If $x_{i_1} > x_{i_2}$, we can extend any decreasing subsequence ending with x_{i_1} by appending x_{i_2} at the end, so $b_{i_2} > b_{i_1}$, a contradiction.

Q.E.D.

Exercise 3.5 (Chapter 1.3.3., Problem 65 in [Bog04])**.** *Show that if p is a prime, $p \neq 2, 5$, then among the first* 51 *powers of p, there is one whose decimal expression ends in* 01*.*

Solution: Since $p \neq 2$, all the powers of p are odd. There are 50 possible two-digit odd endings to a number $2k+1$ for $k = 0, \dots, 49$. So there are two different integers $1 \leq k_1 < k_2 \leq 51$ such that the decimal ending of p^{k_1} and p^{k_2} is the same. Therefore $p^{k_2} - p^{k_2} = p^{k_1}\left(p^{k_2-k_1} - 1\right)$ ends in 00 i.e. it is divisible by 100. The only prime divisors of 100 are 2 and 5, so this means $p^{k_2-k_1} - 1$ is divisible by 100, therefore it ends in 00. Therefore $p^{k_2-k_1}$ ends in 01, and $k_2 - k_1 < 51$.

Definition 3.6. *A **partition** of a set S is a collection $\{P_i\}_{i=1}^n$ of subsets of S such that*

1. *$P_i \neq \emptyset$ for all i,*

2. *$P_i \cap P_j = \emptyset$ for $i \neq j$,*

3. *$\bigcup_{i=1}^n P_i = S$.*

*The P_i are called the **parts** (or blocks) of the partition.*

Note that if $\{P_i\}_{i=1}^n$ is a partition of S, then $\sum_{i=1}^n |P_i| = |S|$.

Theorem 3.7 (Generalized pigeonhole principle)**.** *If S is a set with $|S| > kn$ and $\{P_i\}_{i=1}^n$ is a partition of S, then there exists an i, $1 \leq i \leq n$ such that $|P_i| \geq k+1$.*

Proof. Assume that $|P_i| \leq k$ for all i. Then we have

$$\begin{aligned} |S| &= \sum_{i=1}^n |P_i| \\ &\leq \sum_{i=1}^n k \\ &= kn, \end{aligned}$$

which is a contradiction.

Q.E.D.

Theorem 3.8 (Ramsey's theorem)**.** *Show that in a set of six people, there is a set of at least three people who all know each other, or a set of at least three people none of whom know each other. (We assume that if person 1 knows person 2, then person 2 knows person 1.)*

Proof. Consider person 1. By Theorem 3.7, they either know at least three other people or do not know at least three other people.

If they know three other people, then if any two of the people 1 knows know each other, we are done. Otherwise, the three people 1 knows all do not know each other. The argument is identical if 1 does not know at least 3 people.

Q.E.D.

4 The seven bridges of Königsberg

Figure 1 shows the seven bridges of the historical city of Königsberg in 1736. The citizens of the city were avid

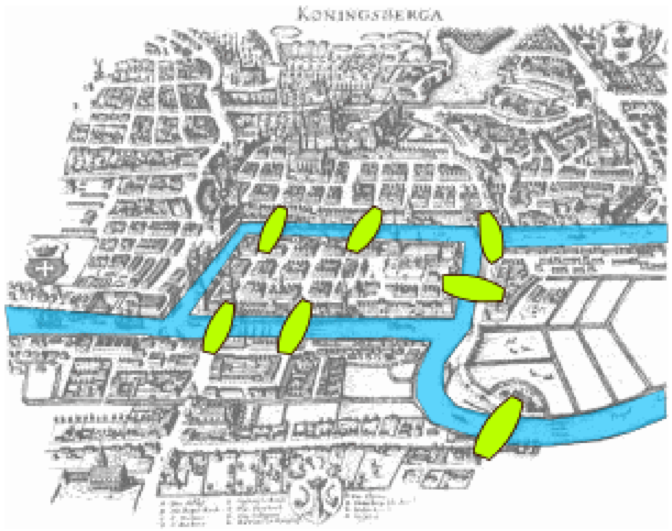


Figure 1: The seven bridges of Königsberg

walkers, and many wondered if it is possible to plan a walk that crosses every bridge of the city exactly once.

Leonhard Euler proved in 1736 that such a walk is impossible. First let's abstract away most of the irrelevant parts of the problem (see Figure 2):

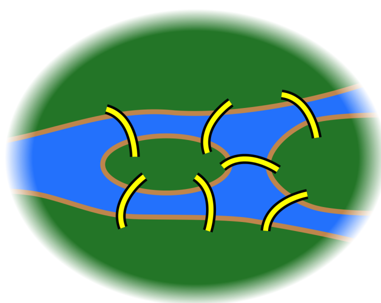


Figure 2: The seven bridges of Königsberg (sketch)

It is not important how we walk within each land mass, so we shrink these to points The structure in Figure 3

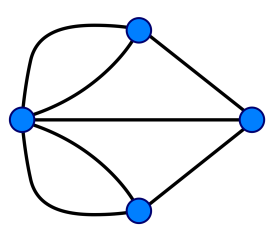


Figure 3: The seven bridges of Königsberg (graph)

is what we'll refer to as a **graph**.

Before getting to precise definitions, let's see Euler's argument for why the walk traversing all seven bridges exactly once is impossible. We'll call the points in Figure 3 **vertices** and the line segments adjacent to the vertices **edges**. What the problem is asking for is a way to traverse all edges exactly once. If, during a walk, we arrive at a vertex and leave it, we use exactly 2 of the edges adjacent to it. Therefore, any vertex visited during the walk must have an even number of edges adjacent to it, except for the start and endpoint. Since all 4 vertices have an odd number of edges adjacent to them, the walk is impossible.

5 Graphs (Chapter 5 in [KT17])

Definition 5.1. *A **graph** G is a pair V, E where V is a set and E is a subset of the 2-element subsets of V.*

Elements of V are the **vertices** and elements of E are the **edges**. Alternatively, we will use the notation $V(G)$ for the vertices and $E(G)$ for the edges of a graph G. We will often suppress the braces indicating that an edge is a 2-subset of vertices, and instead of $\{x, y\} \in E$, we will just write $xy \in E$. Often we will allow multiple edges between the same two vertices (like in the Königsberg bridge problem), the resulting structure is often called a **multigraph**. We will sometimes use the term graph to mean multigraphs, and if we want to emphasize that we are only allowing at most one edge between two vertices we will say that we are considering a **simple graph**. As in the Königsberg bridges, it is often helpful to draw a picture of a graph to try to study it. But since Definition 5.1 does not specify how one should draw this picture, we can end up with very different looking pictures of the same graph, for example, in Figure 4.

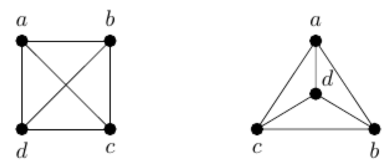


Figure 4: Two drawings of the same graph

6 Graph isomorphisms

You could ask the question

Given two drawings of graphs, how do I know they are drawings of the same graph?

First we should clarify what we mean by *the same graph*.

Definition 6.1. *If $G = (V, E)$ and $H = (W, F)$ are graphs, then we say that G is **isomorphic** to H and write $G \cong H$ if there exists a bijection $f : V \to W$ such that $\{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in F$.*

The question to decide whether two given finite graphs are isomorphic is known as the **Graph isomorphism problem** and it is difficult![1]

Some graphs are easy to identify so they get special names:

Definition 6.2. *The **complete graph** K_n on n vertices is a graph where xy is an edge for all $x, y \in [n]$. The **independent graph** I_n is a graph on n vertices that has no edges.*

Exercise 6.3. *Show that the two graphs in Figure 5 are isomorphic.*

Solution: The isomorphism is given by

$$f(a) = 5, \quad f(b) = 3, \quad f(c) = 1, \quad f(d) = 6, \quad f(e) = 2, \quad f(h) = 4.$$

In many cases, it is easy to see when two graphs *are not* isomorphic.

Exercise 6.4. *Explain why the two graphs in Figure 6 are not isomorphic.*

[1]If you know about algorithms and computational complexity, then this is a more precise statement: The best algorithm to decide graph isomorphism had complexity $O(2^{\sqrt{n \log n}})$ (where n is the number of vertices of the graph), until 2017, when a new algorigthm was announced with complexity $O(2^{((\log n)^3)})$, which is quite close to being polynomial time.

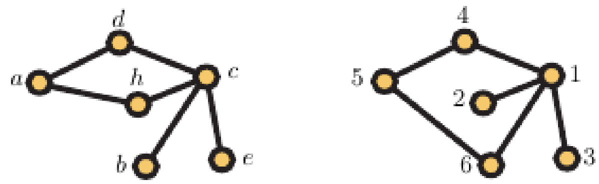


Figure 5: Two isomorphic graphs

Figure 6: Two nonisomorphic graphs

Solution: The graph G has 5 vertices and H has only 3. Isomorphic graphs must have the same number of vertices.

Definition 6.5. *If $G = (V, E)$ and $H = (W, F)$ are graphs we say H is a* ***subgraph*** *of G when $W \subseteq V$ and $F \subseteq E$. We say H is an* ***induced subgraph*** *of G when $W \subseteq V$ and $F = \{\{x, y\} \in E \mid x, y \in W\}$ (see Figure 7 for an example).*

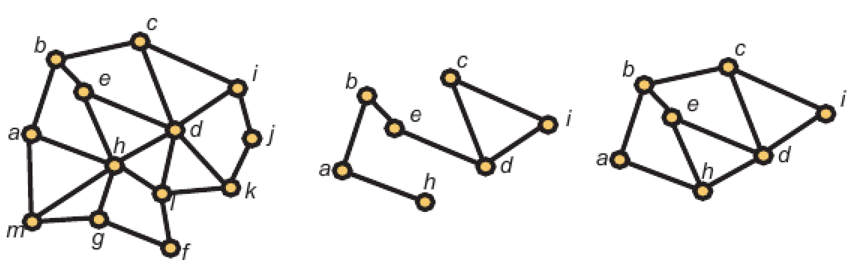


Figure 7: A graph, a subgraph and an induced subgraph

Given two graphs H and G, determining if H is a subgraph of G is a really hard problem.

Definition 6.6. *A sequence $(x_1, \ldots, x_n)$ of vertices in a graph is called a* ***walk*** *when $x_i x_{i+1}$ is an edge. If the vertices in a walk are all distinct, then the walk is called a* ***path****. When $n \geq 3$, a path $(x_1, \ldots, x_n)$ is called a* ***cycle*** *if $x_n x_1$ is also an edge.*

Exercise 6.7. *Explain why the two graphs in Figure 8 are not isomorphic.*

Solution: The graph G contains a 4-cycle as a subgraph, and H does not.

Exercise 6.8. *Explain why the two graphs in Figure 9 are not isomorphic.*

Solution: The vertex c is adjacent to four other vertices in the first graph. There is no vertex in the second graph adjacent to four other vertices.

This leads us to the following definition:

Definition 6.9. *The* ***degree*** *of a vertex v in a graph G, denoted $\deg_G(v)$ is the number of edges incident to it.*

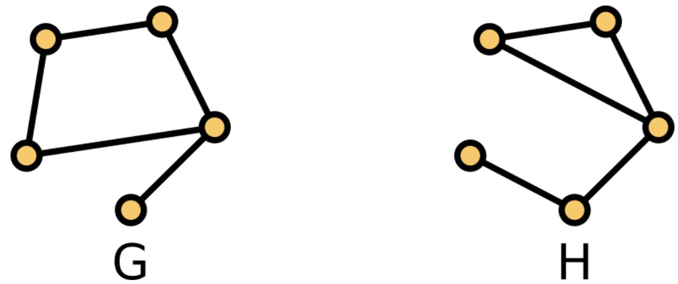


Figure 8: Two nonisomorphic graphs

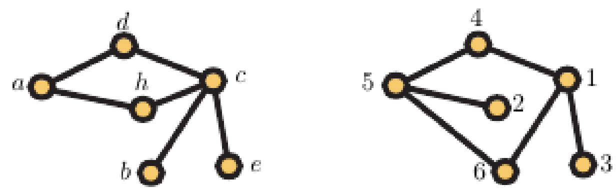


Figure 9: Two nonisomorphic graphs

We could now restate the argument we gave in the solution to Exercise 6.8 like this: For the first graph, $\deg(c) = 4$ and the second graph has no vertex of degree 4.

Theorem 6.10. *Degree is invariant under isomorphism. That is, if $f : G \to H$ is a graph isomorphism, then for any vertex $x \in V(G)$, we have*

$$\deg_G(x) = \deg_H(f(x)).$$

Proof. The number $\deg_G(x)$ is the number of elements in the set of vertices adjacent to x. Let $A = \{y \in V(G) \mid xy \in E(G)\}$. Since f is an isomorphism, we have

$$\{f(y) \in V(H) \mid y \in A\} = \{y' \in V(H) \mid f(x)y' \in V(H)\},$$

therefore

$$\deg_H(f(x)) = |\{y' \in V(H) \mid f(x)y' \in E(H)\}| = |f(A)| = |A| = \deg_G(x).$$

Q.E.D.

This leads us to an easy to check criterion for graph isomorphism

Theorem 6.11. *Two isomorphic graphs must have the same degree sequence. That is, if we list the sequence of the degrees of the vertices of the two graphs in (weakly) decreasing order, the two sequences must be the same.*

Proof. Under a graph isomorphism, any vertex must be mapped to a vertex with the same degree.

Q.E.D.

7 The Handshaking Lemma

If there are many people shaking hands, the total number of hands shaken is twice the number of all handshakes.

Theorem 7.1 (Handshaking Lemma, Theorem 5.1 in [KT17])**.** *Let $\deg_G(v)$ denote the degree of vertex v in a graph $G = (V, E)$. Then*

$$\sum_{v \in V} \deg_G(v) = 2|E|.$$

Proof. We will give a combinatorial proof. For the left hand side at every vertex we count the number of edges incident to that vertex. For the right hand side, notice that this way we counted each edge twice, as every edge is incident to two vertices.

Q.E.D.

Theorem 7.1 seems obvious, but it has at least one useful corollary.

Corollary 7.2 (Corollary 5.2 in [KT17])**.** *Every graph has an even number of vertices of odd degree.*

Proof. The right hand side of the equation in Theorem 7.1 is even, so on the left hand side, we must have an even number of odd numbers.

Q.E.D.

8 Forests and Trees

Definition 8.1. *A graph that contains no cycles is called a* ***forest****. We call a graph* ***connected*** *when there is a path from x to y in G for every pair x, y of vertices. A connected forest is called a* ***tree****.*

Trees have some really important applications, as they are "minimal connected graphs" in some sense.

Definition 8.2. *If $G = (V, E)$ is a graph, then $T = (V, F)$ with $F \subseteq E$ is a* ***spanning tree*** *of G if T is a tree (note that the vertex set of both graphs is the same).*

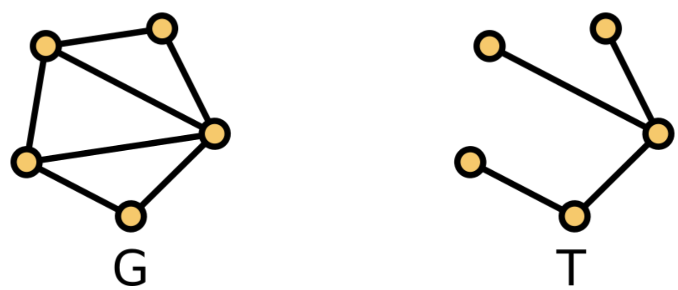


Figure 10: A graph and a spanning tree

Later in the class we will study algorithms to find spanning trees.

9 Eulerian Graphs (Chapter 5.3 in [KT17])

In this section, by "graph", we will mean a multigraph.

Definition 9.1. *A walk on a graph is called an* ***Euler walk*** *if it traverses every edge exactly once. An Euler walk is called an* ***Euler circuit*** *or* ***Euler cycle*** *if the walk finishes at the same vertex where it started.*

In the previous lecture we have seen that a graph can not have an Euler walk if more than two of the edges have odd degrees. A similar argument shows that a graph can only have an Euler circuit if all the vertices have even degrees. It is also clear that in order to have either an Euler walk or Euler circuit, the graph needs to be connected.

Theorem 9.2 (Theorem 5.4. in [KT17])**.** *A connected graph G has an Euler circuit if and only if every vertex has even degree.*

The proof is a bit subtle, and we have to make sure we are doing a rigorous job, but the idea is simple. Since all vertices have an even degree, if we start a walk and traverse edges in some order, we can never get "stuck", since whenever we enter a vertex (hence using up one of the edges incident to it) we can always "leave" (using another edge). So the only place where we can end up without more edges to traverse is the starting vertex. Our path may not be long enough, but the key idea here is that the if we remove the edges we traversed from the graph, the remaining graph still satisfy the property that all degrees are even, so we can repeat the procedure.

Example 9.3. *Before seeing the proof, let's see an example of this idea in action. Consider the graph in figure 11. At first we take an arbitrary path, starting at vertex 1, like in figure 12.*

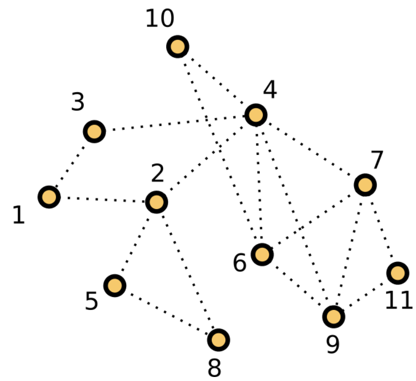
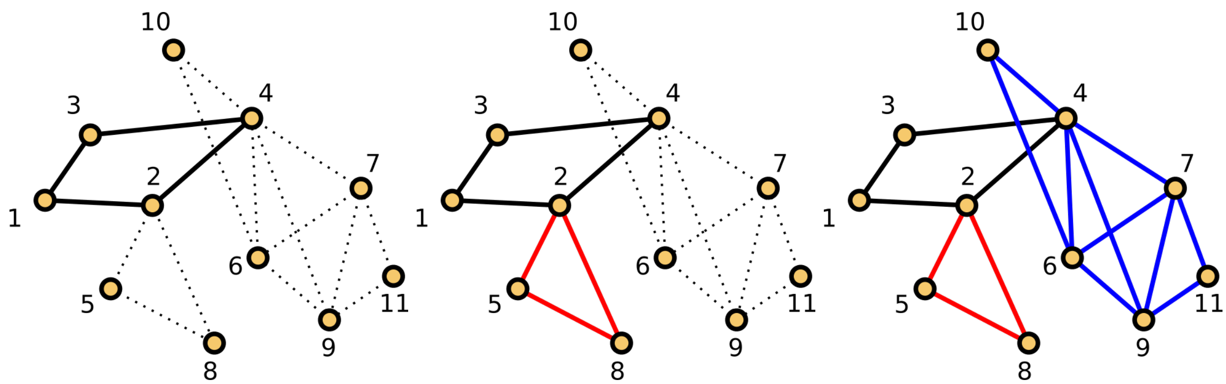


Figure 11: An Eulerian graph



(a) Tracing the first cycle (b) Adding the second cycle (c) Finishing with the third cycle

Figure 12: Finding an Euler circuit

Proof of Theorem 9.2. We already know that the conditions on degrees and connectedness are necessary, so we proceed to prove the converse. Assume that $G = (V, E)$ is a connected graph with every vertex having even degree. We will proceed by induction on the number of edges $n = |E|$. As a base case, we have the graph with one vertex and no edges, this graph has an Euler circuit (the empty walk). Assume for induction that all connected graphs on at most $n-1$ vertices with vertices having even degrees have an Euler circuit. Consider the case where $|E| = n$. Start at any vertex v_0 and traverse edges in any order. Since every vertex has even degree, we can always leave any vertex we entered, so at some point our walk will get us back to v_0. If we traversed all edges, we have found an Euler circuit and we are done. Otherwise, let $W = (v_0, v_1, \ldots v_k = v_0)$ be the walk and let G' be the graph G with the edges from the walk W removed. The graph G' may be disconnected, let its connected components be $G'_1, G'_2, \ldots, G'_m$. Because G is connected, at least one vertex in each of the G'_is appears in W. Let $w_{i,1}$ be a vertex that appears in W and is contained in G'_i. Any component G'_i still only has vertices with even degrees, as for any vertex of G we have removed an even number of edges. Therefore, by the induction hypothesis, G_i has an Euler circuit. Note that if a graph has an Euler circuit starting at some vertex, it will have an Euler circuit starting at any vertex (we can cyclically shift the circuit). Therefore we may assume that G'_i has an Euler circuit of the form

$$(w_{i,1}, w_{i,2}, \ldots, w_{i,k_i} = w_{i,k_i}).$$

Now we will patch these Euler circuits together with W. After possibly reordering the components, we may assume that W contains the starting vertices of the G'_is in the order $w_{1,1}, w_{2,1}, w_{3,1}, \ldots, w_{m,1}$. Define a new walk as follows:

$$\begin{aligned}(v_0, v_1, \ldots v_{i_1} = w_{1,1}, w_{1,2}, \ldots, w_{1,k_1} = v_{i_1}, v_{i_1+1}, \ldots \\ \ldots, v_{i_2} = w_{2,1}, w_{2,2}, \ldots, w_{2,k_2} = v_{i_2}, v_{i_2+1}, \ldots \\ \ldots, v_{i_m} = w_{m,1}, w_{m,2}, \ldots, w_{m,k_m} = v_{i_m}, v_{i_m+1}, \ldots, v_k = v_0).\end{aligned}$$

Our walk now uses every edge in the graph exactly once.

Q.E.D.

Exercise 9.4. *Under what conditions does a graph have an Euler walk (not necessarily a circuit)?*

References

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