MAT344 Week 8

2019/Oct/27

1 This week

This week, we are talking about

1. Networks
2. The max flow-min cut theorem
3. Inclusion-Exclusion

2 Recap

Last week we talked about

1. Planar graphs
2. Minimal spanning trees

3 Network flows (Ch. 13 in [KT17])

If we want to model flows in a network using graphs, we would like to be able to identify a "direction" of the flow. Or, you might want to model streets in a city as a graph, but some of the streets may be one way streets. This is the motivation for *directed graphs.*

Definition 3.1. *A **directed graph**, or **digraph** G is a pair (V, E) where V is a set (the set of vertices of G) and $E \subseteq V \times V$ is a set of* ordered *pairs (x, y) of vertices of G. If G is a directed graph, sometimes we will refer to the edge set E as the set of **arcs**, to distinguish them from undirected edges.*

When drawing directed graphs, we will indicate the direction by drawing an arrow on the edge. We can also assign weights to the edges of a directed graph. We can also assign weights to the edges of a directed graph the same way we did for ordinary graphs.

We will mainly use directed graphs to study flows in networks. The standard terminology in this area is to refer to the weights as **capacities** and denote them $c(e)$. In most of the examples we will be considering, there will be a **source** vertex s and a **sink, or target** vertex t. All edges incident with the source are oriented away from it and all edges incident with the sink are oriented towards it.

Definition 3.2. *A **flow** in a network is a function f which assigns to each directed edge $e = (x, y)$ a non-negative value $f(e) = f(x, y) \leq c(e)$ such that for every vertex which is neither the source nor the sink,*

$$\sum_x f(x, y) = \sum_x f(y, x),$$

*i.e. the amount leaving y is equal to the amount entering y. The amount of flow leaving the source is called **value** or **strength** of a flow and is denoted by $|f|$.*

Example 3.3. *The flow in Figure 1 illustrates a flow. There are two numbers on each edge, the first is the capacity, the second one is the value of the flow through that edge.*

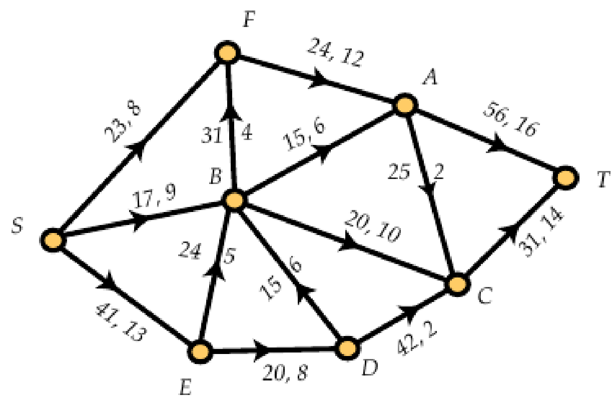


Figure 1: A network flow

Theorem 3.4. *For any flow f in a network G, the total flow out of the source is equal to the total flow into the sink.*

Proof. Consider the sum

$$S = \sum_{x \in V \setminus \{t\}} \left(\sum_{y \in V} f(x, y) - \sum_{y \in V} f(y, x) \right).$$

Since flow is conserved everywhere except the source and the sink, the value

$$\left(\sum_{y \in V} f(x, y) - \sum_{y \in V} f(y, x) \right)$$

is 0 for all x except s and t. In particular,

$$S = \sum_{y \in V} f(s, y),$$

as there is no inflow into the source.

We can rewrite the summation as

$$S = \sum_{x \in V \setminus \{t\}} \sum_{y \in V} f(x, y) - \sum_{x \in V \setminus \{t\}} \sum_{y \in V} f(y, x)$$

every arc $e = (x, y)$ appears in both sums unless $y = t$ (and there are no arcs with $x = t$), so it contributes 0 to the total value. So the entire sum is equal to $\sum_{x \in V \setminus \{t\}} f(x, t)$.

$$\sum_{y \in V} f(s, y) = S = \sum_{y \in V} f(y, t).$$

Q.E.D.

Flows clearly exist in any network, we can just let $f(e) = 0$ for any edge, but this is not a particularly useful flow in applications. One of our objectives is to find a flow of maximum strength through a network. What are some upper bounds we can find for the strength of a flow? Two easy upper bounds are the sum of the capacities of all edges leaving the source, and the sum of the capacities entering the sink.

Definition 3.5. *A **cut** in a network is a pair (X, Y) of disjoint subsets of the vertex set V such that $X \cup Y = V, s \in X, t \in Y$. The **capacity** $c(X, Y)$ of the cut is the sum of the capacities of the edges directed from X to Y (i.e. edges $e = (x, y)$ with $x \in X$ and $y \in Y$).*

Figure 2 shows a cut in a network.

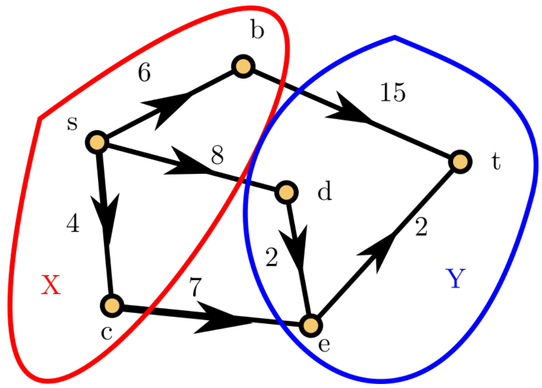


Figure 2: A cut in a network

Proposition 3.6. *The capacity of any cut is an upper bound for the strength of any flow. Moreover, the strength of a flow f can be computed as $f(X, Y) - f(Y, X)$, where $f(A, B)$ denotes the sum of the values of f on all edges directed from A to B.*

So the maximum value of the flow is bounded above by any cut, in particular, a cut of minimal capacity. In the network we are considering, the cut in Figure 3 is minimal.

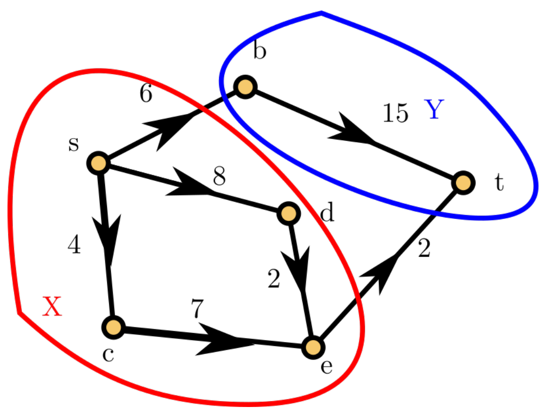


Figure 3: A minimal cut with $c(X, Y) = 8$

And Figure 4 shows a flow with $|f| = 8$.

In this case, we know that the flow in Figure 4 is maximal, because it's equal to the upper bound (the value of the minimal cut).

Theorem 3.7 (Ford-Fulkerson, Theorem 13.8 in [KT17])**.** *Let $G = (V, E)$ be a network. Then let v_0 be the maximum value of a flow, and let c_0 be the minimum capacity of a cut. Then*

$$v_0 = c_0.$$

4 The Marriage Theorem

As an application, we will prove a famous theorem of graph theory using the max flow/min cut theorem. More applications of network flows to combinatorics are in Chapter 14 of [KT17].

Definition 4.1. *Let $G = (V, E)$ $V = X \cup Y$ be a bipartite graph. A* ***matching*** *is a subset $E_1 \subseteq E$ such that no vertex is incident with more than one edge in E_1. A* ***complete matching*** *from X to Y is a matching such that every vertex in X is incident with an edge in E_1.*

For a graph $G = (V, E)$ and a subset $A \subseteq V$ we define $N(A) = \{v \in V | v \text{ is adjacent to a vertex in } A\}$. It's sometimes called the neighborhood of A.

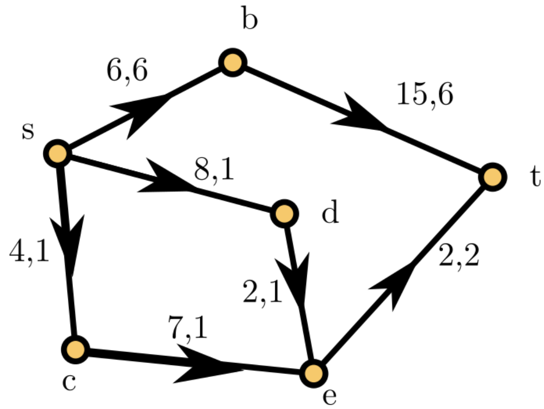


Figure 4: A flow with $|f| = 8$

Theorem 4.2 (Hall's Marriage Theorem, Theorem 14.2 in [KT17])**.** *A bipartite graph $G = (V, E)$ with bipartition $V = X \cup Y$ has a complete matching if and only if for any subset $S \subseteq X$, we have $|S| \leq |N(S)|$.*

Proof. If the condition fails for some subset then clearly a matching is impossible, so the condition is necessary.

To prove that it is also sufficient, consider a network as in Figure 5

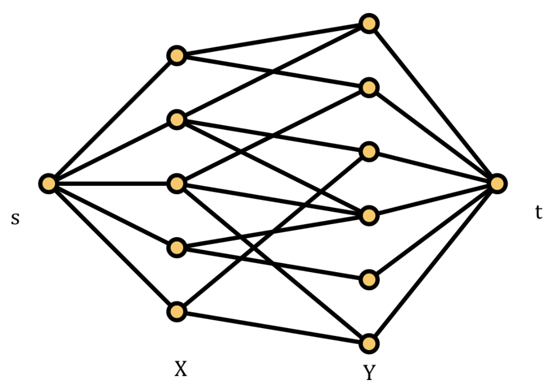


Figure 5: The network from a bipartite graph

Orient all edges from left to right, and assign capacity 1 to all edges from the sink to X, infinity to all edges between X and Y and 1 to all edges from Y to t. A minimum cut in this network will have capacity $\leq |X|$, since the cut $\{s\} \cup (X \cup Y \cup \{t\})$ has capacity $|X|$.

If the minimum cut has capacity $|X|$, then there exists a maximum flow whose values are integers. Since each vertex in $|X|$ has inflow 1 and integral outflow, and each vertex in $|Y|$ has at most 1 outflow, so can have at most 1 inflow, all edges between X and Y that have positive flow must have flow value exactly 1. These edges then define a complete matching from X and Y.

So if there is a flow (necessarily maximal) of strength $|X|$, we have a complete matching. If the maximum flow has strength strictly less than $|X|$, there must be a minimum cut of that capacity. Assume such a minimum cut exists. Since the edges between X and Y have infinite capacity, they can not be a part on any minimum cut. Therefore a minimum cut can not involve any edges that are between X and Y. This means that if a minimum cut contains any subset $A \subseteq X$, it must contain $N(A)$ as well, otherwise it would contain an infinite-capacity edge. By the same argument, there can not be edges from $X \setminus A$ to $N(A)$. Therefore we can compute the capacity of the minimum cut as

$$|X| - |A| + |N(A)|.$$

(see Figure 6 for an example where this fails)

But we assumed that for any subset $A \subseteq X$, $|N(A)| \geq |A|$, so this is a contradiction. Therefore there is no cut with capacity strictly less than $|X|$.

Q.E.D.

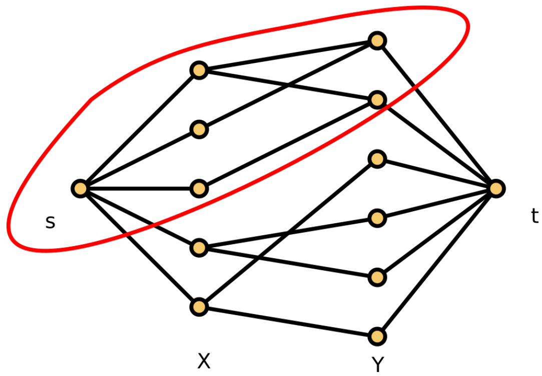


Figure 6: The minimum cut identifying the subset failing the condition

5 Inclusion-Exclusion (Chapter 7 in [KT17])

The first counting technique we discussed was the *addition principle*, which says that if A and B are disjoint sets, then

$$|A \cup B| = |A| + |B|.$$

This is great, but not of much use if the two sets are not disjoint.

If A and B share elements (that is, if $A \cap B \neq \emptyset$), then $A + B$ is an overcount for $|A \cup B|$. We count every element of A once, and every element of B once. This results in us having counted elements both in A and B *twice*, so the exact count is

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Notice how this formula generalizes the addition principle. Inclusion-Exclusion is the generalization of this simple observation to the case when more sets are involved.

Example 5.1. *What happens when we have 3 sets? Let* $A = \{1, 2, 3, 4\}, B = \{2, 4, 5\}, C = \{2, 5, 6\}$. *Then* $|A| = 4, |B| = |C| = 3$ *and*

$$A \cup B \cup C = \{1, 2, 3, 4, 5, 6\},$$

so $|A \cup B \cup C| = 6$. *The count* $|A| + |B| + |C| = 10$ *is an overcount. We counted* $A \cap B = \{2, 4\}$, $A \cap C = \{2\}$ *and* $B \cap C = \{2, 5\}$ *twice. If we subtract the sizes of these pairwise intersections, we get* $10 - 2 - 1 - 2 = 5$, *which is still not correct. So far we counted the triple intersection* $A \cap B \cap C = \{2\}$ *three times (as part of* $A, B,$ *and* C*), then subtracted it three times (as part of* $(A \cap B), (A \cap C)$, *and* $(B \cap C)$. *So we still need to add it back once. Figure 7 shows a figure of a Venn diagram of the situation.*

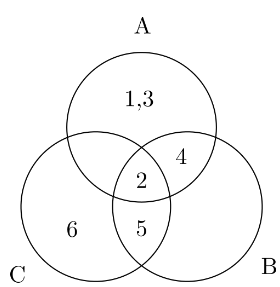
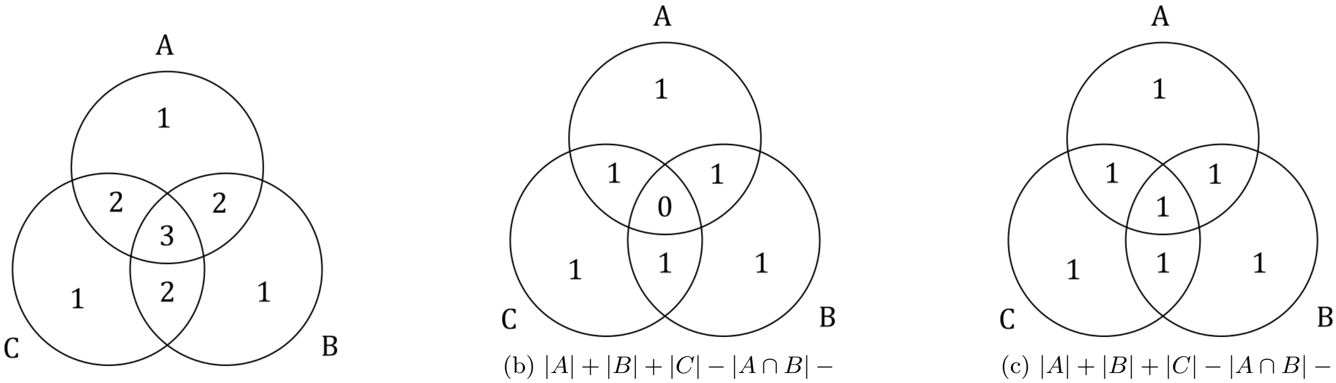


Figure 7: A Venn diagram

This leads us to the following formula

Proposition 5.2 (Inclusion-Exclusion for three sets)**.** *Let* A, B, C *be sets, then*

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$



(a) $|A| + |B| + |C|$

(b) $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$

(c) $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Figure 8: Inclusion-Exclusion on three sets

Proof. Let us draw a Venn diagram (see Figure 8) that illustrates how many times the elements in each intersection are counted. Notice how at each step, elements in one more "layer" of sets are counted correctly (with multiplicity one).

Q.E.D.

Exercise 5.3 (Example 2.1.1 in [Gui18])**.** *Find the number of solutions to the equation*

$$x_1 + x_2 + x_3 = 7$$

with $0 \le x_1 \le 2, 0 \le x_2 \le 4, 0 \le x_3 \le 3$.

Solution: Recall that if we didn't have the upper bounds on the variables, we could model this with a bars and stars computation. We have 7 stars and we want to separate them into 3 piles (so we need 2 bars). The number of solutions would be $\binom{7+3-1}{3-1} = \binom{9}{2}$. We can interpret this number as an overcount, and try to subtract the number of solutions which violate the conditions.

For example, how many of the $\binom{9}{2}$ solutions have $3 \le x_1$? We know how to turn this into a bars and stars problem, and the answer would be $\binom{4+3-1}{3-1} = \binom{6}{2}$. Similarly, $\binom{2+3-1}{3-1}$ of the solutions have $5 \le x_2$, and $\binom{3+3-1}{3-1}$ of the solutions have $4 \le x_2$. If we subtract these we get

$$\binom{9}{2} - \binom{6}{2} - \binom{4}{2} - \binom{5}{2}.$$

What does this number represent? We counted the solutions to $x_1 + x_2 + x_3 = 7$ in nonnegative integers, and subtracted one for each solution where $x_1 \ge 3, x_2 \ge 5$, or $x_3 \ge 4$. Do we have the correct count? We should be careful here, since for example we counted the solution $x_1 = 3, x_2 = 0, x_3 = 4$ once and subtracted it *twice*. First when we considered solutions with $x_1 \ge 3$ and a second time when we considered solutions with $x_3 \ge 4$. There is just one solution with $x_1 \ge 3$ and $x_3 \ge 4$, so we add this back. We should also consider other pairs of variables, but there are no solutions with $x_2 \ge 5$ and $x_1 \ge 3$, and similarly there are no solutions with $x_2 \ge 5$ and $x_3 \ge 4$, (or with all three conditions violated), so the total count is

$$\binom{9}{2} - \binom{6}{2} - \binom{4}{2} - \binom{5}{2} + 1$$

In general, it is useful to think of the above examples as a set X and a family $\mathcal{P} = \{P_1, \ldots, P_m\}$ of **properties**. What we mean by a property is that for every element $x \in X$ and every property P_i, x either satisfies property P_i or it does not. Inclusion-exclusion is about determining the number of elements that satisfy *none* of the properties. For example, in Exercise 5.3, the property P_1 was satisfied by a solution to $x_1 + x_2 + x_3 = 7$ if $x_1 \ge 3$.

Theorem 5.4 (Theorem 2.1.2 and Corollary 2.1.3 in [Gui18], (Principle of Inclusion-Exclusion))**.** *If $A_i \subseteq X$ for $1 \le i \le n$ then*

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n}(-1)^{k+1} \sum_{\{i_1,i_2,\ldots,i_k\}\subseteq[n]} \left|\bigcap_{j=1}^{k} A_{i_j}\right|.$$

Let A_i^c denote the complement (in X) of A_i. Then an alternative way of stating the principule of inclusion-exclusion is

$$\left|\bigcap_{i=1}^{n} A_i^c\right| = |X| + \sum_{k=1}^{n}(-1)^{k} \sum_{\{i_1,i_2,\ldots,i_k\}} \left|\bigcap_{j=1}^{k} A_{i_j}\right|.$$

Proof. We will prove the second formulation (the first one is equivalent). We need to show that every element of $\cap_{i=1}^{n} A_i^c$ is counted once by the right hand side and every other element of X is counted zero times. If $x \in \cap_{i=1}^{n} A_i^c$, then for every A_i, $x \notin A_i$, so x is in none of the sets involving A_i, and is counted exactly once by $|X|$.

If $x \notin A_i^c$, then on the RHS it is counted once by $|X|$, and it is counted for some of the values $\{i_1, i_2, \ldots, i_k\}$, $1 \le m \le k$, if x is not in the remaining sets A_j (for $j \in [n] \setminus \{i_1, i_2, \ldots, i_k\}$). Then x is counted zero times by any term involving A_j with $j \notin \{i_1, i_2, \ldots, i_k\}$, either with a plus or minus sign, by each term involving only the sets $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$. Let's count in how many ways does this happen.

- There is the 1 term $|X|$, resulting in a count of $+1$
- There are k terms of the form $-|A_{i_m}|$, which results in a count of $-k$,
- There are $\binom{k}{2}$ terms of the form $|A_{i_l} \cap A_{i_m}|$, resulting in a count of $+\binom{k}{2}$
- In general, there are $\binom{k}{r}$ terms of the form $(-1)^r|A_{i_{s_1}} \cap A_{i_{s_2}} \cap \ldots A_{i_{s_r}}|$, resulting in a count of $(-1)^r\binom{k}{r}$.

Adding these up, we see that the number of times x is counted on the RHS is

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i},$$

which we know equals zero.

Q.E.D.

Exercise 5.5 (Exercise 10.18. 8) in [Mor17])**.** *At a small university, there are 90 students that are taking either* Calculus or *Linear Algebra (or both). If the Calculus class has 70 students and the Linear Algebra class has 35 students, then how many students are taking both Calculus* and *linear algebra?*

Solution: Let A_1 be the set of students taking Calculus, and A_2 be the set of students taking linear algebra. The problem is asking for $|A_1 \cap A_2|$. From the problem statement, we know that

- $|A_1 \cup A_2| = 90$,
- $|A_1| = 70$,
- $|A_2| = 35$,

Theorem 5.4 tells us that

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|,$$

or, equivalently,

$$|A_1 \cap A_2| = |A_1| + |A_2| - |A_1 \cup A_2| = 70 + 35 - 90 = 15,$$

so, 15 students are taking both Calculus and Linear Algebra.

6 Counting surjections (Chapter 7.3 in [KT17])

Example 6.1 (Example 7.3. in [KT17])**.** *Let m and n be fixed positive integers and let X consist of all functions from $[n]$ to $[m]$. Then for each $i = 1, 2, \ldots, m$ and each function $f \in X$, we say that f is in A_i if there is no j such that $f(j) = i$. In other words, i is not in the image or output of the function f. For example, if $n = 5$ and $m = 3$, then the function f given by*

i	1	2	3	4	5
$f(i)$	2	3	2	2	3

We will use Theorem 5.4. Let X be the set of all functions from $[n]$ to $[m]$. For a subset $S \subseteq [m]$ of size k, we claim that

$$\left| \bigcap_{i \in S} A_i \right| = (m-k)^n.$$

This is true because a function f that is in $\bigcap_{i \in S} A_i$ is a string of length n from an alphabet consisting of $m - k$ letters.

Then by Theorem 5.4, the number $S(n, m)$ of surjections from $[n]$ to $[m]$ is

Theorem 6.2 (Theorem 7.8. in [KT17])**.**

$$S(n,m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n.$$

(you might recognize this formula from the Tutorials this week)

Exercise 6.3 (Chapter 7, Exercise 15 in [KT17])**.** *A teacher has 10 books (all different) that she wants to distribute to four students, ensuring that each of them gets at least one book. In how many ways can she do this?*

Note: You may think that we could answer this by first giving one book to each one of them (there are $10 \cdot 9 \cdot 8 \cdot 7$ ways of doing this, and then distributing the rest in 4^6 ways. But this is an overcount, and it is not so easy to see how much we overcounted each case by.

Solution: This is equivalent to counting surjections from [10] to [4], so the answer is

$$S(10,4) = \sum_{k=0}^{4} (-1)^k \binom{4}{k} (4-k)^{10} = 4^{10} - 4 \cdot 3^{10} + 6 \cdot 2^{10} - 4.$$

7 Counting derangements (Chapter 7.4 in [KT17])

Example 7.1 (Example 7.4. in [KT17])**.** *Let m be a fixed positive integer and let X consist of all bijections from $[m]$ to $[m]$. Note that these are in bijection with permutations of $[m]$. Let a permutation σ be in A_i if $\sigma(i) = i$. A permutation in $\bigcap A_i^c$ is a **derangement**. For example, the permutation σ is a derangement, while π is not*

i	1	2	3	4
$\sigma(i)$	2	4	1	3

i	1	2	3	4
$\pi(i)$	2	4	3	1

For a k-element subset $S \subseteq [n]$, we claim that

$$\left| \bigcap_{i \in S} A_i \right| = (n-k)!.$$

This is true because we have to keep $\sigma(i) = i$ for $i \in S$, and the remaining $(n - k)$ elements can be permuted arbitrarily. Similarly to 6.2, this leads to

Theorem 7.2 (Theorem 7.10. in [KT17])**.** *For each positive integer n, the number d_n of derangements of $[n]$ is*

$$d_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!.$$

Exercise 7.3 (Theorem 7.11. in [KT17])**.** *n men wearing top hats go to a ball. They check in their top hats with a Hat Check person. Later in the evening, the mischeivous hat check person decides to mix up the hats randomly. What is the probability that all n men receive a hat other than their own? Find the limit as $n \to \infty$.*

Solution: The hats can be redistributed in $n!$ ways, so we are looking for the number

$$\begin{aligned}
\frac{d_n}{n!} &= \frac{\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!}{n!} && (1)\\
&= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(n-k)!}{n!} && (2)\\
&= (-1)^k \frac{1}{k!}. && (3)
\end{aligned}$$

and this is the Taylor series expansion of e^x evaluated at $x = -1$. So the answer is

$$\lim_{n\to\infty} \frac{d_n}{n!} = \frac{1}{e}.$$

References

[Gui18] David Guichard. *Combinatorics and Graph Theory.* Open access, 2018. Available at https://www.whitman.edu/mathematics/cgt_online/book/. 6, 7

[KT17] Mitchel T. Keller and William T. Trotter. *Applied Combinatorics.* Open access, 2017. Available at http://www.rellek.net/appcomb/. 1, 3, 4, 5, 8, 9

[Mor17] Joy Morris. *Combinatorics.* Open access, 2017. Available at http://www.cs.uleth.ca/~morris/Combinatorics/Combinatorics.html. 7