MAT344 Week 5

2019/Oct/7

1 This week

This week, we are talking about

1. Hamilton cycles
2. Graph coloring
3. Planar graphs

2 Recap

Last week we talked about

1. Basics of Graphs
2. Eulerian graphs

3 Hamiltonian Graphs (Chapter 5.3 in [KT17])

Definition 3.1. *A **Hamilton path** is a path in the graph that traverses every vertex exactly once. If $v_0, \dots, v_k$ is a Hamilton path and $v_k v_0$ is also an edge in the graph, then we call this Hamilton path a **Hamilton cycle**. A graph that has a Hamilton cycle is said to be **Hamiltonian**.*

Exercise 3.2. *Explain why the graph in Figure 1 has no Hamilton path.*

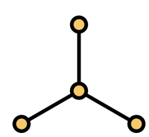

Figure 1: A graph with no Hamilton path

Figure 2 shows the **Petersen graph**, a graph that provides many counterexamples, and a Hamilton path in it.

Exercise 3.3. *Prove that the Petersen graph does not have a Hamilton cycle. (this is not easy!)*

It is very easy to tell when a graph has an Euler circuit (and an Euler walk), one just has to count degrees of vertices. For Hamilton paths and cycles, there is no known easy way of answering the question in general. [1]

However, in some cases we can conclude that the graph has a Hamilton cycle. For example, any complete graph K_n has a Hamilton cycle. Notice that having more edges can never hinder the existence of a Hamilton cycle (unlike in the case of Euler cycles). So most of the Theorems that guarantee the existence of Hamilton cycles are about the graph having "sufficiently many edges".

[1] In the language of complexity theory, finding a Hamilton path on an n-vertex graph is also very difficult, the brute-force check takes $O(n!)$ time if the basic operation is "given a sequence of vertices, check if it is a path in the graph", but the best known algorithm still takes $O(n^2 2^n)$ time.

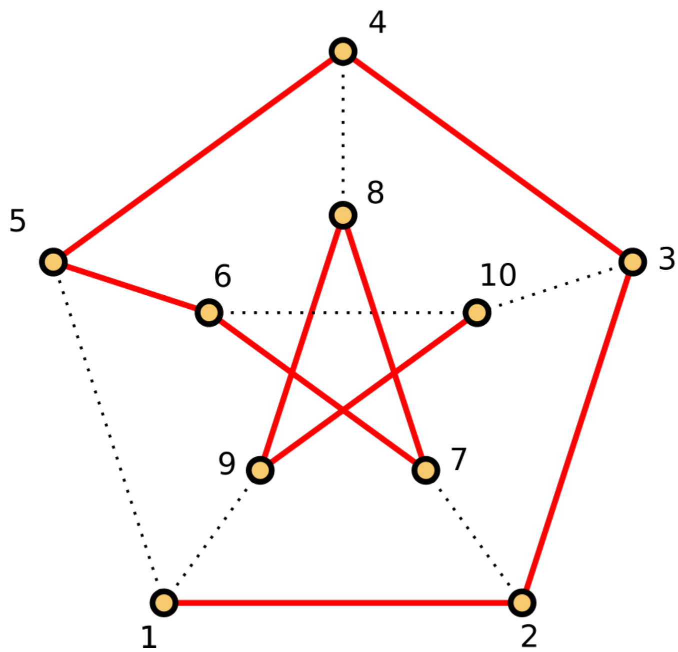

Figure 2: The Petersen graph

Theorem 3.4 (Dirac, 1952, Theorem 5.5. in [KT17]). *If a simple graph G has n vertices with $n \geq 3$ and each vertex v has $\deg_G(v) \geq \lceil \frac{n}{2} \rceil$ then G is Hamiltonian.*

Proof. The proof is tricky, we will proceed in the following steps:

1. We will give a lower bound for the length of the longest path,
2. We will show that we can modify any longest path to a cycle of the same length,
3. Finally we will argue that we can make this cycle into a longer path.

Suppose the Theorem fails, and let n be the smallest positive integer for which there is a graph with each vertex having degree at least $\lceil \frac{n}{2} \rceil$ and there is no Hamiltonian cycle in G. Since the only possible graph on 3 vertices with $\deg_G(v) \geq 2$ is the complete graph, and it has a Hamilton cycle, we may assume $n \geq 4$.

1. Let t be the largest integer for which G has a path $P = (x_1, \ldots, x_t)$ on t vertices. Since the path begins with x_1 and ends with x_t, if any neighbor of x_1 is not already in P, we could attach it to the beginning, resulting in a longer path. So we may assume that all neighbors of x_1 are all in P already. In particular, this shows that $\lceil \frac{n}{2} \rceil < t$. We can also repeat this argument with x_t to argue that all of its neighbors must be in P also.
2. Set up $t - 1 \leq n - 1$ boxes and put the (at least $\lceil \frac{n}{2} \rceil$ many) edges of the form $x_1 x_{i+1}$ into box i, and put the (similarly at least $\lceil \frac{n}{2} \rceil$ many) edges of the form $x_i x_t$ into box i. We have at least n edges to put in at most $n - 1$ boxes, so there is an index i with $1 \leq i < t$ such that both $x_1 x_{i+1}$ and $x_i x_t$ are edges in G. Then we can reverse the end of the path and form

$$C = (x_1, x_2, \ldots, x_i, x_t, x_{t-1}, \ldots, x_{i+2}, x_{i+1}),$$

   which is now a cycle of length t.
3. Since G has no Hamilton cycle, we must have $t < n$, and combining it with our other estimate for t, we get $\lceil \frac{n}{2} \rceil < t < n$. If y is a vertex not contained in C, there must be an x_j adjacent to y, in which case we can form a path in G of length $t + 1$ by starting a path at y, then tracing the cycle from x_j. This contradicts our assumption that t was maximal.

Q.E.D.

A different proof of Theorem 3.4 leads to some other sufficient conditions for the existence of Hamilton cycles. For the proof, see [Mor17], Theorem 13.13 or [Gui18], Theorem 5.3.2.

4 Graph coloring (Chapter 5.4 in [KT17])

Definition 4.1. *If G is a graph and C is a set (the elements of C are often called* colors*), a **proper coloring** of G is a function $f : V(G) \to C$ such that if $xy \in E(G)$, we have $f(x) \neq f(y)$. The smallest integer t for which there is a proper coloring of G with $|C| = t$ is called the **chromatic number** $\chi(G)$ of G. In this case, we say that G is t-colorable.*

Figure 3 shows a 5-coloring of a graph

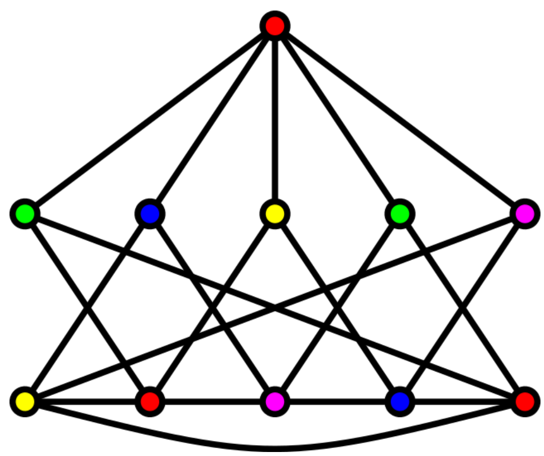

Figure 3: A 5-coloring of a graph

But is this the minimal number of colors we need?

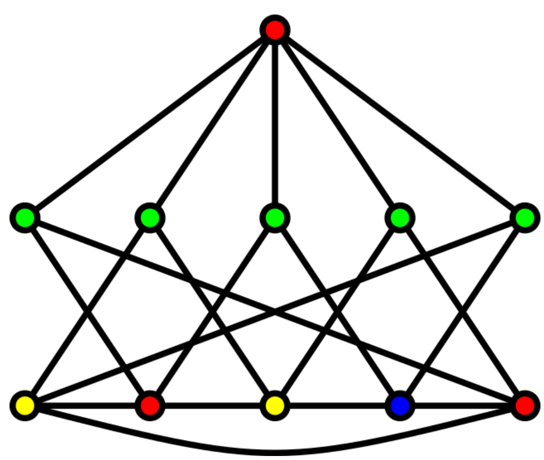

Figure 4: A 4-coloring of the same graph

But is *this* the minimal number of colors? It's not so easy to decide.

Deciding when a graph has a k-coloring is another example of a computationally difficult problem. Like many other difficult problems, in certain special cases it's quite easy.

Theorem 4.2. *A graph is* 1*-colorable if it has no edges.*

Example 4.3. *Prove that a complete graph K_n has $\chi(K_n) = n$.*

Theorem 4.4 (Theorem 5.7 in [KT17])**.** *A graph is* 2*-colorable if and only if it does not contain an odd-length cycle.*

Proof. Clearly the condition is necessary. It suffices to consider connected graphs. Pick a vertex x and define a map $f : V(G) \to [2]$ by the rule

$$f(y) = \begin{cases} 1 \text{ if the shortest path from } x \text{ to } y \text{ is odd} \\ 2 \text{ if the shortest path from } x \text{ to } y \text{ is even} \end{cases}$$

We claim this is a proper coloring. If there are adjacent vertices y and z both colored i (for $i = 1$ or 2), then consider shortest paths $(x = y_0, y_1, \dots, y_{k-1}, y_k = y)$ from x to y and $(x = z_0, z_1, \dots, z_l = z)$ from x to z. Note that if $y_i = z_j$ for some i, j then i must equal j as the paths must be minimal to the vertex $y_i = z_j$. Then $(y_i, y_{i+1}, \dots y_k = y, z = z_l, z_{l-1}, \dots, z_{i+1}, z_i)$ is an odd length cycle.

Q.E.D.

A 2-colorable graph is called a **bipartite** graph. This refers to the idea that we can partition the vertices into 2 subsets A and B such that there are no edges between vertices in A, and no edges between vertices in B. Equivalently, the induced subgraphs of G by A and B are independent, and $A \cup B = V(G)$. This also easily generalizes to n-colorable graphs.

5 Cliques and Chromatic number (Chapter 5.4.2 in [KT17])

Definition 5.1. *A* ***clique*** *in a graph G is a set $K \subseteq V(G)$ such that the subgraph induced by K is the complete graph $K_{|K|}$. The* ***maximum clique size*** *or* ***clique number*** *of a graph G, denoted $\omega(G)$ is the largest t for which there exists a clique K with $|K| = t$.*

Considering example 4.3, we see that

$$\chi(G) \geq \omega(G).$$

But this estimate is not very effective at computing the chromatic number:

Proposition 5.2 (Proposition 5.9 in [KT17])**.** *For every $t \geq 3$, there exists a graph G_t such that $\chi(G_t) = t$ and $\omega(G_t) = 2$.*

6 Planar graphs (Chapter 5.5 in [KT17])

Given a map of certain countries, how many colors do we need to color the map if no two adjacent countries can have the same color? If you try with a couple of maps, you'll notice that four colors seem to be enough. It is also easy to draw a map where 4 colors are necessary, for example, Figure 9

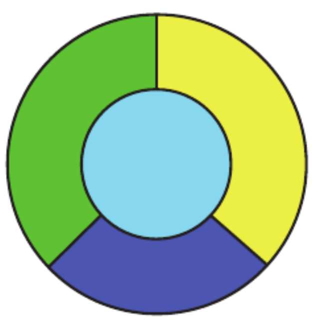

Figure 5: A map that requires four colors

We can easily translate this question into a graph coloring problem by constructing a graph where the vertices represent the countries and there is an edge between two vertices if the two countries share a border, see Figure 6.

How can four colors be enough? Last week we saw that there are graphs with arbitrarily large chromatic number. There must be something special about graphs that we get this way from maps of countries.

Definition 6.1. *A graph if* ***planar*** *if it can be drawn in the plane ($\mathbb{R}^2$) without edges crossing.*

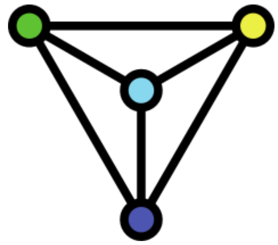

Figure 6: The graph representing the map in Figure 9

Theorem 6.2 (Four color Theorem, Theorem 5.14 in [KT17])**.** *Every planar graph has chromatic number at most four.*

The proof of Theorem 6.2 was a gigantic effort, only finished in 1976, the published paper was pretty much unreadable and it relied on a computer checking many cases, and it contained several flaws (that were later fixed). For more details on this famous Theorem, see [Mor17], section 15C.

Definition 6.1 should give you some discomfort. We defined graphs abstractly, and noticed that it is not so easy to tell when two drawings represent the same graph. Especially if a graph is given to us as a drawing, it may be difficult to say if it is planar or not. It may be possible that this particular drawing has intersecting edges, but if we position the vertices differently we may be able to avoid this.

It also makes it very difficult to prove that a graph *isn't* planar, so we would like to have alternative characterizations of planar graphs.

We already know that certain graphs can't be planar, as Theorem 6.2 implies that if G is planar, then $\chi(G) \leq 4$. So any graph with a chromatic number at least 5 can not be planar. This leads to the following theorem.

Theorem 6.3. *Any graph that contains a copy of K_5 can not be planar.*

Could this be the characterization? Consider the complete bipartite graph $K_{3,3}$ on $3+3$ vertices, shown on Figure 7.

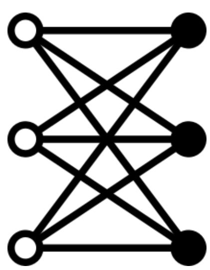

Figure 7: $K_{3,3}$

This graph clearly has $\chi(K_{3,3}) = 2$, but this way of drawing it has a lot of intersections. If we draw the graph slightly differently, for example, as in Figure 8, it does not look that far from being planar.

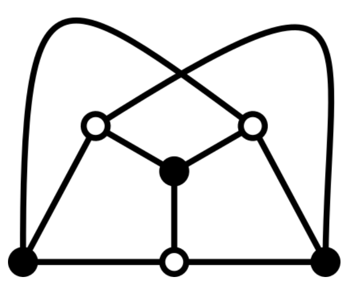

Figure 8: $K_{3,3}$

Exercise 6.4. *Let $\{v_1, v_2, v_3\}$ and $\{u_1, u_2, u_3\}$ be the vertices of the complete bipartite graph $K_{3,3}$. Consider the 4-cycle u_1, v_1, u_2, v_2. This 4-cycle divides the plane into two regions. The remaining two vertices u_3 and v_3 must both lie either inside or outside the 4-cycle, because they are adjacent. In both cases, find a contradiction, thereby showing that $K_{3,3}$ is not planar.*

7 The Euler characteristic

If we want to identify which graphs are planar, we have to look for something that involves the planar drawing. Any drawing of a graph has vertices and edges, but a planar drawing also has

Definition 7.1. *Given a planar drawing of a graph, a **face** of the drawing is a region of the plane bounded by vertices and edges not containing any other vertices and edges.*

What patterns do we notice between the vertices, edges and faces of a planar graph? Let's compute some examples.

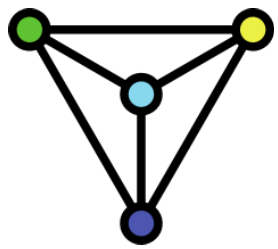

Figure 9: A planar graph

Example 7.2.

For example, the graph on Figure 9 has 4 vertices, 6 edges, and 4 faces (one is the unbouded "outside" face).

The triangle graph K_3 has 3 vertices, 3 edges and 2 faces.

The path graphs P_n have n vertices, $n-1$ edges and one face.

The cycle graphs C_n have n vertices, n edges and 2 faces.

Theorem 7.3 (Euler, Theorem 5.11 in [KT17])**.** *For any planar drawing of a connected graph G, let v, e, f denote the number of vertices, edges and faces in the drawing. Then*

$$v - e + f = 2.$$

To prove the theorem, we need some basic results about graphs:

Theorem 7.4 (Theorem 12.27 in [Mor17])**.** *The following are equivalent for a graph T with n vertices:*

1. *T is a tree.*
2. *T is connected and has $n-1$ edges.*
3. *T has no cycles and has $n-1$ edges.*
4. *T is connected, but deleting any edge leaves a disconnected graph.*

Proof of Theorem 7.3. Let $V(G), E(G)$, and $F(G)$ denote the set of vertices, edges and faces of G. Let $v = |V(G)|, e = |E(G)|, f = |F(G)|$. Since the graph is connected, if it has v vertices, it has at least $v-1$ edges (by Theorem 7.4). We will use induction on $e - v$, with the base case being $e - v = -1$. In this case, G is a tree (using Theorem 7.4 again), and therefore contains no cycles. Therefore the number of regions in the planar drawing is 1, and in this case, we have

$$v - e + f = v - (v-1) + 1 = 2,$$

so the base case holds.

Now assume for induction that any connected planar graph with $e - v < n$ satisfies $v - e + f = 2$.

For the inductive step, suppose that $e - v = n$, with $n > -1$. Then G must have a cycle (using Theorem 7.4 again). Remove one edge from one of the cycles in G. Call the resulting connected graph G'. Then $|V(G')| = |V(G)| = v$, $|E(G')| = |E(G)| - 1 = e - 1$. We want to show that $|F(G')| = |F(G)| - 1 = f - 1$. Since the edge we removed was part of a cycle, it used to separate two regions in the drawing, and now those two regions are joined into one, and the other regions are unchanged, therefore $F(G') = f - 1$. So we have $|E(G')| - |V(G')| = e - 1 - v < n$, so applying the inductive hypothesis for the graph G', we see that

$$\begin{aligned} 2 &= |V(G')| - |E(G')| + |F(G')| \\ &= v - (e - 1) + (f - 1) \\ &= v - e + f \end{aligned}$$

Q.E.D.

References

[Gui18] David Guichard. *Combinatorics and Graph Theory*. Open access, 2018. Available at https://www.whitman.edu/mathematics/cgt_online/book/. 2

[KT17] Mitchel T. Keller and William T. Trotter. *Applied Combinatorics*. Open access, 2017. Available at http://www.rellek.net/appcomb/. 1, 2, 3, 4, 5, 6

[Mor17] Joy Morris. *Combinatorics*. Open access, 2017. Available at http://www.cs.uleth.ca/~morris/Combinatorics/Combinatorics.html. 2, 5, 6